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SIMPLE PROOFS OF TWO THEOREMS ON MINIMAL SURFACES

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To the memory of J. Karamata

1. INTRODUCTION

We will give simple proofs of the following uniqueness theorems on minimal surfaces:

THEOREM 1 (Bernstein). *Let* z = f (x, y) *be a minimal surface in euclidean three-space defined for all* x, y. *Then* f (x, y) *is a linear function.*

THEOREM 2. *A closed minimal surface of genus zero on the three-sphere must be totally geodesic and is hence a great sphere.*

Theorem 2 has been proved by Almgren [1] and Calabi [2].

2. PROOF OF THEOREM 1

Let

$$W = \left(1 + f_x^2 + f_y^2\right)^{\frac{1}{2}} \geqq 1 . \tag{1}$$

The proof is based on the identity

$$\Delta \log\left(1 + \frac{1}{W}\right) = K , \tag{2}$$

where Δ is the Laplacian relative to the induced riemannian metric of the minimal surface M and K is its Gaussian curvature.

Suppose (2) be true. Let ds be the element of arc on M. Introduce the conformal metric

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$$(3) \qquad d\sigma = \left(1 + \frac{1}{W}\right) ds\,.$$

If p, q are isothermal coordinates on M, so that

$$(4) \qquad ds^2 = \lambda^2 (dp^2 + dq^2)\,,$$

we have

$$(5) \qquad K = -\frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2}\right)\log \lambda\,,$$

$$\Delta = \frac{1}{\lambda^2}\left(\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial q^2}\right).$$

Applying this to the metric $d\sigma$, we find immediately that its gaussian curvature is zero, or that the metric is flat.

On the other hand, it is clear that

$$(6) \qquad ds \leqq d\sigma \leqq 2\, ds\,.$$

It follows that the metric $d\sigma$ on M is complete, for it dominates ds and ds is complete. We have therefore on M a complete flat riemannian metric $d\sigma$. By a well-known theorem, M, with the metric $d\sigma$, is isometric to the (ξ, η)-plane with its standard flat metric, i.e.,

$$(7) \qquad d\sigma^2 = d\xi^2 + d\eta^2\,.$$

Since $K \leqq 0$, we have, from (2) and (5),

$$(8) \qquad \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\log\left(1 + \frac{1}{W}\right) \leqq 0\,.$$

The function $\log\left(1+\frac{1}{W}\right)$, considered as a function in the (ξ, η)-plane, is therefore superharmonic. It is also clearly non-negative. By a well-known theorem on superharmonic functions ([3], p. 130) it must be a constant. Equation (2) then gives $K = 0$, which implies that M is a plane.

The proof of (2) is a standard calculation. It will be proved at the end of § 4 as a special case of a more general formula.

An advantage of this proof is the fact that, unlike many other known proofs, complex function theory is not used.

3. Proof of Theorem 2

Let S^3 be the unit sphere in the euclidean 4-space E^4. By an orthonormal frame in E^4 is meant an ordered set of vectors e_α, $0 \leqq \alpha \leqq 3$, satisfying

$$(e_\alpha, e_\beta) = \delta_{\alpha\beta}\,, \qquad 0 \leqq \alpha, \beta, \gamma \leqq 3\,, \tag{9}$$

where the left-hand side is the scalar product of the vectors in question. The space of all orthonormal frames can be identified with the group $SO(4)$. We introduce in $SO(4)$ the Maurer-Cartan forms $\omega_{\alpha\beta}$ according to the equations

$$de_\alpha = \sum_\beta \omega_{\alpha\beta}\, e_\beta \tag{10}$$

or

$$\omega_{\alpha\beta} = (de_\alpha, e_\beta)\,. \tag{11}$$

It follows from (9) that

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0\,. \tag{12}$$

Exterior differentiation of (10) gives the Maurer-Cartan structure equations of $SO(4)$, which are

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}\,. \tag{13}$$

There is a fibering

$$SO(4) \to S^3 = SO(4)\,/\,SO(3)\,, \tag{14}$$

with the projection defined by sending the frame $e_0\, e_1\, e_2\, e_3$ to the unit vector e_0.

Suppose a smooth surface

$$M \to S^3 \tag{15}$$

be described by the vector e_0. We restrict to frames such that e_3 is the unit normal vector to M at e_0. There are two choices for e_3, any one of which is called an orientation of M. Suppose M be oriented. Then the frames are defined up to a rotation of the vectors e_1, e_2 in the tangent plane. In other words, our restricted family of frames is a circle bundle over M, for which the structure equations (13) are valid.

