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# SIMPLE PROOFS OF TWO THEOREMS ON MINIMAL SURFACES

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*To the memory of J. Karamata*

## 1. INTRODUCTION

We will give simple proofs of the following uniqueness theorems on minimal surfaces:

THEOREM 1 (Bernstein). *Let* $z = f(x, y)$ *be a minimal surface in euclidean three-space defined for all* $x, y$. *Then* $f(x, y)$ *is a linear function.*

THEOREM 2. *A closed minimal surface of genus zero on the three-sphere must be totally geodesic and is hence a great sphere.*

Theorem 2 has been proved by Almgren [1] and Calabi [2].

## 2. PROOF OF THEOREM 1

Let

$$W = \left(1 + f_x^2 + f_y^2\right)^{\frac{1}{2}} \geqq 1 . \tag{1}$$

The proof is based on the identity

$$\Delta \log\left(1 + \frac{1}{W}\right) = K , \tag{2}$$

where $\Delta$ is the Laplacian relative to the induced riemannian metric of the minimal surface $M$ and $K$ is its Gaussian curvature.

Suppose (2) be true. Let $ds$ be the element of arc on $M$. Introduce the conformal metric

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(3) $$d\sigma = \left(1 + \frac{1}{W}\right) ds .$$

If $p$, $q$ are isothermal coordinates on $M$, so that

(4) $$ds^2 = \lambda^2 (dp^2 + dq^2) ,$$

we have

(5) $$K = -\frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2}\right) \log \lambda ,$$

$$\Delta = \frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2}\right) .$$

Applying this to the metric $d\sigma$, we find immediately that its gaussian curvature is zero, or that the metric is flat.

On the other hand, it is clear that

(6) $$ds \leqq d\sigma \leqq 2\, ds .$$

It follows that the metric $d\sigma$ on $M$ is complete, for it dominates $ds$ and $ds$ is complete. We have therefore on $M$ a complete flat riemannian metric $d\sigma$. By a well-known theorem, $M$, with the metric $d\sigma$, is isometric to the $(\xi, \eta)$-plane with its standard flat metric, i.e.,

(7) $$d\sigma^2 = d\xi^2 + d\eta^2 .$$

Since $K \leqq 0$, we have, from (2) and (5),

(8) $$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) \log\left(1 + \frac{1}{W}\right) \leqq 0 .$$

The function $\log\left(1 + \frac{1}{W}\right)$, considered as a function in the $(\xi, \eta)$-plane, is therefore superharmonic. It is also clearly non-negative. By a well-known theorem on superharmonic functions ([3], p. 130) it must be a constant. Equation (2) then gives $K = 0$, which implies that $M$ is a plane.

The proof of (2) is a standard calculation. It will be proved at the end of § 4 as a special case of a more general formula.

An advantage of this proof is the fact that, unlike many other known proofs, complex function theory is not used.