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2. Dominated variation

We start from the following

Definition. *A monotone function U varies dominatedly if*

$$\limsup \frac{U(2t)}{U(t)} < \infty \qquad \textit{in case} \qquad U\uparrow$$

$$(2.1)$$

$$\limsup \frac{U(t/2)}{U(t)} < \infty \qquad \textit{in case} \qquad U\downarrow .$$

This leads immediately to the

Criterion. *A non-decreasing U varies dominatedly if there exist constants* C, γ, *and* t_0 *such that*

$$(2.2) \qquad \frac{U(tx)}{U(t)} < Cx^{\gamma} \qquad x > 1\,, \; t > t_0\,.$$

For non-increasing U the same criterion applies with $x>1$ replaced by $x<1$.

Proof. The sufficiency is obvious. Assume (2.1) and choose t_0 and C such that

$$(2.3) \qquad \frac{U(2t)}{U(t)} < C \qquad \text{for} \qquad t > t_0\,.$$

Put $\gamma = \mathrm{Log}_2\, C$. For $x>1$ define n by $2^{n-1}<x\leqslant 2^n$. A repeated application of (2.3) then shows that the left side in (2.2) is $\leqslant C^n \leqslant Cx^{\gamma}$.

Dominated variation of U may be described by saying that *the measures associated with $U(t\cdot)/U(t)$ form a sequentially compact family* in the sense that every sequence contains a subsequence converging on finite intervals to a finite measure. As in the case of slowly varying functions, the occurrence of limit measures that vanish identically on $(0, \infty)$ introduces some lack of symmetry. The supplementary condition (4.1) is designed to avoid this anomaly.