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MULTIPLIERS OF UNIFORM CONVERGENCE

by Ronald DeVore

1. Introduction. If A and B are two classes of 2π -periodic integrable functions we say that (λ_k) is a multiplier sequence from A into B and we write $(\lambda_k) \in (A, B)$ if whenever

$$\sum_{0}^{\infty} \left(a_{n} \cos nx + b_{n} \sin nx \right)$$

is the Fourier series of a function in A

$$\sum_{0}^{\infty} \lambda_{n} (a_{n} \cos nx + b_{n} \sin nx)$$

is the Fourier series of a function in B. Let C denote the class of 2π -periodic continuous functions and C_F the subclass of those functions in C whose Fourier series converges uniformly. Karamata [1] has shown that $(\lambda_k) \in (C, C_F)$ if and only if

(1.1)
$$\int_{0}^{2\pi} |\Lambda_{n}(t)| dt = O(1) \quad (n \to \infty)$$

where

$$\Lambda_n(t) = \sum_{0}^{n} \lambda_k \cos kt.$$

This theorem contains as a special case an earlier result of Tomić [2] who showed that if (λ_k) is monotone decreasing and convex (i.e. $\Delta^2 \lambda_k = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2} \ge 0$) or more generally quasi-convex (i.e. $\sum_{k=0}^{\infty} (k+1) |\Delta^2 \lambda_k| < \infty$) then $(\lambda_k) \in (C, C_F)$ if and only if $\lambda_n \log n = O(1)$ $(n \to \infty)$.

It is interesting to see to what extent condition (1.1) can be relaxed if we restrict our attention to a sub-class of C determined by some structural property. For example, let ω be a modulus of continuity and C_{ω} the sub-class of C consisting of those functions whose modulus of continuity ω (f, h) satisfies

$$\omega\left(f,h\right) \; = \; O\left(\omega\left(h\right)\right) \quad \ (h \! \to \! 0) \; .$$

Then Tomić [3] has shown that for a quasi-convex sequence (λ_k) to be in (C_{ω}, C_F) it is sufficient that

(1.2)
$$\omega\left(\frac{1}{n}\right)\lambda_n\log n = o(1) \qquad (n\to\infty).$$

Also Bojanic [4] has shown that sufficient conditions for (λ_k) to be in (C_{ω}, C_F) are

(1.3)
$$\int_{0}^{2\pi} \left| \sum_{n=0}^{\infty} \Lambda_{k}(t) \right| dt = O(n) \quad (n \to \infty)$$

and

(1.4)
$$\omega\left(\frac{1}{n}\right)\int_{0}^{2\pi}|\Lambda_{n}(t)|dt = o(1) \quad (n\to\infty).$$

Of course, condition (1.3) is equivalent to (λ_k) being a Fourier Stieljes sequence which in particular characterizes the class of multipliers (C, C).

No necessary conditions have been given for (λ_k) to be in (C_{ω}, C_F) and sufficient conditions have been restricted to quasi-convex and Fourier-Stieljes sequences. In order to obtain necessary and sufficient conditions for (λ_k) to be in (C_{ω}, C_F) , it is natural to attempt to make C_{ω} a Banach space in which trigonometric polynomials are dense and then invoke the Banach-Steinhaus theorem as Karamata did in characterizing (C, C_F) . The most natural norm is to define for $f \in C_{\omega}$

$$||f||_{\omega} = \max\left(||f||_{\infty}, \sup_{h>0} \frac{\omega(f,h)}{\omega(h)}\right)$$

where $||f||_{\infty}$ is the usual supremum norm.

