

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	15 (1969)
Heft:	1: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	ONE-SIDED ANALOGUES OF KARAMATA's REGULAR VARIATION
Autor:	Feller, William
Kapitel:	3. One-sided version of the Karamata relations
DOI:	https://doi.org/10.5169/seals-43209

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 08.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

3. ONE-SIDED VERSION OF THE KARAMATA RELATIONS

From now on U will stand for a non-decreasing function and $p > 0$ will be a fixed number such that the integral U_p converges. Only the case $U(\infty) = \infty$ is of practical interest. We adhere to the notation (1.5) for R_U and put

$$(3.1) \quad \underline{r} = \liminf R_U(t), \quad \bar{r} = \limsup R_U(t).$$

We shall also use the notation

$$(3.2) \quad \mathcal{I}_U(t) = \int_t^{\infty} y^{-p-1} U(y) dy.$$

THEOREM 1. *For U to vary dominatedly it is necessary and sufficient that $\bar{r} < \infty$. Similarly U_p varies dominatedly iff $\underline{r} > 0$.*

More precisely: The relation (2.2) with $\gamma < p$ entails

$$(3.3) \quad R_U(t) \leq A \quad t > t_0$$

with

$$(3.4) \quad A = \frac{Cp}{p-\gamma} - 1.$$

Conversely, (3.3) implies (2.2) with

$$(3.5) \quad C = A + 1, \quad \gamma = \frac{A}{A+1} p.$$

In like manner, if

$$(3.6) \quad R_U(t) \geq \eta > 0, \quad t > t_0$$

then

$$(3.7) \quad \frac{U_p(tx)}{U_p(t)} \geq K x^{-q}, \quad x > 1, \quad t > t_0$$

with

$$(3.8) \quad K = \frac{\eta}{\eta+1}, \quad q = \frac{p}{\eta+1}.$$

Conversely, if (3.7) holds with $q < p$ then

$$(3.9) \quad r \geqslant \frac{K(p-q)}{Kq + (1-K)p}.$$

(Note that necessarily $K \leqslant 1$ as can be seen letting $x \rightarrow 1$ in (3.7). On replacing t by tx^{-1} it is seen that (3.7) not only asserts dominated variation of U_p , but implies uniformity away from the origin.)

PROOF. (i) Using integration by parts and the notation (3.2) it is seen that the definition (1.2) of U_p leads to the identity

$$(3.10) \quad p \mathcal{I}_U(t) = U_p(t) + t^{-p} U(t)$$

valid at all points of continuity. If (2.2) holds with $\gamma < p$ we conclude for $t > t_0$

$$(3.11) \quad U_p(t) + t^{-p} U(t) \leqslant Cp \cdot U(t) \int_t^\infty y^{-p-1} (y/t)^\gamma dy = \\ = C \frac{p}{p-\gamma} t^{-p} U(t)$$

and so (3.3) holds with A defined in (3.4).

(ii) Assume (3.3). Then by (3.10)

$$(3.12) \quad pt^p \mathcal{I}_U(t) \leqslant (A+1) U(t)$$

or

$$(3.13) \quad \frac{s^{-p-1} U(s)}{\mathcal{I}_U(s)} \geqslant \frac{p}{A+1} \cdot \frac{1}{s} \quad s > t_0.$$

Integrating between t and $tx > t$ we get

$$(3.14) \quad \log \frac{\mathcal{I}_U(t)}{\mathcal{I}_U(tx)} \geqslant \frac{p}{A+1} \log x, \quad t > t_0.$$

Thus from (3.12)

$$(3.15) \quad (A+1) t^{-p} U(t) \geqslant p \mathcal{I}_U(t) \geqslant p \mathcal{I}_U(tx) \cdot x^{p/(A+1)}$$

and by the definition (3.2)

$$(3.16) \quad p \mathcal{I}_U(tx) \geqslant U(tx) \cdot (tx)^{-p}.$$

Accordingly, (2.2) holds with C and γ given in (3.5). (This part of the theorem was proved slightly differently in [2].)

(iii) Assume (3.6). As in the last part we conclude

$$(3.17) \quad \log \frac{\mathcal{I}_U(t)}{\mathcal{I}_U(tx)} \leq \frac{p}{\eta + 1} \log x, \quad x > 1, \quad t > t_0.$$

A repeated use of (3.10) now shows that

$$(3.18) \quad \begin{aligned} U_p(t) &\leq p\mathcal{I}_U(t) \leq p\mathcal{I}_U(tx) \cdot x^{p/(\eta+1)} = \\ &= x^{p/(\eta+1)} [U_p(tx) + (tx)^{-p} U(tx)]. \end{aligned}$$

From (3.6) with t replaced by tx it is seen that the expression within brackets is $<(1+\eta^{-1}) U_p(tx)$, and so the assertion concerning (3.7) is true.

(iv) Assume (3.7) with $q < p$. From the definition (1.2) of U_p we get by Fubini's theorem

$$(3.19) \quad p \int_0^t y^{p-1} U_p(y) dy = U(t) + t^p U_p(t)$$

which proves that the integral on the left converges for all $t > 0$. Let B stand for the value of the left side when $t = t_0$. For $y > t_0$ we can apply (3.7) to conclude

$$(3.20) \quad \begin{aligned} U(t) + t^p U_p(t) &\leq B + pK^{-1} U_p(t) \int_{t_0}^t y^{p-1} (t/y)^q dy < \\ &< B + \frac{p}{p-q} K^{-1} t^p U_p(t). \end{aligned}$$

Divide this inequality by $U(t)$ and let $t \rightarrow \infty$. If $U(t) \rightarrow \infty$ we get the assertion (3.8). If $U(t)$ remains bounded there is nothing to be proved because (3.7) implies that $t^p U_p(t)$ increases at least as fast as t^{p-q} , and hence $\underline{r} = \infty$ whenever U is bounded.

NOTE. Our result lack the perfect symmetry of the original Karamata relations. Starting from (2.2) we get (3.3)-(3.4). However, when we apply the converse with these given values we get (2.2) in the weaker form with γ replaced by a constant $\gamma' > \gamma$. Examples given in [2] show that, in an obvious sense, this result is the best possible.