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Autor: Feller, William
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4. OTHER CONDITIONS

By theorem 1 both U and U_p vary dominatedly whenever $\underline{r} > 0$ and $\bar{r} < \infty$. The next theorem gives even simpler criteria for dominated variation that remain applicable in the limiting situations $\underline{r} = 0$ and $\bar{r} = \infty$.

THEOREM 3. *If there exists a number $\xi > 1$ such that*

$$(4.1) \quad \liminf \frac{U(\xi t)}{U(t)} > 1$$

then U_p varies dominatedly. Similarly the relation

$$(4.2) \quad \liminf \frac{U_p(t/\xi)}{U_p(t)} > 1$$

implies the dominated variation of U .

PROOF. Clearly

$$(4.3) \quad \frac{t^p U_p(t)}{U(t)} \geq \frac{t^p [U_p(t) - U_p(t\xi)]}{U(t)} \geq \xi^{-p} \frac{U(t\xi) - U(t)}{U(t)}.$$

When the right side is bounded away from 0 this implies $\underline{r} > 0$, and so U_p varies dominatedly by theorem 1.

We can go a step further. If, besides (4.1), it is known that U varies dominatedly with exponent $\gamma < p$, then $t^{-p} U(t)/U_p(t)$ is bounded away from 0, and hence the second inequality in (4.3) implies that

$$(4.4) \quad \liminf \frac{U_p(t) - U_p(t\xi)}{U_p(t)} > 0.$$

This is equivalent to (4.2).

Similarly

$$(4.5) \quad \begin{aligned} \frac{U_p(t) - U_p(t\xi)}{U_p(t\xi)} &\leq t^{-p} \frac{U(t\xi) - U(t)}{U_p(t\xi)} = \\ &= \frac{\xi^p}{R_U(t\xi)} \frac{U(t\xi) - U(t)}{U(t\xi)}. \end{aligned}$$

The second fraction on the right does not exceed 1, and so (4.2) ensures that $R_U(t\xi)$ remains bounded, and hence that U is of dominated variation.

Again, if it is known that R_U is bounded away from 0 then (4.5) shows that (4.2) implies (4.1).

We have thus proved the

COROLLARY. *If U is of dominated variation with exponent $\gamma < p$ then (4.1) implies (4.2). Similarly, if U_p is of dominated variation with exponent $-q$ where $q < p$, then (4.2) entails (4.1). (In each case both functions are of dominated variation.)*

5. RATIO LIMIT THEOREMS

Let U and V be non-decreasing unbounded functions, and suppose that L is slowly varying (= regularly varying with exponent 0).

DEFINITION. *We shall say that U and V are L -equivalent and write*

$$(5.1) \quad V \leftrightarrow UL$$

if the ratio UL/V tends to 1 at all points of continuity.

More precisely, it is required that for each $\varepsilon > 0$ and fixed $\lambda > 1$

$$(5.2) \quad (1 - \varepsilon) L(t) U(t/\lambda) \leq V(t) \leq (1 + \varepsilon) L(t) U(t\lambda)$$

for all t sufficiently large.

THEOREM 4. *Let U be of dominated variation. In order that there exist a slowly varying function L such that (5.1) holds it is necessary and sufficient that*

$$(5.3) \quad R_U(t) - R_V(t) \rightarrow 0 \quad \text{boundedly.}$$

Needless to say, R_V and \mathcal{J}_V are defined by analogy with R_U in (1.5) and \mathcal{J}_U in (3.2).

PROOF. (a) *Necessity.* Assume (5.1) and suppose that U satisfies the basic inequality (2.2). Obviously the slow variation of L implies that for t sufficiently large and all $x > 1$

$$(5.4) \quad \frac{V(tx)}{V(t)} < C' x^{\gamma'}$$

for any pair of constants $C' > C$ and $\gamma' > \gamma$. Thus V is of dominated variation, and since $p > \gamma$ the function V_p exists.