

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	15 (1969)
Heft:	1: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	SIMPLE PROOFS OF TWO THEOREMS ON MINIMAL SURFACES
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Kapitel:	2. Proof of Theorem 1
DOI:	https://doi.org/10.5169/seals-43204

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SIMPLE PROOFS OF TWO THEOREMS ON MINIMAL SURFACES

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To the memory of J. Karamata

1. INTRODUCTION

We will give simple proofs of the following uniqueness theorems on minimal surfaces:

THEOREM 1 (Bernstein). *Let $z = f(x, y)$ be a minimal surface in euclidean three-space defined for all x, y . Then $f(x, y)$ is a linear function.*

THEOREM 2. *A closed minimal surface of genus zero on the three-sphere must be totally geodesic and is hence a great sphere.*

Theorem 2 has been proved by Almgren [1] and Calabi [2].

2. PROOF OF THEOREM 1

Let

$$(1) \quad W = \left(1 + f_x^2 + f_y^2 \right)^{\frac{1}{2}} \geq 1 .$$

The proof is based on the identity

$$(2) \quad \Delta \log \left(1 + \frac{1}{W} \right) = K ,$$

where Δ is the Laplacian relative to the induced riemannian metric of the minimal surface M and K is its Gaussian curvature.

Suppose (2) be true. Let ds be the element of arc on M . Introduce the conformal metric

*) Work done under partial support of NSF grant GP 8623.

$$(3) \quad d\sigma = \left(1 + \frac{1}{W}\right) ds.$$

If p, q are isothermal coordinates on M , so that

$$(4) \quad ds^2 = \lambda^2 (dp^2 + dq^2),$$

we have

$$(5) \quad K = -\frac{1}{\lambda^2} \left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2} \right) \log \lambda,$$

$$\Delta = \frac{1}{\lambda^2} \left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2} \right).$$

Applying this to the metric $d\sigma$, we find immediately that its gaussian curvature is zero, or that the metric is flat.

On the other hand, it is clear that

$$(6) \quad ds \leq d\sigma \leq 2ds.$$

It follows that the metric $d\sigma$ on M is complete, for it dominates ds and ds is complete. We have therefore on M a complete flat riemannian metric $d\sigma$. By a well-known theorem, M , with the metric $d\sigma$, is isometric to the (ξ, η) -plane with its standard flat metric, i.e.,

$$(7) \quad d\sigma^2 = d\xi^2 + d\eta^2.$$

Since $K \leq 0$, we have, from (2) and (5),

$$(8) \quad \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \log \left(1 + \frac{1}{W} \right) \leq 0.$$

The function $\log \left(1 + \frac{1}{W} \right)$, considered as a function in the (ξ, η) -plane, is therefore superharmonic. It is also clearly non-negative. By a well-known theorem on superharmonic functions ([3], p. 130) it must be a constant. Equation (2) then gives $K = 0$, which implies that M is a plane.

The proof of (2) is a standard calculation. It will be proved at the end of § 4 as a special case of a more general formula.

An advantage of this proof is the fact that, unlike many other known proofs, complex function theory is not used.