The normed space $(C_{\omega}, ||\cdot||_{\omega})$ is a Banach space. However, trigonometric polynomials are not dense in $(C_{\omega}, ||\cdot||_{\omega})$. For if $\omega(h) \neq O(h)$ $(h \rightarrow 0)$, then whenever (T_n) is a sequence of trigonometric polynomials which converge in $||\cdot||_{\omega}$ to f, f satisfies

$$\omega(f, h) = o(\omega(h)) \quad (h \to 0).$$

In the case that $\omega(h) = O(h)$ $(h \to 0)$, then a sequence of trigonometric polynomials (T_n) converge in $||\cdot||_{\omega}$ if and only both T_n an T'_n converge uniformly and therefore f is the limit of the sequence (T_n) only if f is contin-

uously differentiable. Accordingly, when $\omega(h) \neq O(h)$ $(h \rightarrow 0)$, we define c_{ω} as the class of those functions in C_{ω} for which

$$\omega(f, h) = o(\omega(h)) \quad (h \to 0)$$

and when $\omega(h) = O(h)$ $(h \to 0)$ we define c_{ω} as the class of all continuously differentiable functions. c_{ω} is then a closed subspace of C_{ω} and it is easy to see that if $f \in c_{\omega}$, the Fejer sums of f

$$\sigma_n(f) = \int_0^{2\pi} f(t) F_n(t-x) dt$$

with

$$F_n(t) = \frac{1}{2\pi (n+1)} \left(\frac{\sin (n+1) \frac{1}{2} t}{\sin \frac{1}{2} t} \right)^2$$

converges in $||\cdot||_{\omega}$ to f. Thus, c_{ω} is precisely the closure of the class of trigonometric polynomials in $||\cdot||_{\omega}$. It therefore appears some what more natural to consider the class c_{ω} rather than the class C_{ω} in terms of problems involving multiplier sequences. For we then have

PROPOSITION 1. The sequence $(\lambda_k) \in (c_{\omega}, C_F)$ if and only if

$$||| \Lambda_n |||_{\omega} \equiv \sup_{\substack{f \in c_{\omega} \\ ||f||_{\omega} \leq 1}} || \int_{0}^{2\pi} f(t) \Lambda_n (t-x) dt ||_{\infty} = O(1) \quad (n \to \infty).$$

This is an immediate application of the Banach-Steinhaus theorem [5, p. 60] and the fact that the operators

$$L_n(f)(x) = \int_0^{2\pi} f(t) \Lambda_n(t-x) dt$$

converge in $\|\cdot\|_{\infty}$ for each trigonometric polynomial T.

We shall find it convenient to use the following proposition which follows immediately from the fact that any function f in C_{ω} with $||f||_{\omega} \leq 1$ is the uniform limit of sequence of functions from the unit ball of $(c_{\omega}, ||\cdot||_{\omega})$ (e.g. $\sigma_n(f)$ provides such a sequence of functions).

PROPOSITION 2. If Λ (t) is an integrable function then

$$||| \Lambda |||_{\omega} = \sup_{\substack{f \in C_{\omega} \\ ||f||_{\omega} \leq 1}} || \int_{0}^{2\pi} f(t) \Lambda_{n}(t-x) dt ||_{\infty}$$

In section 2, we shall consider quasi-convex sequences and show that in this case $(\lambda_k) \in (c_\omega, C_F)$ if and only if

$$\lambda_n \omega\left(\frac{1}{n}\right) \log n = O(1) \quad (n \to \infty).$$

In section 3, we shall give a necessary condition that (λ_k) be in (c_{ω}, C_F) with no restrictions on (λ_k) . We shall show that $(\lambda_k) \in (c_{\omega}, C_F)$ only if

$$\omega\left(\frac{1}{n}\right)\int_{0}^{2\pi}|\Lambda_{n}(t)|dt=O(1)\quad (n\to\infty).$$

