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3. ONE-SIDED VERSION OF THE KARAMATA RELATIONS

From now on U will stand for a non-decreasing function and $p > 0$ will be a fixed number such that the integral U_p converges. Only the case $U(\infty) = \infty$ is of practical interest. We adhere to the notation (1.5) for R_U and put

$$(3.1) \quad \underline{r} = \liminf R_U(t), \quad \bar{r} = \limsup R_U(t).$$

We shall also use the notation

$$(3.2) \quad \mathcal{I}_U(t) = \int_t^{\infty} y^{-p-1} U(y) dy.$$

THEOREM 1. *For U to vary dominatedly it is necessary and sufficient that $\bar{r} < \infty$. Similarly U_p varies dominatedly iff $\underline{r} > 0$.*

More precisely: The relation (2.2) with $\gamma < p$ entails

$$(3.3) \quad R_U(t) \leq A \quad t > t_0$$

with

$$(3.4) \quad A = \frac{Cp}{p-\gamma} - 1.$$

Conversely, (3.3) implies (2.2) with

$$(3.5) \quad C = A + 1, \quad \gamma = \frac{A}{A+1}p.$$

In like manner, if

$$(3.6) \quad R_U(t) \geq \eta > 0, \quad t > t_0$$

then

$$(3.7) \quad \frac{U_p(tx)}{U_p(t)} \geq K x^{-q}, \quad x > 1, \quad t > t_0$$

with

$$(3.8) \quad K = \frac{\eta}{\eta + 1}, \quad q = \frac{p}{\eta + 1}.$$

Conversely, if (3.7) holds with $q < p$ then

$$(3.9) \quad r \geq \frac{K(p-q)}{Kq + (1-K)p}.$$

(Note that necessarily $K \leq 1$ as can be seen letting $x \rightarrow 1$ in (3.7). On replacing t by tx^{-1} it is seen that (3.7) not only asserts dominated variation of U_p , but implies uniformity away from the origin.)

PROOF. (i) Using integration by parts and the notation (3.2) it is seen that the definition (1.2) of U_p leads to the identity

$$(3.10) \quad p \mathcal{J}_U(t) = U_p(t) + t^{-p} U(t)$$

valid at all points of continuity. If (2.2) holds with $\gamma < p$ we conclude for $t > t_0$

$$(3.11) \quad U_p(t) + t^{-p} U(t) \leq Cp \cdot U(t) \int_t^\infty y^{-p-1} (y/t)^\gamma dy = \\ = C \frac{p}{p-\gamma} t^{-p} U(t)$$

and so (3.3) holds with A defined in (3.4).

(ii) Assume (3.3). Then by (3.10)

$$(3.12) \quad pt^p \mathcal{J}_U(t) \leq (A+1) U(t)$$

or

$$(3.13) \quad \frac{s^{-p-1} U(s)}{\mathcal{J}_U(s)} \geq \frac{p}{A+1} \cdot \frac{1}{s} \quad s > t_0.$$

Integrating between t and $tx > t$ we get

$$(3.14) \quad \log \frac{\mathcal{J}_U(t)}{\mathcal{J}_U(tx)} \geq \frac{p}{A+1} \log x, \quad t > t_0.$$

Thus from (3.12)

$$(3.15) \quad (A+1) t^{-p} U(t) \geq p \mathcal{J}_U(t) \geq p \mathcal{J}_U(tx) \cdot x^{p/(A+1)}$$

and by the definition (3.2)

$$(3.16) \quad p \mathcal{J}_U(tx) \geq U(tx) \cdot (tx)^{-p}.$$

Accordingly, (2.2) holds with C and γ given in (3.5). (This part of the theorem was proved slightly differently in [2].)

(iii) Assume (3.6). As in the last part we conclude

$$(3.17) \quad \log \frac{\mathcal{J}_U(t)}{\mathcal{J}_U(tx)} \leq \frac{p}{\eta + 1} \log x, \quad x > 1, t > t_0.$$

A repeated use of (3.10) now shows that

$$(3.18) \quad \begin{aligned} U_p(t) &\leq p\mathcal{J}_U(t) \leq p\mathcal{J}_U(tx) \cdot x^{p/(\eta+1)} = \\ &= x^{p/(\eta+1)} [U_p(tx) + (tx)^{-p} U(tx)]. \end{aligned}$$

From (3.6) with t replaced by tx it is seen that the expression within brackets is $< (1 + \eta^{-1}) U_p(tx)$, and so the assertion concerning (3.7) is true.

(iv) Assume (3.7) with $q < p$. From the definition (1.2) of U_p we get by Fubini's theorem

$$(3.19) \quad p \int_0^t y^{p-1} U_p(y) dy = U(t) + t^p U_p(t)$$

which proves that the integral on the left converges for all $t > 0$. Let B stand for the value of the left side when $t = t_0$. For $y > t_0$ we can apply (3.7) to conclude

$$(3.20) \quad \begin{aligned} U(t) + t^p U_p(t) &\leq B + pK^{-1} U_p(t) \int_{t_0}^t y^{p-1} (t/y)^q dy < \\ &< B + \frac{p}{p-q} K^{-1} t^p U_p(t). \end{aligned}$$

Divide this inequality by $U(t)$ and let $t \rightarrow \infty$. If $U(t) \rightarrow \infty$ we get the assertion (3.8). If $U(t)$ remains bounded there is nothing to be proved because (3.7) implies that $t^p U_p(t)$ increases at least as fast as t^{p-q} , and hence $\underline{r} = \infty$ whenever U is bounded.

NOTE. Our result lack the perfect symmetry of the original Karamata relations. Starting from (2.2) we get (3.3)-(3.4). However, when we apply the converse with these given values we get (2.2) in the weaker form with γ replaced by a constant $\gamma' > \gamma$. Examples given in [2] show that, in an obvious sense, this result is the best possible.