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MULTIPLIERS OF UNIFORM CONVERGENCE

by Ronald DeVORE

1. *Introduction.* If A and B are two classes of 2π -periodic integrable functions we say that (λ_k) is a multiplier sequence from A into B and we write $(\lambda_k) \in (A, B)$ if whenever

$$\sum_0^{\infty} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a function in A

$$\sum_0^{\infty} \lambda_n (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a function in B . Let C denote the class of 2π -periodic continuous functions and C_F the subclass of those functions in C whose Fourier series converges uniformly. Karamata [1] has shown that $(\lambda_k) \in (C, C_F)$ if and only if

$$(1.1) \quad \int_0^{2\pi} |A_n(t)| dt = O(1) \quad (n \rightarrow \infty)$$

where

$$A_n(t) = \sum_0^n \lambda_k \cos kt.$$

This theorem contains as a special case an earlier result of Tomić [2] who showed that if (λ_k) is monotone decreasing and convex (i.e. $\Delta^2 \lambda_k = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2} \geq 0$) or more generally quasi-convex (i.e. $\sum_0^{\infty} (k+1) |\Delta^2 \lambda_k| < \infty$) then $(\lambda_k) \in (C, C_F)$ if and only if $\lambda_n \log n = O(1)$ ($n \rightarrow \infty$).

It is interesting to see to what extent condition (1.1) can be relaxed if we restrict our attention to a sub-class of C determined by some structural property. For example, let ω be a modulus of continuity and C_{ω} the sub-class of C consisting of those functions whose modulus of continuity $\omega(f, h)$ satisfies

$$\omega(f, h) = O(\omega(h)) \quad (h \rightarrow 0).$$

Then Tomić [3] has shown that for a quasi-convex sequence (λ_k) to be in (C_ω, C_F) it is sufficient that

$$(1.2) \quad \omega \left(\frac{1}{n} \right) \lambda_n \log n = o(1) \quad (n \rightarrow \infty).$$

Also Bojanic [4] has shown that sufficient conditions for (λ_k) to be in (C_ω, C_F) are

$$(1.3) \quad \int_0^{2\pi} \left| \sum_0^n A_k(t) \right| dt = O(n) \quad (n \rightarrow \infty)$$

and

$$(1.4) \quad \omega \left(\frac{1}{n} \right) \int_0^{2\pi} |A_n(t)| dt = o(1) \quad (n \rightarrow \infty).$$

Of course, condition (1.3) is equivalent to (λ_k) being a Fourier Stieljes sequence which in particular characterizes the class of multipliers (C, C) .

No necessary conditions have been given for (λ_k) to be in (C_ω, C_F) and sufficient conditions have been restricted to quasi-convex and Fourier-Stieljes sequences. In order to obtain necessary and sufficient conditions for (λ_k) to be in (C_ω, C_F) , it is natural to attempt to make C_ω a Banach space in which trigonometric polynomials are dense and then invoke the Banach-Steinhaus theorem as Karamata did in characterizing (C, C_F) . The most natural norm is to define for $f \in C_\omega$

$$\|f\|_\omega = \max \left(\|f\|_\infty, \sup_{h>0} \frac{\omega(f, h)}{\omega(h)} \right)$$

where $\|f\|_\infty$ is the usual supremum norm.

The normed space $(C_\omega, \|\cdot\|_\omega)$ is a Banach space. However, trigonometric polynomials are not dense in $(C_\omega, \|\cdot\|_\omega)$. For if $\omega(h) \neq O(h)$ ($h \rightarrow 0$), then whenever (T_n) is a sequence of trigonometric polynomials which converge in $\|\cdot\|_\omega$ to f , f satisfies

$$\omega(f, h) = o(\omega(h)) \quad (h \rightarrow 0).$$

In the case that $\omega(h) = O(h)$ ($h \rightarrow 0$), then a sequence of trigonometric polynomials (T_n) converge in $\|\cdot\|_\omega$ if and only both T_n and T'_n converge uniformly and therefore f is the limit of the sequence (T_n) only if f is contin-