The condition that e_3 is a normal vector at e_0 implies

(16) $$\omega_{03} = 0 .$$

Taking its exterior derivative and using (13), we get

$$\omega_{01} \wedge \omega_{13} + \omega_{02} \wedge \omega_{23} = 0 .$$

Since M is an immersed surface, we have $\omega_{01} \wedge \omega_{02} \neq 0$ and Cartan's lemma allows us to set

(17) $$\omega_{13} = a\omega_{01} + b\omega_{02} , \qquad \omega_{23} = b\omega_{01} + c\omega_{02} .$$

The condition for a minimal surface is the vanishing of the mean curvature:

(18) $$a + c = 0 .$$

Let

(19) $$\alpha = \omega_{01} + i\omega_{02} , \qquad \beta = \omega_{13} + i\omega_{23} .$$

The structure equations (13) give

$$d\alpha = - i\omega_{12} \wedge \alpha ,$$

(20) $$d\beta = - i\omega_{12} \wedge \beta .$$

Under a rotation of e_1 e_2 both α and β will be multiplied by the same complex number of absolute value 1. It follows that

(21) $$\alpha \wedge \bar{\beta} , \qquad \alpha\bar{\beta} ,$$

which are exterior and ordinary two-forms respectively, are globally defined on our oriented surface M.

Suppose from now on that M is a minimal surface. Condition (18) can be written

(22) $$\beta = A\bar{\alpha} , \qquad A = a + ib .$$

In this case the first form in (21) vanishes identically, while

(23) $$\alpha\bar{\beta} = \bar{A}\alpha^2 .$$

Taking the exterior derivative of (22) and using (20), we get

(24) $$dA + 2iA\omega_{12} \equiv 0 , \qquad \text{mod } \bar{\alpha} .$$

The induced riemannian metric on M has an underlying complex structure which makes M into a Riemann surface. We wish to show that the form in (23) is a quadratic differential in the sense of Riemann surfaces. For this purpose let z be a local complex coordinate on M, so that

$$\alpha = \lambda dz . \tag{25}$$

Then we have, locally,

$$\alpha\bar{\beta} = (\lambda^2 \bar{A})\, dz^2 .$$

Exterior differentiation of (25) and use of (20) give

$$d\lambda + i\lambda\omega_{12} \equiv 0 , \qquad \mathrm{mod}\; dz .$$

Combining with (24), we get

$$\frac{\partial}{\partial \bar{z}} (\lambda^2 \bar{A}) = 0$$

i.e., the coefficient of dz^2 in $\alpha\bar{\beta}$ is holomorphic.

Since M is of genus zero, the quadratic differential must vanish. This implies $A = 0$ and that M is a great sphere.

The proof given above is not essentially different from those of Almgren and Calabi. The main idea of using the quadratic differential in surface theory goes back to H. Hopf. The formalism developed in this proof should also be useful in the study of other problems on surfaces in S^3.

4. A formula on non-parametric minimal hypersurfaces in euclidean space

Instead of proving formula (2) we will establish a more general formula for a non-parametric minimal hypersurface in the euclidean $(n+1)$-space E^{n+1}, which seems to have an independent interest.

Suppose x: $M \to E^{n+1}$ be an immersion of an n-dimensional manifold M in E^{n+1}. We consider orthonormal frames $x\, e_1 \ldots e_{n+1}$ in E^{n+1}, such that $x \in M$ and e_{n+1} is the unit normal vector to M at x. We have then

$$\begin{aligned} dx &= \sum_i \omega_i e_i , \\ de_i &= \sum_k \omega_{ik} e_k + \omega_{i,n+1} e_{n+1} , \qquad 1 \leqq i, j, k, l \leqq n , \\ de_{n+1} &= -\sum_i \omega_{i,n+1} e_i , \end{aligned} \tag{26}$$

with

$$\omega_{ik} + \omega_{ki} = 0 \tag{27}$$

and

$$\omega_{i,n+1} = \sum_k h_{ik}\,\omega_k\,, \qquad h_{ik} = h_{ki}\,. \tag{28}$$

The quadratic differential form

$$\prod = \sum_i \omega_i\,\omega_{i,n+1} = \sum_{i,k} h_{ik}\,\omega_i\,\omega_k \tag{29}$$

is the second fundamental form of M and the condition for M to be a minimal hypersurface is

$$\sum_i h_{ii} = 0\,. \tag{30}$$

Exterior differentiation of (26) gives the structure equations

$$\begin{aligned}
d\omega_i &= \sum_j \omega_j \wedge \omega_{ji}\,,\\
d\omega_{i,n+1} &= \sum_j \omega_{ij} \wedge \omega_{j,n+1}\,,\\
d\omega_{ik} &= \sum_j \omega_{ij} \wedge \omega_{jk} - \omega_{i,n+1} \wedge \omega_{k,n+1}\,.
\end{aligned} \tag{31}$$

The ω_{ik} are connection forms of the riemannian metric induced on M. If we define its curvature by the equation

$$d\omega_{ik} = \sum_j \omega_{ij} \wedge \omega_{jk} - \frac{1}{2}\sum_{j,l} R_{ikjl}\,\omega_j \wedge \omega_l\,, \tag{32}$$

where R_{ikjl} satisfy the symmetry relations

$$\begin{aligned}
R_{ikjl} &= -R_{kijl} = -R_{iklj}\,,\\
R_{ikjl} &+ R_{ijlk} + R_{ilkj} = 0\,,
\end{aligned} \tag{33}$$

the R_{ikjl} in this case of a hypersurface are expressible in terms of the h_{ik} by

$$R_{ikjl} = h_{ij}\,h_{kl} - h_{il}\,h_{jk}\,. \tag{34}$$

Taking the exterior derivative of (28) and using the second equation of (31), we get

$$\sum_k \Big(dh_{ik} + \sum_j h_{jk}\,\omega_{ji} + \sum_j h_{ij}\,\omega_{jk}\Big) \wedge \omega_k = 0\,.$$