It is easy to see that this condition is in general not sufficient. For example, if $\omega(h) = h$, then simple integration by parts (see theorem 4.2) shows that

$$||| \Lambda_n |||_{\omega} = \int_0^{2\pi} |\int_0^t \Lambda_n(x) dx | dt + O(1) \quad (n \to \infty)$$

thus, if we let

$$\lambda_n = \begin{cases} n, n = 2^k \\ o, n \neq 2^k \end{cases} \quad k = 0, 1, 2, \dots$$

then

$$\int_{0}^{2\pi} |\Lambda_{n}(t)| dt = \int_{0}^{2\pi} |\sum_{0}^{\lceil \log_{2} n \rceil} 2^{k} \cos 2^{k} t | dt = O(n) \quad (n \to \infty).$$

Whereas,

$$\int_{0}^{2\pi} \left| \int_{0}^{t} \Lambda_{n}(x) dx \right| dt = \int_{0}^{2\pi} \left| \sum_{0}^{\lfloor \log_{2} n \rfloor} \sin 2^{k} t \right| dt$$

and it follows from a theorem of Helson [6] that

$$\int_{0}^{2\pi} \left| \int_{0}^{t} \Lambda_{n}(x) dx \right| dt \neq O(1) \quad (n \to \infty).$$

In section 4, we shall examine sufficient conditions for (λ_k) to be in (c_{ω}, C_F) . First we shall obtain the result analogous to that of Bojanic. In particular, using the necessary condition given in Section 3, we shall prove that if (λ_k) is a Stieltjes sequence then $(\lambda_k) \in (c_{\omega}, C_F)$ if and only if

$$\omega\left(\frac{1}{n}\right)\int_{0}^{2\pi}|\Lambda_{n}(t)|dt=O(1) \qquad (N\to\infty)$$

Finally, we shall give a sufficient condition for (λ_k) to be in (c_{ω}, C_F) with no restrictions on (λ_k) . We shall show that $(\lambda_k) \in (c_{\omega}, C_F)$ if

(1.5)
$$\omega\left(\mu_{n}\right) \int_{0}^{2\pi} |\Lambda_{n}(t)| dt = O(1)$$

where

$$\mu_n = \frac{\int\limits_0^{2\pi} \left| \int\limits_0^t \Lambda_n(x) dx \right| dt}{\int\limits_0^{2\pi} \left| \Lambda_n(t) \right| dt}.$$

This condition is also necessary in the case that $\omega(h) = O(h) (h \to 0)$. However, it is generally not necessary. For example, if F(x) is the classical Lebesgue function (see [7, p. 195]), then $F(x) - \frac{x}{2\pi}$ is continuous, of bounded variation, and its Fourier coefficients are not $o\left(\frac{1}{n}\right)(n\to\infty)$. Thus, if (λ_k) is the sequence of Fourier-Stieljes coefficients of $d\left(F(t) - \frac{t}{2\pi}\right)$ we have using the theorem of Dirichlet-Jordan [7, p. 57] that

$$\lim_{n\to\infty} \int_{0}^{2\pi} |\sum_{k=0}^{n} \frac{\lambda_{k}}{k} \sin kt | dt = \int_{0}^{2\pi} |F(t) - \frac{t}{2\pi}| dt > 0.$$

while by the result of Helson [6]

$$\int_{0}^{2\pi} \left| \sum_{k=0}^{n} \lambda_{k} \cos kt \right| dt \neq O(1) \quad (n \to \infty).$$

Also,

$$\int_{0}^{2\pi} \left| \sum_{k=0}^{n} \lambda_{k} \cos kt \right| dt = O(\log n) \quad (n \to \infty)$$

since it is a Fourier-Stieljes series. So that, if we choose ω to satisfy the conditions

$$\omega\left(\frac{1}{n}\right)\int_{0}^{2\pi} |\sum_{k=0}^{n} \lambda_{k} \cos kt| dt = O(1) \quad (n \to \infty)$$

and

$$\omega(\mu_n) \int_0^{2\pi} |\sum_{k=0}^n \lambda_k \cos kt| dt \neq O(1) \qquad (n \to \infty)$$

with

$$\mu_n = \frac{\int\limits_0^{2\pi} \left| \sum\limits_0^{2n} \frac{\lambda_k}{k} \sin kt \right| dt}{\int\limits_0^{2\pi} \left| \sum\limits_0^{n} \lambda_k \cos kt \right| dt}$$

we see that (1.5) is in general not necessary.