uously differentiable. Accordingly, when $\omega(h) \neq O(h)$ ($h \rightarrow 0$), we define c_ω as the class of those functions in C_ω for which

$$\omega(f, h) = o(\omega(h)) \quad (h \rightarrow 0)$$

and when $\omega(h) = O(h)$ ($h \rightarrow 0$) we define c_ω as the class of all continuously differentiable functions. c_ω is then a closed subspace of C_ω and it is easy to see that if $f \in c_\omega$, the Fejer sums of f

$$\sigma_n(f) = \int_0^{2\pi} f(t) F_n(t-x) dt$$

with

$$F_n(t) = \frac{1}{2\pi(n+1)} \left(\frac{\sin(n+1)\frac{1}{2}t}{\sin\frac{1}{2}t} \right)^2$$

converges in $\|\cdot\|_\omega$ to f . Thus, c_ω is precisely the closure of the class of trigonometric polynomials in $\|\cdot\|_\omega$. It therefore appears some what more natural to consider the class c_ω rather than the class C_ω in terms of problems involving multiplier sequences. For we then have

PROPOSITION 1. *The sequence $(\lambda_k) \in (c_\omega, C_F)$ if and only if*

$$\| \| A_n \| \|_\omega \equiv \sup_{\substack{f \in c_\omega \\ \|f\|_\omega \leq 1}} \left\| \int_0^{2\pi} f(t) A_n(t-x) dt \right\|_\omega = O(1) \quad (n \rightarrow \infty).$$

This is an immediate application of the Banach-Steinhaus theorem [5, p. 60] and the fact that the operators

$$L_n(f)(x) = \int_0^{2\pi} f(t) A_n(t-x) dt$$

converge in $\|\cdot\|_\omega$ for each trigonometric polynomial T .

We shall find it convenient to use the following proposition which follows immediately from the fact that any function f in C_ω with $\|f\|_\omega \leq 1$ is the uniform limit of sequence of functions from the unit ball of $(c_\omega, \|\cdot\|_\omega)$ (e.g. $\sigma_n(f)$ provides such a sequence of functions).

PROPOSITION 2. *If $A(t)$ is an integrable function then*

$$\| \| A \| \|_\omega = \sup_{\substack{f \in C_\omega \\ \|f\|_\omega \leq 1}} \left\| \int_0^{2\pi} f(t) A_n(t-x) dt \right\|_\omega$$

In section 2, we shall consider quasi-convex sequences and show that in this case $(\lambda_k) \in (c_\omega, C_F)$ if and only if

$$\lambda_n \omega \left(\frac{1}{n} \right) \log n = O(1) \quad (n \rightarrow \infty).$$

In section 3, we shall give a necessary condition that (λ_k) be in (c_ω, C_F) with no restrictions on (λ_k) . We shall show that $(\lambda_k) \in (c_\omega, C_F)$ only if

$$\omega \left(\frac{1}{n} \right) \int_0^{2\pi} |A_n(t)| dt = O(1) \quad (n \rightarrow \infty).$$

It is easy to see that this condition is in general not sufficient. For example, if $\omega(h) = h$, then simple integration by parts (see theorem 4.2) shows that

$$||| A_n |||_\omega = \int_0^{2\pi} \left| \int_0^t A_n(x) dx \right| dt + O(1) \quad (n \rightarrow \infty)$$

thus, if we let

$$\lambda_n = \begin{cases} n, & n = 2^k \\ 0, & n \neq 2^k \end{cases} \quad k = 0, 1, 2, \dots$$

then

$$\int_0^{2\pi} |A_n(t)| dt = \int_0^{2\pi} \left| \sum_0^{[\log 2n]} 2^k \cos 2^k t \right| dt = O(n) \quad (n \rightarrow \infty).$$

Whereas,

$$\int_0^{2\pi} \left| \int_0^t A_n(x) dx \right| dt = \int_0^{2\pi} \left| \sum_0^{[\log 2n]} \sin 2^k t \right| dt$$

and it follows from a theorem of Helson [6] that

$$\int_0^{2\pi} \left| \int_0^t A_n(x) dx \right| dt \neq O(1) \quad (n \rightarrow \infty).$$

