Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 15 (1969)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SIMPLE PROOFS OF TWO THEOREMS ON MINIMAL SURFACES
Autor: Chern, Shiing-shen
Kapitel: 2. Proof of Theorem 1
DOI: https://doi.org/10.5169/seals-43204

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 07.03.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

SIMPLE PROOFS OF TWO THEOREMS ON MINIMAL SURFACES

Shiing-shen CHERN *)

To the memory of J. Karamata

1. INTRODUCTION

We will give simple proofs of the following uniqueness theorems on minimal surfaces:

THEOREM 1 (Bernstein). *Let* $z = f(x, y)$ *be a minimal surface in euclidean three-space defined for all* x, y. *Then* $f(x, y)$ *is a linear function.*

THEOREM 2. *A closed minimal surface of genus zero on the three-sphere must be totally geodesic and is hence a great sphere.*

Theorem 2 has been proved by Almgren [1] and Calabi [2].

2. PROOF OF THEOREM 1

Let

$$W = \left(1 + f_x^2 + f_y^2\right)^{\frac{1}{2}} \geqq 1 . \tag{1}$$

The proof is based on the identity

$$\Delta \log\left(1 + \frac{1}{W}\right) = K , \tag{2}$$

where Δ is the Laplacian relative to the induced riemannian metric of the minimal surface M and K is its Gaussian curvature.

Suppose (2) be true. Let ds be the element of arc on M. Introduce the conformal metric

*) Work done under partial support of NSF grant GP 8623.

$$d\sigma = \left(1 + \frac{1}{W}\right) ds . \tag{3}$$

If p, q are isothermal coordinates on M, so that

$$ds^2 = \lambda^2 (dp^2 + dq^2) , \tag{4}$$

we have

$$K = -\frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2}\right) \log \lambda , \tag{5}$$

$$\Delta = \frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2}\right) .$$

Applying this to the metric $d\sigma$, we find immediately that its gaussian curvature is zero, or that the metric is flat.

On the other hand, it is clear that

$$ds \leqq d\sigma \leqq 2\, ds . \tag{6}$$

It follows that the metric $d\sigma$ on M is complete, for it dominates ds and ds is complete. We have therefore on M a complete flat riemannian metric $d\sigma$. By a well-known theorem, M, with the metric $d\sigma$, is isometric to the (ξ, η)-plane with its standard flat metric, i.e.,

$$d\sigma^2 = d\xi^2 + d\eta^2 . \tag{7}$$

Since $K \leqq 0$, we have, from (2) and (5),

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) \log\left(1 + \frac{1}{W}\right) \leqq 0 . \tag{8}$$

The function $\log\left(1 + \frac{1}{W}\right)$, considered as a function in the (ξ, η)-plane, is therefore superharmonic. It is also clearly non-negative. By a well-known theorem on superharmonic functions ([3], p. 130) it must be a constant. Equation (2) then gives $K = 0$, which implies that M is a plane.

The proof of (2) is a standard calculation. It will be proved at the end of § 4 as a special case of a more general formula.

An advantage of this proof is the fact that, unlike many other known proofs, complex function theory is not used.