Although, we give necessary and sufficient conditions for (λ_k) to be in (c_{ω}, C_F) in the case that (λ_k) is quasi-convex or a Stieljes sequence in general no conditions that are both necessary and sufficient are known.

2. Quasi-convex sequences. We consider first the simplest case of quasi convex sequences. If we apply Abel summation twice we find

$$\Lambda_{n}(t) = \sum_{0}^{n} (k+1) \Delta^{2} \lambda_{k} F_{k}(t) + n \Delta \lambda_{n-1} F_{n}(t) + \lambda_{n} D_{n}(t)$$

where D_n is the Dirichlet kernel

$$D_n(t) = \frac{1}{2\pi} \frac{\sin((n+\frac{1}{2})t)}{\sin\frac{1}{2}t}.$$

From the quasi-convexity and the fact that $\int_{0}^{2\pi} |F_{n}(t)| dt = 1$, we have

$$|||\sum_{0}^{n} (k+1) \Delta^{2} \lambda_{k} F_{k}|||_{\omega} \leq \int_{0}^{2\pi} |\sum_{0} (k+1) \Delta^{2} \lambda_{k} F_{k}(t)| dt = O(1) \quad (n \to \infty)$$

for any modulus of continuity ω . Thus

$$(2.1) ||| \Lambda_n |||_{\omega} = O(1) + ||| n \Delta \lambda_{n-1} F_n + \lambda_n D_n |||_{\omega} (n \to \infty)$$

It follows from standard estimates that there exist positive constants C_1 , C_2 such that

(2.2)
$$C_1 \omega \left(\frac{1}{n}\right) \log n \leq ||D_n||_{\omega} \leq C_2 \omega \left(\frac{1}{n}\right) \log n.$$

This result is contained in theorems (3.1) and (4.1) so we shall not supply an independent proof.

The main result of this section is

THEOREM 2.1. If (λ_k) is a quasi-convex sequence then $(\lambda_k) \in (c_\omega, C_F)$ if and only if

(2.1)
$$\lambda_n \omega\left(\frac{1}{n}\right) \log n = O(1) \quad (n \to \infty).$$

Proof: We first consider the case when (λ_n) is a bounded sequence. Then by a result of Tomic [3]

$$n \Delta \lambda_{n-1} = o(1).$$

Thus from (2.1) we have

$$||| \Lambda_n |||_{\omega} = O(1) + ||| \lambda_n D_n |||_{\omega}$$

and the theorem follows immediately from the inequalities (2.2).

We shall now show that the case (λ_k) unbounded does not arise. Tomić [3] has shown that if (λ_k) is quasi convex and unbounded then

(2.3)
$$\lambda_n = An + B + o(1) \quad (n \to \infty)$$

and

(2.4)
$$n \Delta \lambda_{n-1} = -An + o\left(\frac{1}{n}\right). \quad (n \to \infty)$$

thus if

$$\lambda_n \omega \left(\frac{1}{n}\right) \log n = O(1) \quad (n \to \infty)$$

we must have

$$\frac{\lambda_n}{n}\log n = O(1) \quad (n \to \infty)$$

and therefor (λ_n) cannot satisfy (2.3) and the conditions (2.1) and (λ_k) unbounded are not compatible. Secondly, if (λ_k) is unbounded then by virtue of (2.1)

$$||| \Lambda_n |||_{\omega} = O(1) + ||| n \Delta \lambda_{n-1} F_n + \lambda_n D_n |||_{\omega}$$

and thus by (2.2) (2.3), and (2.4) we must have

(2.5)
$$||| \Lambda_n |||_{\omega} \ge An - A C_2 n \omega \left(\frac{1}{n}\right) \log n .$$

For $\omega(h) = h$, (2.5) fails and thus $(\lambda_k) \notin (c_{\omega}, C_F)$ for any ω . Thus, (λ_k) unbounded and $(\lambda_k) \in (c_{\omega}, C_F)$ are also incompatible.