In section 4, we shall examine sufficient conditions for (λ_k) to be in (c_ω, C_F) . First we shall obtain the result analogous to that of Bojanic. In particular, using the necessary condition given in Section 3, we shall prove that if (λ_k) is a Stieltjes sequence then $(\lambda_k) \in (c_\omega, C_F)$ if and only if

$$\omega \left(\frac{1}{n} \right) \int_0^{2\pi} |A_n(t)| dt = O(1) \quad (N \rightarrow \infty)$$

Finally, we shall give a sufficient condition for (λ_k) to be in (c_ω, C_F) with no restrictions on (λ_k) . We shall show that $(\lambda_k) \in (c_\omega, C_F)$ if

$$(1.5) \quad \omega(\mu_n) \int_0^{2\pi} |A_n(t)| dt = O(1)$$

where

$$\mu_n = \frac{\int_0^{2\pi} \left| \int_0^t A_n(x) dx \right| dt}{\int_0^{2\pi} |A_n(t)| dt}.$$

This condition is also necessary in the case that $\omega(h) = O(h)$ ($h \rightarrow 0$). However, it is generally not necessary. For example, if $F(x)$ is the classical Lebesgue function (see [7, p. 195]), then $F(x) - \frac{x}{2\pi}$ is continuous, of bounded variation, and its Fourier coefficients are not $o\left(\frac{1}{n}\right)$ ($n \rightarrow \infty$). Thus, if (λ_k) is the sequence of Fourier-Stieljes coefficients of $d\left(F(t) - \frac{t}{2\pi}\right)$ we have using the theorem of Dirichlet-Jordan [7, p. 57] that

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} \left| \sum_0^n \frac{\lambda_k}{k} \sin kt \right| dt = \int_0^{2\pi} \left| F(t) - \frac{t}{2\pi} \right| dt > 0.$$

while by the result of Helson [6]

$$\int_0^{2\pi} \left| \sum_0^n \lambda_k \cos kt \right| dt \neq O(1) \quad (n \rightarrow \infty).$$

Also,

$$\int_0^{2\pi} \left| \sum_0^n \lambda_k \cos kt \right| dt = O(\log n) \quad (n \rightarrow \infty)$$

since it is a Fourier-Stieljes series. So that, if we choose ω to satisfy the conditions

$$\omega\left(\frac{1}{n}\right) \int_0^{2\pi} \left| \sum_0^n \lambda_k \cos kt \right| dt = O(1) \quad (n \rightarrow \infty)$$

and

$$\omega(\mu_n) \int_0^{2\pi} \left| \sum_0^n \lambda_k \cos kt \right| dt \neq O(1) \quad (n \rightarrow \infty)$$

with

$$\mu_n = \frac{\int_0^{2\pi} \left| \sum_0^{2n} \frac{\lambda_k}{k} \sin kt \right| dt}{\int_0^{2\pi} \left| \sum_0^n \lambda_k \cos kt \right| dt}$$

we see that (1.5) is in general not necessary.

Although, we give necessary and sufficient conditions for (λ_k) to be in (c_ω, C_F) in the case that (λ_k) is quasi-convex or a Stieljes sequence in general no conditions that are both necessary and sufficient are known.

2. *Quasi-convex sequences.* We consider first the simplest case of quasi convex sequences. If we apply Abel summation twice we find

$$A_n(t) = \sum_0^n (k+1) \Delta^2 \lambda_k F_k(t) + n \Delta \lambda_{n-1} F_n(t) + \lambda_n D_n(t)$$

where D_n is the Dirichlet kernel

$$D_n(t) = \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})t)}{\sin \frac{1}{2}t}.$$

