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## 4. Other conditions

By theorem 1 both $U$ and $U_p$ vary dominatedly whenever $\underline{r}>0$ and $\bar{r}<\infty$. The next theorem gives even simpler criteria for dominated variation that remain applicable in the limiting situations $\underline{r}=0$ and $\bar{r}=\infty$.

Theorem 3. *If there exists a number $\xi>1$ such that*

$$\text{(4.1)} \qquad \liminf \frac{U(\xi t)}{U(t)} > 1$$

*then $U_p$ varies dominatedly. Similarly the relation*

$$\text{(4.2)} \qquad \liminf \frac{U_p(t/\xi)}{U_p(t)} > 1$$

*implies the dominated variation of $U$.*

Proof. Clearly

$$\text{(4.3)} \qquad \frac{t^p U_p(t)}{U(t)} \geqslant \frac{t^p\left[U_p(t)-U_p(t\xi)\right]}{U(t)} \geqslant \xi^{-p}\,\frac{U(t\xi)-U(t)}{U(t)}\,.$$

When the right side is bounded away from 0 this implies $\underline{r}>0$, and so $U_p$ varies dominatedly by theorem 1.

We can go a step further. If, besides (4.1), it is known that $U$ varies dominatedly with exponent $\gamma<p$, then $t^{-p}\,U(t)/U_p(t)$ is bounded away from 0, and hence the second inequality in (4.3) implies that

$$\text{(4.4)} \qquad \liminf \frac{U_p(t)-U_p(t\xi)}{U_p(t)} > 0\,.$$

This is equivalent to (4.2).

Similarly

$$\text{(4.5)} \qquad \frac{U_p(t)-U_p(t\xi)}{U_p(t\xi)} \leqslant t^{-p}\,\frac{U(t\xi)-U(t)}{U_p(t\xi)} = \\ = \frac{\xi^p}{R_U(t\xi)}\,\frac{U(t\xi)-U(t)}{U(t\xi)}\,.$$

The second fraction on the right does not exceed 1, and so (4.2) ensures that $R_U(t\xi)$ remains bounded, and hence that $U$ is of dominated variation.

Again, if it is known that $R_U$ is bounded away from 0 then (4.5) shows that (4.2) implies (4.1).

We have thus proved the

COROLLARY. *If $U$ is of dominated variation with exponent $\gamma<p$ then (4.1) implies (4.2). Similarly, if $U_p$ is of dominated variation with exponent $-q$ where $q<p$, then (4.2) entails (4.1).* (In each case both functions are of dominated variation.)

## 5. RATIO LIMIT THEOREMS

Let $U$ and $V$ be non-decreasing unbounded functions, and suppose that $L$ is slowly varying (= regularly varying with exponent 0).

DEFINITION. *We shall say that $U$ and $V$ are $L$-equivalent and write*

$$V \leftrightarrow UL \tag{5.1}$$

*if the ratio $UL/V$ tends to 1 at all points of continuity.*

More precisely, it is required that for each $\varepsilon>0$ and fixed $\lambda>1$

$$(1-\varepsilon)\, L(t)\, U(t/\lambda) \leqslant V(t) \leqslant (1+\varepsilon)\, L(t)\, U(t\lambda) \tag{5.2}$$

for all $t$ sufficiently large.

THEOREM 4. *Let $U$ be of dominated variation. In order that there exist a slowly varying function* L *such that (5.1) holds it is necessary and sufficient that*

$$R_U(t) - R_V(t) \to 0 \qquad \textit{boundedly.} \tag{5.3}$$

Needless to say, $R_V$ and $\mathcal{J}_V$ are defined by analogy with $R_U$ in (1.5) and $\mathcal{J}_U$ in (3.2).

PROOF. (a) *Necessity.* Assume (5.1) and suppose that $U$ satisfies the basic inequality (2.2). Obviously the slow variation of $L$ implies that for $t$ sufficiently large and all $x>1$

$$\frac{V(tx)}{V(t)} < C'\, x^{\gamma'} \tag{5.4}$$

for any pair of constants $C'>C$ and $\gamma'>\gamma$. Thus $V$ is of dominated variation, and since $p>\gamma$ the function $V_p$ exists.