This allows us to put

$$dh_{ik} + \sum_j h_{jk}\,\omega_{ji} + \sum_j h_{ij}\,\omega_{jk} = \sum_j h_{ikj}\,\omega_j\,, \tag{35}$$

where h_{ikj} is symmetric in any two of the indices i, k, j. It follows that *for a minimal hypersurface the contraction of* h_{ikj} *with respect to any two indices is zero*. The left-hand side of (35) is the covariant differential of h_{ik}.

Let u be a real-valued smooth function on M. Then we have

$$du = \sum_i u_i\,\omega_i\,, \tag{36}$$

$$Du_i = du_i + \sum_j u_j\,\omega_{ji} = \sum_j u_{ij}\,\omega_j\,, \quad u_{ij} = u_{ji}\,, \tag{37}$$

where Du_i is the covariant differential of the gradient vector u_i. The square of the gradient of u and the Laplacian of u are respectively defined by

$$(\text{grad } u)^2 = \sum_i u_i^2\,, \tag{38}$$

$$\Delta u = \sum_i u_{ii}\,. \tag{39}$$

If $\varphi(u)$ is a smooth function of u, we have

$$d\,\varphi(u) = \varphi'(u)\,d\,u\,,$$

$$D(\varphi'(u)\,u_i) = \sum_k (\varphi'(u)\,u_{ik} + \varphi''(u)\,u_i\,u_k)\,\omega_k\,,$$

so that

$$\Delta\varphi(u) = \varphi'(u)\,\Delta u + \varphi''(u)\,(\text{grad } u)^2\,. \tag{40}$$

From now on suppose M be a minimal hypersurface, so that the condition (30) is fulfilled. The Ricci curvature is given by

$$R_{ij} = \sum_k R_{ikjk} = -\sum_k h_{ik}\,h_{jk}\,, \tag{41}$$

which is negative semi-definite. The scalar curvature is

$$R = -\sum_{i,k} h_{ik}^2 \leqq 0\,. \tag{42}$$

For $n = 2$ we have $R = 2K$, K being the gaussian curvature.

Now let $a_1, \ldots, a_{n+1}$ be a fixed orthonormal frame in E^{n+1}. We can write

$$x = \sum_i x_i\,a_i + z a_{n+1}\,, \quad 1 \leqq i, k \leqq n\,, \tag{43}$$

and a non-parametric hypersurface will be defined by the equation

$$z = z(x_1, \ldots, x_n) . \tag{44}$$

Let

$$(a_A, e_B) = v_{AB} , \qquad 1 \leqq A, B \leqq n+1 , \tag{45}$$

where the left-hand side stands for the scalar product of the vectors in question and (v_{AB}) is a properly orthogonal matrix. In particular, $v_{A,n+1}$ are the components of the unit normal vector e_{n+1} with respect to the fixed frame a_A. If we put

$$p_i = \frac{\partial z}{\partial x_i}, \qquad W = \left(1 + \sum_i p_i^2\right)^{\frac{1}{2}} \geqq 1 , \tag{46}$$

we have

$$v_{i,n+1} = \frac{p_i}{W}, \qquad v_{n+1,n+1} = -\frac{1}{W} . \tag{47}$$

For simplicity we will write $v = v_{n+1,n+1}$. We wish to establish the formula

$$\Delta v = R v . \tag{48}$$

In fact, we have, by (45) and (26),

$$dv = dv_{n+1,n+1} = (a_{n+1}, de_{n+1}) = - \sum_{i,k} v_{n+1,i} h_{ik} \omega_k ,$$

and, by (37),

$$D\left(- \sum_i v_{n+1,i} h_{ik}\right) = - v \sum_{i,j} h_{ik} h_{ij} \omega_j - \sum_{i,j} v_{n+1,i} h_{ikj} \omega_j .$$

Formula (48) then follows from the definition of the Laplacian.

Formula (48) has the interesting consequence that on a minimal hypersurface the corresponding equation (48), with v as an unknown function, has a negative solution. In general, I do not know whether on a complete simply-connected non-compact riemannian manifold with negative semi-definite Ricci curvature the equation (48) has a negative solution other than constants; in the latter case we will have $R = 0$. If the answer to this question is no, it will give a proof of the n-dimensional Bernstein conjecture.

Formula (2) now follows as an easy consequence. Suppose therefore $n = 2$. In this case we have, for a minimal surface,

$$\sum_i h_{ij} h_{ik} = - \frac{1}{2} R \delta_{jk} = - K \delta_{jk} , \tag{49}$$

so that

$$(\mathrm{grad}\, v)^2 = -K(1-v^2). \tag{50}$$

Formula (2) then follows immediately from (40).

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