From the quasi-convexity and the fact that $\int_0^{2\pi} |F_n(t)| dt = 1$, we have

$$||| \sum_0^n (k+1) \Delta^2 \lambda_k F_k |||_\omega \leq \int_0^{2\pi} \left| \sum_0^n (k+1) \Delta^2 \lambda_k F_k(t) \right| dt = O(1) \quad (n \rightarrow \infty)$$

for any modulus of continuity ω . Thus

$$(2.1) \quad ||| A_n |||_\omega = O(1) + ||| n \Delta \lambda_{n-1} F_n + \lambda_n D_n |||_\omega \quad (n \rightarrow \infty)$$

It follows from standard estimates that there exist positive constants C_1, C_2 such that

$$(2.2) \quad C_1 \omega\left(\frac{1}{n}\right) \log n \leq ||| D_n |||_\omega \leq C_2 \omega\left(\frac{1}{n}\right) \log n.$$

This result is contained in theorems (3.1) and (4.1) so we shall not supply an independent proof.

The main result of this section is

THEOREM 2.1. *If (λ_k) is a quasi-convex sequence then $(\lambda_k) \in (c_\omega, C_F)$ if and only if*

$$(2.1) \quad \lambda_n \omega \left(\frac{1}{n} \right) \log n = O(1) \quad (n \rightarrow \infty).$$

Proof: We first consider the case when (λ_n) is a bounded sequence. Then by a result of Tomić [3]

$$n \Delta \lambda_{n-1} = o(1).$$

Thus from (2.1) we have

$$||| A_n |||_\omega = O(1) + ||| \lambda_n D_n |||_\omega$$

and the theorem follows immediately from the inequalities (2.2).

We shall now show that the case (λ_k) unbounded does not arise. Tomić [3] has shown that if (λ_k) is quasi convex and unbounded then

$$(2.3) \quad \lambda_n = An + B + o(1) \quad (n \rightarrow \infty)$$

and

$$(2.4) \quad n \Delta \lambda_{n-1} = -An + o\left(\frac{1}{n}\right). \quad (n \rightarrow \infty)$$

thus if

$$\lambda_n \omega \left(\frac{1}{n} \right) \log n = O(1) \quad (n \rightarrow \infty)$$

we must have

$$\frac{\lambda_n}{n} \log n = O(1) \quad (n \rightarrow \infty)$$

and therefor (λ_n) cannot satisfy (2.3) and the conditions (2.1) and (λ_k) unbounded are not compatible. Secondly, if (λ_k) is unbounded then by virtue of (2.1)

$$||| A_n |||_\omega = O(1) + ||| n \Delta \lambda_{n-1} F_n + \lambda_n D_n |||_\omega$$

and thus by (2.2) (2.3), and (2.4) we must have

$$(2.5) \quad ||| A_n |||_\omega \geq An - A C_2 n \omega \left(\frac{1}{n} \right) \log n.$$

For $\omega(h) = h$, (2.5) fails and thus $(\lambda_k) \notin (c_\omega, C_F)$ for any ω . Thus, (λ_k) unbounded and $(\lambda_k) \in (c_\omega, C_F)$ are also incompatible.

3. *A necessary condition for (λ_k) to be in (c_ω, C_F) .* In this section, we shall give a necessary condition for (λ_k) to be in (c_ω, C_F) . Our main result is the following theorem.

THEOREM 3.1. *There exists an absolute constant $C > 0$ such that for any trigonometric polynomial T of degree n we have*

$$||| T |||_\omega \geq C \omega \left(\frac{1}{n} \right) \int_0^{2\pi} |T| dt \quad n = 1, 2, \dots$$

An immediate corollary of this theorem and Proposition 1 is

COROLLARY 3.1. *A necessary condition for the sequence (λ_k) to be in (c_ω, C_F) is that*

$$\omega \left(\frac{1}{n} \right) \int_0^{2\pi} |A_n| dt = O(1), \quad (n \rightarrow \infty)$$

We shall need some preliminary results concerning representations of trigonometric polynomials. Let $x_k = \frac{2k\pi}{3n}$, $k = 0, 1, 2, \dots, 3n-1$. Then if T is a trigonometric polynomial of degree n , we have (see [8, p. 33])

$$(3.1) \quad T(x) = \frac{2}{3n} \sum_0^{3n-1} T(x_k) K_n(x - x_k)$$

where

$$(3.2) \quad K_n(t) = \frac{1}{\pi} \frac{\sin(\frac{3n}{2}t) \sin(\frac{n}{2}t)}{2n (\sin \frac{t}{2})^2}.$$

