Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 15 (1969)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SIMPLE PROOFS OF TWO THEOREMS ON MINIMAL SURFACES
Autor: Chern, Shiing-shen
Kapitel: 4. A FORMULA ON NON-PARAMETRIC MINIMAL HYPERSURFACES IN EUCLIDEAN SPACE
DOI: https://doi.org/10.5169/seals-43204

the form in (23) is a quadratic differential in the sense of Riemann surfaces. For this purpose let z be a local complex coordinate on M, so that

$$\alpha = \lambda dz . \tag{25}$$

Then we have, locally,

$$\alpha\beta = (\lambda^2 \bar{A})\, dz^2 .$$

Exterior differentiation of (25) and use of (20) give

$$d\lambda + i\lambda\omega_{12} \equiv 0 , \qquad \operatorname{mod} dz .$$

Combining with (24), we get

$$\frac{\partial}{\partial \bar{z}} (\lambda^2 \bar{A}) = 0$$

i.e., the coefficient of dz^2 in $\alpha\beta$ is holomorphic.

Since M is of genus zero, the quadratic differential must vanish. This implies $A = 0$ and that M is a great sphere.

The proof given above is not essentially different from those of Almgren and Calabi. The main idea of using the quadratic differential in surface theory goes back to H. Hopf. The formalism developed in this proof should also be useful in the study of other problems on surfaces in S^3.

4. A FORMULA ON NON-PARAMETRIC MINIMAL HYPERSURFACES IN EUCLIDEAN SPACE

Instead of proving formula (2) we will establish a more general formula for a non-parametric minimal hypersurface in the euclidean $(n+1)$-space E^{n+1}, which seems to have an independent interest.

Suppose $x\colon M \to E^{n+1}$ be an immersion of an n-dimensional manifold M in E^{n+1}. We consider orthonormal frames $x\, e_1 \ldots e_{n+1}$ in E^{n+1}, such that $x \in M$ and e_{n+1} is the unit normal vector to M at x. We have then

$$\begin{aligned} dx &= \sum_i \omega_i e_i , \\ de_i &= \sum_k \omega_{ik} e_k + \omega_{i,n+1} e_{n+1} , \qquad 1 \leqq i,j,k,l \leqq n , \\ de_{n+1} &= -\sum_i \omega_{i,n+1} e_i , \end{aligned} \tag{26}$$

with

(27) $$\omega_{ik} + \omega_{ki} = 0$$

and

(28) $$\omega_{i,n+1} = \sum_k h_{ik}\,\omega_k\,, \qquad h_{ik} = h_{ki}\,.$$

The quadratic differential form

(29) $$\prod = \sum_i \omega_i\,\omega_{i,n+1} = \sum_{i,k} h_{ik}\,\omega_i\,\omega_k$$

is the second fundamental form of M and the condition for M to be a minimal hypersurface is

(30) $$\sum_i h_{ii} = 0\,.$$

Exterior differentiation of (26) gives the structure equations

(31) $$\begin{aligned} d\omega_i &= \sum_j \omega_j \wedge \omega_{ji}\,, \\ d\omega_{i,n+1} &= \sum_j \omega_{ij} \wedge \omega_{j,n+1}\,, \\ d\omega_{ik} &= \sum_j \omega_{ij} \wedge \omega_{jk} - \omega_{i,n+1} \wedge \omega_{k,n+1}\,. \end{aligned}$$

The ω_{ik} are connection forms of the riemannian metric induced on M. If we define its curvature by the equation

(32) $$d\omega_{ik} = \sum_j \omega_{ij} \wedge \omega_{jk} - \frac{1}{2}\sum_{j,l} R_{ikjl}\,\omega_j \wedge \omega_l\,,$$

where R_{ikjl} satisfy the symmetry relations

(33) $$\begin{aligned} R_{ikjl} &= -R_{kijl} = -R_{iklj}\,, \\ R_{ikjl} &+ R_{ijlk} + R_{ilkj} = 0\,, \end{aligned}$$

the R_{ikjl} in this case of a hypersurface are expressible in terms of the h_{ik} by

(34) $$R_{ikjl} = h_{ij}\,h_{kl} - h_{il}\,h_{jk}\,.$$

Taking the exterior derivative of (28) and using the second equation of (31), we get

$$\sum_k \left(dh_{ik} + \sum_j h_{jk}\,\omega_{ji} + \sum_j h_{ij}\,\omega_{jk}\right) \wedge \omega_k = 0\,.$$

This allows us to put

(35) $$dh_{ik} + \sum_j h_{jk}\,\omega_{ji} + \sum_j h_{ij}\,\omega_{jk} = \sum_j h_{ikj}\,\omega_j\,,$$

where h_{ikj} is symmetric in any two of the indices i, k, j. It follows that *for a minimal hypersurface the contraction of* h_{ikj} *with respect to any two indices is zero*. The left-hand side of (35) is the covariant differential of h_{ik}.

