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AN APPLICATION OF STOCHASTIC PROCESS SEPARABILITY

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To the memory of J. Karamata

Let R be a set and let $\mathscr{B}$ be a σ-algebra of subsets of R; that is, let $(R, \mathscr{B})$ be a measurable space. (If R is a topological space we always choose $\mathscr{B}$ as the σ-algebra generated by the open sets.) Let $(\Omega, \mathscr{F})$ be a second measurable space and let P be a measure defined on $\mathscr{F}$ with $P\{\Omega\} = 1$. Let T be an interval on the real line and for each point t in T let $x(t)$ be a measurable function from $(\Omega, \mathscr{F})$ into $(R, \mathscr{B})$. In probability terminology $x(t)$ is a random variable and the family $\{x(t), t \in T\}$ of random variables is a stochastic process with state space R. If $x(t, \omega)$ is the value of $x(t)$ at the point ω of Ω, the function $x(., \omega)$ from T into R is a sample function of the stochastic process.

Now suppose that R is a topological space and that one wishes to discuss such concepts as continuity of sample functions, concepts which involve values of sample functions on uncountable subsets of T. Such discussions lead to nonmeasurable sets of Ω, that is sets not in $\mathscr{F}$, unless further restrictions are imposed. If R is a compact metric space the usual way out of the difficulty is to suppose that the process is separable, that is to suppose that there is a countable dense subset T_0 of T, a " separability set " with the property that for almost every ω the graph of the sample function $x(., \omega)$ is in the closure of the graph of the restriction of the sample function to T_0 (see [I], or, in a more sophisticated version [III]. It is proved in the study of separable stochastic processes that if $\{x(t), t \in T\}$ is a stochastic process defined on $(\Omega, \mathscr{F}, P)$ with a compact metric state space there is a separable stochastic process $\{x_0(t), t \in T\}$, defined on $(\Omega, \mathscr{F}, P)$, satisfying

$$P\{x(t) = x_0(t)\} = 1$$

for all t in T. Thus the distributions of countable sets of the random variables are not affected by the shift from $x(t)$ to $x_0(t)$. Many conditions on sample functions of a separable process involving uncountable subsets of T, such as sample function continuity at a point or on an interval, can be defined in terms of sample function values on the countable separability set, and in this way it is shown that the conditions determine subsets of Ω in $\mathscr{F}$, so that the corresponding probabilities are defined.

The purpose of this paper is to point out that the shift from a process to a separable version of the process is more than just a trick to make certain conditions lead to measurable sets. It is also a tool to be used in compactification problems. The following problem is common. One is given a stochastic process $\{x(t), t \in T\}$ with measurable state space $(R_0, \mathscr{B}_0)$, where R_0 is either not provided with a topology or at least is not provided with a compact metric topology, and one wishes to immerse R_0 in a compact metric space and them replace the process by a separable version of the process. (This is the context in which separability was first introduced: R_0 was the set $(-\infty, \infty)$ in the usual topology.) Suppose that R can be mapped in a one to one way on to a Borel subset of a compact metric space R (and that the map is continuous if R_0 is topological). Then we identify R_0 with its image and the given stochastic process can be considered as a stochastic process with state space R. If the given process is now replaced by a separable version the change will yield a process whose individual random variables will each have its values almost certainly in R_0, but the price of separability is that in general the sample functions of the separable process have values in $R - R_0$ as well as R_0. In applications R_0 is given and R is chosen by some compactification procedure. The character of the sample functions of the final separable process depend radically on the choice of R. When $R_0 = (-\infty, \infty)$ one has in the past usually chosen R as the one point compactification of the line, or as the extended real line $[-\infty, \infty]$. The following procedure provides a compactification (in the context of Markov processes) which is adapted to the character of the given stochastic process. This compactification is designed to yield separable processes with sample functions having nice continuity properties.

Ler $(R_0, \mathscr{B}_0)$ be a measurable space, and let p be a Markov process transition function on $(R_0, \mathscr{B}_0)$. That is if $t>0$ and if ξ is in R_0, $p(t, \xi, \cdot)$ is a probability measure of sets in $\mathscr{B}_0$ such that $p(t, ., A)$ is $\mathscr{B}_0$-measurable if A is in $\mathscr{B}_0$ and that the Chapman-Kolmogorov equation

$$(1) \qquad p(s+t, \xi, A) = \int p(t, \eta, A)\, p(s, \xi, d\eta)$$

is satisfied for strictly positive s and t. Suppose now that $\xi \dashrightarrow f(\xi)$ maps R_0 in a one to one way onto a Borel subset of a compact metric space R, and that if A is a Borel subset of R, $f^{-1}(A) \in \mathscr{B}_0$. Then R_0 can be identified with its image in R and a Markov process with state space R_0 and the given transition function can be looked on as having state space R. Such a process has a separable version with state space R and the same probability distributions of countable sets of the random variables. The separable process is therefore Markovian, and has the original transition function in a slightly loose sense. If R_0 is a topological space f is supposed continuous If this procedure is to be useful in investigating the fine structure of the separable process it must be possible to extend the transition function from R_0 to R, or at least to a Borel subset of R carrying the process (after it has been made separable).