Also [8, p. 33]

$$(3.3) \quad \int_0^{2\pi} |T(x)| dx \leq \frac{1}{n} \sum_0^{3n-1} |T(x_k)|.$$

Now to the proof of theorem (3.1). Let $0 < \delta < \frac{1}{4}$. We wish to estimate

$$\int_{-\frac{\pi\delta}{3n}}^{\frac{\pi\delta}{3n}} K_n(t) dt$$

from below. We have for $|t| \leq \frac{\pi\delta}{3n}$

$$K_n(t) \geq \frac{1}{\pi} \left(\frac{\left(\frac{2}{\pi}\right) \left(\frac{3nt}{2}\right) \left(\frac{2}{\pi}\right) \left(\frac{nt}{2}\right)}{2n \left(\frac{t}{2}\right)^2} \right) = \frac{6}{\pi^3} n.$$

So that,

$$(3.4) \quad \int_{-\frac{\pi\delta}{3n}}^{\frac{\delta\pi}{3n}} K_n(t) dt \geq \frac{6n}{\pi^3} \cdot \frac{2\pi\delta}{3n} = \frac{4}{\pi^2} \delta.$$

Secondly, for $k \neq 0$ we estimate $\int_{x_k - \frac{\pi}{3n}}^{x_k + \frac{2\pi\delta}{3n}} K_n(t) dt$ from above. For

$|t - x_k| \leq \frac{2\pi\delta}{3n}$, we have

$$K_n(t) \leq \frac{\sin \frac{\delta\pi}{2}}{2n \left(\frac{2\pi}{3n} \left(k - \frac{1}{2}\right)\right)^2} \leq \frac{\delta\pi}{4n} \frac{1}{\left(\frac{2\pi}{3n} \left(k - \frac{1}{2}\right)\right)^2} = \frac{9\delta}{8\pi} \frac{n}{\left(k - \frac{1}{2}\right)^2}.$$

Thus

$$(3.5) \quad \int_{x_k - \frac{2\pi\delta}{3n}}^{x_k + \frac{2\pi}{3n}} |K_n(t)| dt \leq \frac{4\delta\pi}{3n} \cdot \frac{9\delta}{8\pi} \frac{n}{\left(k - \frac{1}{2}\right)^2} = \frac{3}{2} \frac{\delta^2}{\left(k - \frac{1}{2}\right)^2}.$$

Let $g_\delta(x)$ be the 2π -periodic continuous function which has the value one on the interval $\left[\frac{-\pi\delta}{3n}, \frac{\pi\delta}{3n}\right]$ has the value zero on $[-\pi, \pi] - \left[\frac{-2\pi\delta}{3n}, \frac{2\pi\delta}{3n}\right]$ and is linear on the intervals $\left[\frac{-\pi\delta}{3n}, \frac{-\pi\delta}{3n}\right]$ and $\left[\frac{\pi\delta}{3n}, \frac{2\pi\delta}{3n}\right]$.

The function

$$\bar{g}_\delta(x) = \omega \left(\frac{\delta\pi}{3n}\right) \sum_{k=0}^{3n-1} \text{Sgn}(T(x_k)) g_\delta(x - x_k)$$

is in C_ω and $\|\bar{g}_\delta\|_\omega \leq 1$. Also,

$$\begin{aligned} T(x_k) \int_0^{2\pi} \bar{g}_\delta(x) K_n(x-x_k) dx &\geq \omega\left(\frac{\delta\pi}{3n}\right) |T(x_k)| \int_{x_k - \frac{\pi\delta}{3n}}^{x_k + \frac{\pi\delta}{3n}} |K_n(x-x_k)| dx \\ &\quad - \omega\left(\frac{\delta\pi}{3n}\right) |T(x_k)| \sum_{\substack{j=0 \\ j \neq k}}^{3n-1} \int_{x_j - \frac{2\pi\delta}{3n}}^{x_j + \frac{2\pi\delta}{3n}} |K_n(x-x_k)| dx \end{aligned}$$