3. A necessary condition for (λ_k) to be in (c_{ω}, C_F) . In this section, we shall give a necessary condition for (λ_k) to be in (c_{ω}, C_F) . Our main result is the following theorem.

THEOREM 3.1. There exists an absolute constant C>0 such that for any trigonometric polynomial T of degree n we have

$$|||T|||_{\omega} \ge C\omega \left(\frac{1}{n}\right) \int_{0}^{2\pi} |T| dt \qquad n = 1, 2, \dots$$

An immediate corollary of this theorem and Proposition 1 is

COROLLARY 3.1. A necessary condition for the sequence (λ_k) to be in (c_{ω}, C_F) is that

$$\omega\left(\frac{1}{n}\right)\int_{0}^{2\pi}|\Lambda_{n}|dt=O(1), \quad (n\to\infty)$$

We shall need some preliminary results concerning representations of trigonometric polynomials. Let $x_k = \frac{2k\pi}{3n}$, k = 0, 1, 2, ..., 3n-1. Then if T is a trigonometric polynomial of degree n, we have (see [8, p. 33])

(3.1)
$$T(x) = \frac{2}{3n} \sum_{k=0}^{3n-1} T(x_k) K_n(x - x_k)$$

where

(3.2)
$$K_n(t) = \frac{1}{\pi} \frac{\sin\left(\frac{3n}{2}t\right)\sin\left(\frac{n}{2}t\right)}{2n\left(\sin\frac{t}{2}\right)^2}.$$

Also [8, p. 33]

(3.3)
$$\int_{0}^{2\pi} |T(x)| dx \leq \frac{1}{n} \sum_{k=0}^{3n-1} |T(x_{k})|.$$

Now to the proof of theorem (3.1). Let $0 < \delta < \frac{1}{4}$. We wish to estimate

$$\int_{\frac{-\pi\delta}{3n}}^{\frac{\pi\delta}{3n}} K_n(t) dt$$

from below. We have for $|t| \le \frac{\pi \delta}{3n}$

$$K_n(t) \ge \frac{1}{\pi} \left(\frac{\left(\frac{2}{\pi}\right) \left(\frac{3nt}{2}\right) \left(\frac{2}{\pi}\right) \left(\frac{nt}{2}\right)}{2n \left(\frac{t}{2}\right)^2} \right) = \frac{6}{\pi^3} n.$$

So that,

(3.4)
$$\int_{\frac{-\pi\delta}{3n}}^{\frac{\delta\pi}{3n}} K_n(t) dt \ge \frac{6n}{\pi^3} \cdot \frac{2\pi\delta}{3n} = \frac{4}{\pi^2} \delta.$$

Secondly, for $k \neq 0$ we estimate $\int_{X_k - \frac{\pi}{3n}}^{X_k + \frac{2\pi\delta}{3n}} K_n(t) dt$ from above. For

$$|t-x_k| \leq \frac{2\pi\delta}{3n}$$
, we have

$$K_n(t) \leq \frac{\sin\frac{\delta\pi}{2}}{2n\left(\frac{2\pi}{3n}(k-\frac{1}{2})\right)^2} \leq \frac{\delta\pi}{4n} \frac{1}{\left(\frac{2\pi}{3n}(k-\frac{1}{2})\right)^2} = \frac{9\delta}{8\pi} \frac{n}{(k-\frac{1}{2})^2}.$$

Thus

(3.5)
$$\int_{x_k - \frac{2\pi\delta}{3n}} |K_n(t)| dt \le \frac{4\delta\pi}{3n} \cdot \frac{9\delta}{8\pi} \frac{n}{(k - \frac{1}{2})^2} = \frac{3}{2} \frac{\delta^2}{(k - \frac{1}{2})^2}.$$

