Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 15 (1969)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ONE-SIDED ANALOGUES OF KARAMATA's REGULAR VARIATION
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Kapitel: 3. One-sided version of the Karamata relations
DOI: https://doi.org/10.5169/seals-43209

3. One-sided version of the Karamata relations

From now on U will stand for a non-decreasing function and $p>0$ will be a fixed number such that the integral U_p converges. Only the case $U(\infty) = \infty$ is of practical interest. We adhere to the notation (1.5) for R_U and put

$$(3.1) \qquad \underline{r} = \liminf R_U(t), \qquad \bar{r} = \limsup R_U(t).$$

We shall also use the notation

$$(3.2) \qquad \mathscr{I}_U(t) = \int_t^\infty y^{-p-1} U(y)\, dy.$$

Theorem 1. *For U to vary dominatedly it is necessary and sufficient that $\bar{r}<\infty$. Similarly U_p varies dominatedly iff $\underline{r}>0$.*

More precisely: The relation (2.2) with $\gamma<p$ entails

$$(3.3) \qquad R_U(t) \leqslant A \qquad t > t_0$$

with

$$(3.4) \qquad A = \frac{Cp}{p-\gamma} - 1.$$

Conversely, (3.3) implies (2.2) with

$$(3.5) \qquad C = A + 1, \qquad \gamma = \frac{A}{A+1} p.$$

In like manner, if

$$(3.6) \qquad R_U(t) \geqslant \eta > 0, \qquad t > t_0$$

then

$$(3.7) \qquad \frac{U_p(tx)}{U_p(t)} \geqslant K x^{-q}, \qquad x > 1,\ t > t_0$$

with

$$(3.8) \qquad K = \frac{\eta}{\eta+1}, \qquad q = \frac{p}{\eta+1}.$$

Conversely, if (3.7) holds with $q<p$ then

$$r \geq \frac{K(p-q)}{Kq + (1-K)p}. \tag{3.9}$$

(Note that necessarily $K \leq 1$ as can be seen letting $x \to 1$ in (3.7). On replacing t by tx^{-1} it is seen that (3.7) not only asserts dominated variation of U_p, but implies uniformity away from the origin.)

PROOF. (i) Using integration by parts and the notation (3.2) it is seen that the definition (1.2) of U_p leads to the identity

$$p\mathcal{I}_U(t) = U_p(t) + t^{-p}U(t) \tag{3.10}$$

valid at all points of continuity. If (2.2) holds with $\gamma<p$ we conclude for $t>t_0$

$$U_p(t) + t^{-p}U(t) \leq Cp \cdot U(t)\int_t^\infty y^{-p-1}(y/t)^\gamma\,dy = \tag{3.11}$$

$$= C\frac{p}{p-\gamma}t^{-p}U(t)$$

and so (3.3) holds with A defined in (3.4).

(ii) Assume (3.3). Then by (3.10)

$$pt^p\mathcal{I}_U(t) \leq (A+1)\,U(t) \tag{3.12}$$

or

$$\frac{s^{-p-1}U(s)}{\mathcal{I}_U(s)} \geq \frac{p}{A+1}\cdot\frac{1}{s} \qquad s > t_0\,. \tag{3.13}$$

Integrating between t and $tx>t$ we get

$$\log\frac{\mathcal{I}_U(t)}{\mathcal{I}_U(tx)} \geq \frac{p}{A+1}\log x\,, \qquad t > t_0\,. \tag{3.14}$$

Thus from (3.12)

$$(A+1)\,t^{-p}U(t) \geq p\mathcal{I}_U(t) \geq p\mathcal{I}_U(tx)\cdot x^{p/(A+1)} \tag{3.15}$$

and by the definition (3.2)

$$p\mathcal{I}_U(tx) \geq U(tx)\cdot(tx)^{-p}\,. \tag{3.16}$$

Accordingly, (2.2) holds with C and γ given in (3.5). (This part of the theorem was proved slightly differently in [2].)

(iii) Assume (3.6). As in the last part we conclude

$$\log \frac{\mathscr{I}_U(t)}{\mathscr{I}_U(tx)} \leqslant \frac{p}{\eta+1} \log x\,, \qquad x > 1\,,\ t > t_0\,. \tag{3.17}$$

A repeated use of (3.10) now shows that

$$\begin{aligned} U_p(t) &\leqslant p\mathscr{I}_U(t) \leqslant p\mathscr{I}_U(tx)\cdot x^{p/(\eta+1)} = \\ &= x^{p/(\eta+1)}\left[U_p(tx) + (tx)^{-p}\, U(tx)\right]. \end{aligned} \tag{3.18}$$

From (3.6) with t replaced by tx it is seen that the expression within brackets is $<(1+\eta^{-1})\, U_p(tx)$, and so the assertion concerning (3.7) is true.

(iv) Assume (3.7) with $q<p$. From the definition (1.2) of U_p we get by Fubini's theorem

$$p\int_0^t y^{p-1}\, U_p(y)\, dy = U(t) + t^p\, U_p(t) \tag{3.19}$$

which proves that the integral on the left converges for all $t>0$. Let B stand for the value of the left side when $t = t_0$. For $y>t_0$ we can apply (3.7) to conclude

$$\begin{aligned} U(t) + t^p\, U_p(t) &\leqslant B + pK^{-1}\, U_p(t) \int_{t_0}^t y^{p-1}\, (t/y)^q\, dy < \\ &< B + \frac{p}{p-q}\, K^{-1}\, t^p\, U_p(t)\,. \end{aligned} \tag{3.20}$$

Divide this inequality by $U(t)$ and let $t\to\infty$. If $U(t)\to\infty$ we get the assertion (3.8). If $U(t)$ remains bounded there is nothing to be proved because (3.7) implies that $t^p\, U_p(t)$ increases at least as fast as t^{p-q}, and hence $\underline{r} = \infty$ whenever U is bounded.

NOTE. Our result lack the perfect symmetry of the original Karamata relations. Starting from (2.2) we get (3.3)-(3.4). However, when we apply the converse with these given values we get (2.2) in the weaker form with γ replaced by a constant $\gamma' > \gamma$. Examples given in [2] show that, in an obvious sense, this result is the best possible.