which by virtue of (3.4) and (3.5) is

$$\begin{aligned} &\geq \omega\left(\frac{\delta\pi}{3n}\right) |T(x_k)| \left(\frac{4}{\pi^2} \delta - \frac{3}{2} \delta^2 \sum_{\substack{j=0 \\ j \neq k}}^{3n-1} \frac{1}{(j-k-\frac{1}{2})^2} \right) \\ &\geq \omega\left(\frac{\delta\pi}{3n}\right) |T(x_k)| \left(\frac{4}{\pi^2} \delta - \frac{3}{2} \delta^2 \sum_{j=0}^{\infty} \frac{1}{(j-\frac{1}{2})^2} \right) \end{aligned}$$

Thus if we choose $\delta_0 > 0$ such that

$$\left(\frac{4}{\pi^2} \delta_0 - \frac{3}{2} \delta_0^2 \sum_{j=0}^{\infty} \frac{1}{(j-\frac{1}{2})^2} \right) = C_0 > 0$$

We have, using the elementary properties of a modulus of continuity that

$$T(x_k) \int_0^{2\pi} \bar{g}_{\delta_0}(x) K_n(x-x_k) dx \geq C \omega\left(\frac{1}{n}\right) |T(x_k)| \quad k = 0, 1, 2, \dots, 3n-1$$

where C is an absolute positive constant. Finally,

$$\begin{aligned} \int_0^{2\pi} \bar{g}_{\delta_0}(x) T(x) dx &= \frac{2}{3n} \sum_{k=0}^{3n-1} T(x_k) \int_0^{2\pi} \bar{g}_{\delta_0}(x) K_n(x-x_k) dx \geq \\ &\geq C \omega\left(\frac{1}{3n}\right) \cdot \frac{2}{3n} \sum_{k=0}^{3n-1} |T(x_k)| \end{aligned}$$

which by virtue of (3.3.) is

$$\geq \frac{2}{3} C \omega\left(\frac{1}{n}\right) \int_0^{2\pi} |T(x)| dx.$$

Thus, using Proposition 2,

$$||| T_n |||_{\omega} \geq \int_0^{2\pi} \bar{g}_{\delta_0}(x) T(x) dx \geq \frac{2}{3} C \omega \left(\frac{1}{n} \right) \int_0^{2\pi} |T(x)| dx$$

and the theorem is proved.

4. *Sufficient conditions for (λ_k) to be in (c_{ω}, C_F) .* We first establish the result analogous to that of Bojanic (1.3) and (1.4). The proof is essentially that of Haršiladze [9].

THEOREM 4. 1. *If (λ_k) is a Stieljes sequence and if*

$$\omega \left(\frac{1}{n} \right) \int_0^{2\pi} |A_n(x)| dx = O(1) \quad (n \rightarrow \infty)$$

then $(\lambda_k) \in (c_{\omega}, C_F)$.

Proof: Let $V_n(f)$ be the de la Vallée Poussin sums of f

$$V_n(f) = \int_0^{2\pi} f(t) (2F_{2n}(t-x) - F_n(t-x)) dt.$$

It is well known [10, p. 92] that

$$(4.1) \quad \|f - V_n(f)\|_{\infty} \leq C \omega \left(f, \frac{1}{n} \right)$$

where C is a constant independent of f and n . Also if T is a trigonometric polynomial of degree n then

$$V_n(T) = T.$$

Thus if $f \in C_{\omega}$, $\|f\|_{\omega} \leq 1$

$$\begin{aligned} \int_0^{2\pi} f(t) A_n(t-x) dt &= \int_0^{2\pi} (f(t) - V_n(f)(t)) A_n(t-x) dt + \\ &+ \int_0^{2\pi} V_n(f)(t) A_n(t-x) dt. \end{aligned}$$