Let $g_{\delta}(x)$ be the 2π -periodic continuous function which has the value one on the interval $\left[\frac{-\pi\delta}{3n}, \frac{\pi\delta}{3n}\right]$ has the value zero on $[-\pi, \pi] - \left[\frac{-2\pi\delta}{3n}, \frac{2\pi\delta}{3n}\right]$ and is linear on the intervals $\left[\frac{-\pi\delta}{3n}, \frac{-\pi\delta}{3n}\right]$ and $\left[\frac{\pi\delta}{3n}, \frac{2\pi\delta}{3n}\right]$.

The function

$$\bar{g}_{\delta}(x) = \omega \left(\frac{\delta \pi}{3n}\right) \sum_{k=0}^{3n-1} Sgn(T(x_k)) g_{\delta}(x - x_k)$$

is in C_{ω} and $||\bar{g}_{\delta}||_{\omega} \leq 1$. Also,

$$T(x_{k}) \int_{0}^{2\pi} \bar{g}_{\delta}(x) K_{n}(x - x_{k}) dx \ge \omega \left(\frac{\delta \pi}{3n}\right) |T(x_{k})| \int_{X_{k} - \frac{\pi \delta}{3n}} |K_{n}(x - x_{k})| dx$$

$$- \omega \left(\frac{\delta \pi}{3n}\right) |T(x_{k})| \sum_{\substack{j=0 \ j \neq k}}^{3n-1} \int_{X_{j} - \frac{2\pi \delta}{3n}} |K_{n}(x - x_{k})| dx$$

which by virtue of (3.4) and (3.5) is

$$\geq \omega \left(\frac{\delta \pi}{3n} \right) | T(x_k) | \left(\frac{4}{\pi^2} \delta - \frac{3}{2} \delta^2 \sum_{\substack{j=0 \ j \neq k}}^{3n-1} \frac{1}{(j-k-\frac{1}{2})^2} \right)$$

$$\geq \omega \left(\frac{\delta \pi}{3n} \right) | T(x_k) | \left(\frac{4}{\pi^2} \delta - \frac{3}{2} \delta^2 \sum_{\substack{j=0 \ j \neq k}}^{\infty} \frac{1}{(j-\frac{1}{2})^2} \right)$$

Thus if we choose $\delta_0 > 0$ such that

$$\left(\frac{4}{\pi^2}\,\delta_0\,-\frac{3}{2}\,\delta_0^2\,\sum_{j=0}^\infty\frac{1}{\left(j-\frac{1}{2}\right)^2}\right)=\,C_0>0$$

We have, using the elementary properties of a modulus of continuity that

$$T(x_{k}) \int_{0}^{2\pi} \bar{g}_{\delta_{0}}(x) K_{n}(x-x_{k}) dx \ge C\omega\left(\frac{1}{n}\right) |T(x_{k})| k = 0, 1, 2, ..., 3n - 1$$

where C is an absolute positive constant. Finally,

$$\int_{0}^{2\pi} \bar{g}_{\delta_{o}}(x) T(x) dx = \frac{2}{3n} \sum_{k=0}^{3n-1} T(x_{k}) \int_{0}^{2\pi} \bar{g}_{\delta_{o}}(x) K_{n}(x - x_{k}) dx \ge$$

$$\geq C \omega \left(\frac{1}{3n}\right) \cdot \frac{2}{3n} \sum_{k=0}^{3n-1} |T(x_{k})|$$

which by virtue of (3.3.) is

$$\geq \frac{2}{3} C \omega \left(\frac{1}{n}\right) \int_{0}^{2\pi} |T(x)| dx.$$

Thus, using Proposition 2,

$$||| T_n |||_{\omega} \ge \int_{0}^{2\pi} \bar{g}_{\delta_0}(x) T(x) dx \ge \frac{2}{3} C \omega \left(\frac{1}{n}\right) \int_{0}^{2\pi} |T(x)| dx$$

and the theorem is proved.