One simple way of carrying out this compactification procedure is the following. If ϕ is bounded, positive, and B_0 measurable and if α is strictly positive define $r(\alpha, \xi, \phi)$ by

$$(2) \qquad r(\alpha, \xi, \phi) = \int_0^\infty e^{-\alpha t}\, dt \int_{R_0} p(t, \xi, d\eta)\, \phi(\eta)/\alpha.$$

If now $\{x(t), t>0\}$ is a Markov process with state space R_0 and the given transition function, the process

$$(3) \qquad \{e^{-\alpha t} r(\alpha, x(t), \phi), t>0\}$$

is a (not necessarily separable) supermartingale. Suppose that it is possible to choose a sequence $\{\phi_n, n\geqq 1\}$ such that (for convenience) $0\leqq\phi_n\leqq 1$ and that the set of functions $\{r(\alpha, ., \phi_n), n\geqq 1, \alpha \text{ rational} >0\}$ separates R_0. Let $\{u_n, n\geqq 1\}$ be this set of functions in some order. The map

$$\xi \dashrightarrow f(\xi) = (u_1(\xi), u_2(\xi), \ldots)$$

maps R_0 in a one to one way into the countable dimensional interval $[0, 1]\times[0, 1]\times\ldots$ in which we adopt one of the standard compact metric topologies, say that in which the distance between $\{a_n\}$ and $\{b_n\}$ is

$$\textstyle\sum_1^\infty |b_n - a_n|\, 2^{-n}.$$

Let R be either this interval or, to avoid superfluous points, let R be the closure of the image of R_0 in this interval. From now on we identify R_0 with this image. The function u_n has a continuous extension to R, and the sequence of these extensions separates R. It is easily seen that $r(\alpha, ., \phi_n)$ has a continuous extension to R even if $\alpha\,(>0)$ is not rational. Now suppose that the given $x(t)$ process is replaced by a separable version relative to R. We use the same notation $\{x(t), t>0\}$ for the separable version as for the original one. For each $\alpha>0$ and n the process (3) is now a separable supermartingale because $r(\alpha, ., \phi_n)$ is continuous on R. Almost no sample function of a separable supermartingale has an oscillatory discontinuity, and it follows readily that almost no sample function of the $x(t)$ process has an oscillatory discontinuity. Thus the immersion of R_0 in R has led to a process with desirable sample function continuity properties, even though the original state space may not be topological. If the original state space R_0 is topological, ϕ_n should be chosen to make each function $r(\alpha, ., \phi_n)$ continuous on R_0 in its given topology.

The procedure just described has been carried through in detail in one case. Let R_0 be a countable set, whose elements are identified with the strictly positive integers, and let $\mathscr{B}_0$ be the class of all subsets of R_0. The transition function is then determined by the point to point transition probability $p(t, i, j)$. We are thus dealing with a continuous parameter Markov chain and the hypotheses on p become

$$
\begin{aligned}
(4) \qquad & p(t,i,j) \geqq 0, \quad \Sigma_j\, p(t,i,j) = 1, \\
& p(s+t,i,k) = \Sigma_j\, p(s,i,j)\, p(t,j,k)
\end{aligned}
$$

to which we add the standard minimal continuity assumption

$$
(5) \qquad \lim_{t\to 0} p(t,i,j) = 1.
$$

The simplest topologization assigns to R_0 the discrete topology and compactifies R_0 by introducing a point at infinity. This topologization is not adapted to the process however, except in special cases. The procedure outlined above is usable, with ϕ_n defined as 1 at n and 0 otherwise. The compact metric space R is then a compactification of the set of strictly positive integers, and, depending on the given transition probabilities, a sequence of integers may converge to an integer. A separable version of the original Markov chain can be obtained in this way whose sample functions are almost all right continuous with left limits; the process has the strong Markov property and is quasi left continuous [2]. The sample functions and their left limits are supported by a G_δ subset S of R to which tne transition function is extended in such a way that the function $(t, \xi) \dashrightarrow p(t, \xi, A)$ is continuous for $t>0$, ξ in S. Note however that it is still true that $x(t)$ is almost surely integer valued for each strictly positive specified value of t. That is, $\Sigma_j\, p(t, \xi, j) = 1$ for $t>0$ and ξ in S, where j runs through the strictly positive integers. On the other hand the distribution $p(t, \xi, .)$ has a limit for $t \to 0$ and the limit distribution is supported by S but not necessarily by the set of integers.

Finally we remark that the compactification procedure for Markov processes suggested in this paper is not the only way that has been proposed. The other methods, of which [4] is the most recent example, use semigroup theory to extend the given transition probability function from the given state space to a larger compact metric state space in which the process is then defined using the extended transition probability function. The method suggested in this paper, which is applicable to a more general situation, proceeds on the contrary from a process to a separable process in the larger space and the extended transition probabilities for the larger space are defined by the separable process.

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