

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 15 (1969)  
**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** SIMPLE PROOFS OF TWO THEOREMS ON MINIMAL SURFACES  
**Autor:** Chern, Shiing-shen  
**Kapitel:** 4. A FORMULA ON NON-PARAMETRIC MINIMAL HYPERSURFACES  
IN EUCLIDEAN SPACE  
**DOI:** <https://doi.org/10.5169/seals-43204>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 22.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

the form in (23) is a quadratic differential in the sense of Riemann surfaces. For this purpose let  $z$  be a local complex coordinate on  $M$ , so that

$$(25) \quad \alpha = \lambda dz .$$

Then we have, locally,

$$\alpha\bar{\beta} = (\lambda^2 \bar{A}) dz^2 .$$

Exterior differentiation of (25) and use of (20) give

$$d\lambda + i\lambda\omega_{12} \equiv 0, \quad \text{mod } dz .$$

Combining with (24), we get

$$\frac{\partial}{\partial \bar{z}} (\lambda^2 \bar{A}) = 0$$

i.e., the coefficient of  $dz^2$  in  $\alpha\bar{\beta}$  is holomorphic.

Since  $M$  is of genus zero, the quadratic differential must vanish. This implies  $A = 0$  and that  $M$  is a great sphere.

The proof given above is not essentially different from those of Almgren and Calabi. The main idea of using the quadratic differential in surface theory goes back to H. Hopf. The formalism developed in this proof should also be useful in the study of other problems on surfaces in  $S^3$ .

#### 4. A FORMULA ON NON-PARAMETRIC MINIMAL HYPERSURFACES IN EUCLIDEAN SPACE

Instead of proving formula (2) we will establish a more general formula for a non-parametric minimal hypersurface in the euclidean  $(n+1)$ -space  $E^{n+1}$ , which seems to have an independent interest.

Suppose  $x: M \rightarrow E^{n+1}$  be an immersion of an  $n$ -dimensional manifold  $M$  in  $E^{n+1}$ . We consider orthonormal frames  $x e_1 \dots e_{n+1}$  in  $E^{n+1}$ , such that  $x \in M$  and  $e_{n+1}$  is the unit normal vector to  $M$  at  $x$ . We have then

$$(26) \quad \begin{aligned} dx &= \sum_i \omega_i e_i, \\ de_i &= \sum_k \omega_{ik} e_k + \omega_{i,n+1} e_{n+1}, \quad 1 \leq i, j, k, l \leq n, \\ de_{n+1} &= - \sum_i \omega_{i,n+1} e_i, \end{aligned}$$

with

$$(27) \quad \omega_{ik} + \omega_{ki} = 0$$

and

$$(28) \quad \omega_{i,n+1} = \sum_k h_{ik} \omega_k, \quad h_{ik} = h_{ki}.$$

The quadratic differential form

$$(29) \quad \Pi = \sum_i \omega_i \omega_{i,n+1} = \sum_{i,k} h_{ik} \omega_i \omega_k$$

is the second fundamental form of  $M$  and the condition for  $M$  to be a minimal hypersurface is

$$(30) \quad \sum_i h_{ii} = 0.$$

Exterior differentiation of (26) gives the structure equations

$$(31) \quad \begin{aligned} d\omega_i &= \sum_j \omega_j \wedge \omega_{ji}, \\ d\omega_{i,n+1} &= \sum_j \omega_{ij} \wedge \omega_{j,n+1}, \\ d\omega_{ik} &= \sum_j \omega_{ij} \wedge \omega_{jk} - \omega_{i,n+1} \wedge \omega_{k,n+1}. \end{aligned}$$

The  $\omega_{ik}$  are connection forms of the riemannian metric induced on  $M$ . If we define its curvature by the equation

$$(32) \quad d\omega_{ik} = \sum_j \omega_{ij} \wedge \omega_{jk} - \frac{1}{2} \sum_{j,l} R_{ikjl} \omega_j \wedge \omega_l,$$

where  $R_{ikjl}$  satisfy the symmetry relations

$$(33) \quad \begin{aligned} R_{ikjl} &= -R_{kijl} = -R_{iklj}, \\ R_{ikjl} + R_{ijlk} + R_{ilkj} &= 0, \end{aligned}$$

the  $R_{ikjl}$  in this case of a hypersurface are expressible in terms of the  $h_{ik}$  by

$$(34) \quad R_{ikjl} = h_{ij} h_{kl} - h_{il} h_{jk}.$$

Taking the exterior derivative of (28) and using the second equation of (31), we get

$$\sum_k (dh_{ik} + \sum_j h_{jk} \omega_{ji} + \sum_j h_{ij} \omega_{jk}) \wedge \omega_k = 0.$$

This allows us to put

$$(35) \quad dh_{ik} + \sum_j h_{jk} \omega_{ji} + \sum_j h_{ij} \omega_{jk} = \sum_j h_{ikj} \omega_j,$$

where  $h_{ikj}$  is symmetric in any two of the indices  $i, k, j$ . It follows that *for a minimal hypersurface the contraction of  $h_{ikj}$  with respect to any two indices is zero*. The left-hand side of (35) is the covariant differential of  $h_{ik}$ .