We have

$$\int_0^{2\pi} \left| \int_0^{2\pi} (2F_{2n}(t) - F_n(t)) A_n(t-x) dt \right| dx = O(1) \quad (n \rightarrow \infty).$$

Since (λ_k) is a Stieltjes sequence. Thus

$$\begin{aligned} \left\| \int_0^{2\pi} f(t) A_n(t-x) dt \right\|_{\infty} &\leq \left\| \int_0^{2\pi} (f(t) - V_n(f)(t)) A_n(t-x) dt \right\|_{\infty} + \\ &+ \|f\|_{\infty} \int_0^{2\pi} \left| \int_0^{2\pi} (2F_{2n}(t) - F_n(t)) A_n(t-x) dt \right| dx \leq \\ &\leq \left\| \int_0^{2\pi} (f(t) - V_n(f)(t)) A_n(t-x) dt \right\|_{\infty} + O(1) \quad (n \rightarrow \infty) \end{aligned}$$

which by virtue of (4.1) is

$$\leq C \omega\left(\frac{1}{n}\right) \int_0^{2\pi} |A_n(t)| dt + O(1) \quad (n \rightarrow \infty).$$

As a corollary of theorem 4.1 and theorem 3.1, we have

COROLLARY 4.1. *A Stieltjes Sequence (λ_k) is in (c_{ω}, C_F) if and only if*

$$\omega\left(\frac{1}{n}\right) \int_0^{2\pi} |A_n(t)| dt = O(1) \quad (n \rightarrow \infty).$$

We shall now give a sufficient condition for (λ_k) to be in (c_{ω}, C_F) which requires no special restriction on (λ_k) .

THEOREM 4.2. *A sufficient condition for (λ_k) to be in (c_{ω}, C_F) is that*

$$(4.2) \quad \omega(\mu_n) \int_0^{2\pi} |A_n(t)| dt = O(1) \quad (n \rightarrow \infty)$$

where

$$\mu_n = \frac{\int_0^{2\pi} \left| \int_0^x A_n(t) dt \right| dx}{\int_0^{2\pi} |A_n(t)| dt} \quad n = 0, 1, 2, \dots$$

If $\omega(h) = h$ then (4.2) is also necessary.

Proof: We consider first the case when $\omega(h) = h$.

If $f \in C_{\omega}$ with $\|f\|_{\omega} \leq 1$ then

$$|f'(x)| \leq 1 \text{ a. e.}$$

So that

$$\left| \int_0^{2\pi} f(t) A_n(t-x) dt \right| = \left| \int_0^{2\pi} f'(t) \overline{A}_n(t-x) dt \right| \leq \int_0^{2\pi} |\overline{A}_n(t)| dt$$

$$\text{with } \overline{A}_n(t) = \int_0^t A_n(u) du.$$

Thus,

$$||| A_n |||_{\omega} \leq \int_0^{2\pi} |\overline{A}_n(t)| dt,$$

the function $g(x) = \frac{1}{2\pi} \operatorname{sgn} \int_0^x A_n(t) dt$ is in C_{ω} and $\|g\|_{\omega} \leq 1$. Also

$$\int_0^{2\pi} g(t) A_n(t) dt = |g(2\pi) A_n(2\pi) - \int_0^{2\pi} |\overline{A}_n(t)| dt| \geq \int_0^{2\pi} |\overline{A}_n(t)| dt - \lambda_0.$$

Thus,

$$\int_0^{2\pi} |\overline{A}_n(t)| dt - \lambda_0 \leq ||| A_n |||_{\omega} \leq \int_0^{2\pi} |\overline{A}_n(t)| dt \quad n = 1, 2, \dots$$

This shows that (4.2) is necessary and sufficient for (λ_k) to be in (c_{ω}, C_F) if $\omega(h) = h$.

Finally in the general case, the inequality

$$\left\| \int_0^{2\pi} f(t) A_n(t-x) dt \right\|_{\omega} \leq \omega(\mu_n) \int_0^{2\pi} |A_n(t)| dt$$

is a simple modification of Lemma 1 of [11] and we will not give its proof.

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