Let u be a real-valued smooth function on M. Then we have

(36) $$du = \sum_i u_i\,\omega_i\,,$$

(37) $$Du_i = du_i + \sum_j u_j\,\omega_{ji} = \sum_j u_{ij}\,\omega_j\,, \quad u_{ij} = u_{ji}\,,$$

where Du_i is the covariant differential of the gradient vector u_i. The square of the gradient of u and the Laplacian of u are respectively defined by

(38) $$(\text{grad } u)^2 = \sum_i u_i^2\,,$$

(39) $$\Delta u = \sum_i u_{ii}\,.$$

If $\varphi(u)$ is a smooth function of u, we have

$$d\,\varphi(u) = \varphi'(u)\,du\,,$$

$$D(\varphi'(u)\,u_i) = \sum_k (\varphi'(u)\,u_{ik} + \varphi''(u)\,u_i\,u_k)\,\omega_k\,,$$

so that

(40) $$\Delta\varphi(u) = \varphi'(u)\,\Delta u + \varphi''(u)\,(\text{grad } u)^2\,.$$

From now on suppose M be a minimal hypersurface, so that the condition (30) is fulfilled. The Ricci curvature is given by

(41) $$R_{ij} = \sum_k R_{ikjk} = -\sum_k h_{ik}\,h_{jk}\,,$$

which is negative semi-definite. The scalar curvature is

(42) $$R = -\sum_{i,k} h_{ik}^2 \leqq 0\,.$$

For $n = 2$ we have $R = 2K$, K being the gaussian curvature.

Now let $a_1, \ldots, a_{n+1}$ be a fixed orthonormal frame in E^{n+1}. We can write

(43) $$x = \sum_i x_i\,a_i + z a_{n+1}\,, \quad 1 \leqq i, k \leqq n\,,$$

and a non-parametric hypersurface will be defined by the equation

(44) $$z = z(x_1, \ldots, x_n).$$

Let

(45) $$(a_A, e_B) = v_{AB}, \qquad 1 \leqq A, B \leqq n+1,$$

where the left-hand side stands for the scalar product of the vectors in question and (v_{AB}) is a properly orthogonal matrix. In particular, $v_{A,n+1}$ are the components of the unit normal vector e_{n+1} with respect to the fixed frame a_A. If we put

(46) $$p_i = \frac{\partial z}{\partial x_i}, \qquad W = \left(1 + \sum_i p_i^2\right)^{\frac{1}{2}} \geqq 1,$$

we have

(47) $$v_{i,n+1} = \frac{p_i}{W}, \qquad v_{n+1,n+1} = -\frac{1}{W}.$$

For simplicity we will write $v = v_{n+1,n+1}$. We wish to establish the formula

(48) $$\Delta v = Rv.$$

In fact, we have, by (45) and (26),

$$dv = dv_{n+1,n+1} = (a_{n+1}, de_{n+1}) = -\sum_{i,k} v_{n+1,i}\, h_{ik}\, \omega_k,$$

and, by (37),

$$D\left(-\sum_i v_{n+1,i}\, h_{ik}\right) = -v \sum_{i,j} h_{ik}\, h_{ij}\, \omega_j - \sum_{i,j} v_{n+1,i}\, h_{ikj}\, \omega_j.$$

Formula (48) then follows from the definition of the Laplacian.

Formula (48) has the interesting consequence that on a minimal hypersurface the corresponding equation (48), with v as an unknown function, has a negative solution. In general, I do not know whether on a complete simply-connected non-compact riemannian manifold with negative semi-definite Ricci curvature the equation (48) has a negative solution other than constants; in the latter case we will have $R = 0$. If the answer to this question is no, it will give a proof of the n-dimensional Bernstein conjecture.

Formula (2) now follows as an easy consequence. Suppose therefore $n = 2$. In this case we have, for a minimal surface,

(49) $$\sum_i h_{ij}\, h_{ik} = -\frac{1}{2} R\delta_{jk} = -K\,\delta_{jk},$$

so that

$$(\mathrm{grad}\, v)^2 = -K(1-v^2)\,. \tag{50}$$

Formula (2) then follows immediately from (40).

BIBLIOGRAPHY

[1] ALMGREN, F. J. Jr. Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. *Ann. of Math.*, 85 (1966), 277-292.

[2] CALABI, E. Minimal immersions of surfaces in euclidean spheres. *J. of Diff. Geom.*, 1 (1967), 111-125.

[3] PROTTER, M. H. and WEINBERGER, H. *Maximum principles in differential equations*, Prentice-Hall, 1967.

(Reçu le 10 septembre 1968).

S. S. Chern
Dep. of Math.
University of California
Berkeley, Ca. 94720.