Let  $u$  be a real-valued smooth function on  $M$ . Then we have

$$(36) \quad du = \sum_i u_i \omega_i,$$

$$(37) \quad Du_i = du_i + \sum_j u_j \omega_{ji} = \sum_j u_{ij} \omega_j, \quad u_{ij} = u_{ji},$$

where  $Du_i$  is the covariant differential of the gradient vector  $u_i$ . The square of the gradient of  $u$  and the Laplacian of  $u$  are respectively defined by

$$(38) \quad (\text{grad } u)^2 = \sum_i u_i^2,$$

$$(39) \quad \Delta u = \sum_i u_{ii}.$$

If  $\varphi(u)$  is a smooth function of  $u$ , we have

$$d\varphi(u) = \varphi'(u) du,$$

$$D(\varphi'(u) u_i) = \sum_k (\varphi'(u) u_{ik} + \varphi''(u) u_i u_k) \omega_k,$$

so that

$$(40) \quad \Delta\varphi(u) = \varphi'(u) \Delta u + \varphi''(u) (\text{grad } u)^2.$$

From now on suppose  $M$  be a minimal hypersurface, so that the condition (30) is fulfilled. The Ricci curvature is given by

$$(41) \quad R_{ij} = \sum_k R_{ikjk} = - \sum_k h_{ik} h_{jk},$$

which is negative semi-definite. The scalar curvature is

$$(42) \quad R = - \sum_{i,k} h_{ik}^2 \leq 0.$$

For  $n = 2$  we have  $R = 2K$ ,  $K$  being the gaussian curvature.

Now let  $a_1, \dots, a_{n+1}$  be a fixed orthonormal frame in  $E^{n+1}$ . We can write

$$(43) \quad x = \sum_i x_i a_i + z a_{n+1}, \quad 1 \leq i, k \leq n,$$

and a non-parametric hypersurface will be defined by the equation

$$(44) \quad z = z(x_1, \dots, x_n).$$

Let

$$(45) \quad (a_A, e_B) = v_{AB}, \quad 1 \leq A, B \leq n+1,$$

where the left-hand side stands for the scalar product of the vectors in question and  $(v_{AB})$  is a properly orthogonal matrix. In particular,  $v_{A,n+1}$  are the components of the unit normal vector  $e_{n+1}$  with respect to the fixed frame  $a_A$ . If we put

$$(46) \quad p_i = \frac{\partial z}{\partial x_i}, \quad W = \left(1 + \sum_i p_i^2\right)^{\frac{1}{2}} \geq 1,$$

we have

$$(47) \quad v_{i,n+1} = \frac{p_i}{W}, \quad v_{n+1,n+1} = -\frac{1}{W}.$$

For simplicity we will write  $v = v_{n+1,n+1}$ . We wish to establish the formula

$$(48) \quad \Delta v = Rv.$$

In fact, we have, by (45) and (26),

$$dv = dv_{n+1,n+1} = (a_{n+1}, de_{n+1}) = -\sum_{i,k} v_{n+1,i} h_{ik} \omega_k,$$

and, by (37),

$$D\left(-\sum_i v_{n+1,i} h_{ik}\right) = -v \sum_{i,j} h_{ik} h_{ij} \omega_j - \sum_{i,j} v_{n+1,i} h_{ikj} \omega_j.$$

Formula (48) then follows from the definition of the Laplacian.

Formula (48) has the interesting consequence that on a minimal hypersurface the corresponding equation (48), with  $v$  as an unknown function, has a negative solution. In general, I do not know whether on a complete simply-connected non-compact riemannian manifold with negative semi-definite Ricci curvature the equation (48) has a negative solution other than constants; in the latter case we will have  $R = 0$ . If the answer to this question is no, it will give a proof of the  $n$ -dimensional Bernstein conjecture.

Formula (2) now follows as an easy consequence. Suppose therefore  $n = 2$ . In this case we have, for a minimal surface,

$$(49) \quad \sum_i h_{ij} h_{ik} = -\frac{1}{2} R \delta_{jk} = -K \delta_{jk},$$

so that

$$(50) \quad (\text{grad } v)^2 = -K(1 - v^2).$$

Formula (2) then follows immediately from (40).

#### BIBLIOGRAPHY

- [1] ALMGREN, F. J. Jr. Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. *Ann. of Math.*, 85 (1966), 277-292.
- [2] CALABI, E. Minimal immersions of surfaces in euclidean spheres. *J. of Diff. Geom.*, 1 (1967), 111-125.
- [3] PROTTER, M. H. and WEINBERGER, H. *Maximum principles in differential equations*, Prentice-Hall, 1967.

(Reçu le 10 septembre 1968).

S. S. Chern  
Dep. of Math.  
University of California  
Berkeley, Ca. 94720.