4. Sufficient conditions for (λ_k) to be in (c_{ω}, C_F) . We first establish the result analogous to that of Bojanic (1.3) and (1.4). The proof is essentially that of Haršiladze [9].

THEOREM 4. 1. If (λ_k) is a Stieljes sequence and if

$$\omega\left(\frac{1}{n}\right)\int_{0}^{2\pi} |\Lambda_{n}(x)| dx = O(1) \quad (n \to \infty)$$

then $(\lambda_k) \in (c_{\omega}, C_F)$.

Proof: Let $V_n(f)$ be the de la Vallée Poussin sums of f

$$V_n(f) = \int_0^{2\pi} f(t) \left(2F_{2n}(t-x) - F_n(t-x) \right) dt.$$

It is well known [10, p. 92] that

$$(4.1) ||f - V_n(f)||_{\infty} \le C \omega \left(f, \frac{1}{n}\right)$$

where C is a constant independent of f and n. Also if T is a trigonometric polynomial of degree n then

$$V_n(T) = T$$
.

Thus if $f \in C_{\omega}$, $||f||_{\omega} \leq 1$

$$\int_{0}^{2\pi} f(t) \Lambda_{n}(t-x) dt = \int_{0}^{2\pi} (f(t) - V_{n}(f)(t)) \Lambda_{n}(t-x) dt + \int_{0}^{2\pi} V_{n}(f)(t) \Lambda_{n}(t-x) dt.$$

We have

$$\int_{0}^{2\pi} |\int_{0}^{2\pi} (2F_{2n}(t) - F_{n}(t)) \Lambda_{n}(t-x) dt | dx = O(1) \quad (n \to \infty).$$

Since (λ_k) is a Stieltjes sequence. Thus

$$||\int_{0}^{2\pi} f(t) \Lambda_{n}(t-x) dt ||_{\infty} \leq ||\int_{0}^{2\pi} (f(t) - V_{n}(f)(t)(\Lambda_{n}(t-x)) dt ||_{\infty} + ||f||_{\infty} \int_{0}^{2\pi} ||\int_{0}^{2\pi} (2F_{2n}(t) - F_{n}(t)(\Lambda_{n}(t-x)) dt || dx \leq ||\int_{0}^{2\pi} (f(t) - V_{n}(f)(t)) \Lambda_{n}(t-x) dt ||_{\infty} + O(1) \quad (n \to \infty)$$

which by virtue of (4.1) is

$$\leq C \omega \left(\frac{1}{n}\right) \int_{0}^{2\pi} |\Lambda_{n}(t)| dt + O(1) \quad (n \to \infty).$$

As a corollary of theorem 4.1 and theorem 3.1, we have

COROLLARY 4.1. A Stieljes Sequence (λ_k) is in (c_{ω}, C_F) if and only if

$$\omega\left(\frac{1}{n}\right)\int_{0}^{2\pi} |\Lambda_{n}(t)| dt = O(1) \quad (n \to \infty).$$

We shall now give a sufficient condition for (λ_k) to be in (c_{ω}, C_F) which requires no special restriction on (λ_k) .

THEOREM 4.2. A sufficient condition for (λ_k) to be in (c_{ω}, C_F) is that

(4.2)
$$\omega(\mu_n) \int_0^{2\pi} |\Lambda_n(t)| dt = O(1) \quad (n \to \infty)$$

where

$$\mu_{n} = \frac{\int_{0}^{2\pi} \int_{0}^{x} \Lambda_{n}(t) dt | dx}{\int_{0}^{2\pi} |\Lambda_{n}(t)| dt} n = 0, 1, 2, \dots.$$

If $\omega(h) = h$ then (4.2) is also necessary.

Proof: We consider first the case when $\omega(h) = h$. If $f \in C_{\omega}$ with $||f||_{\omega} \le 1$ then

$$|f'(x)| \le 1 a. e.$$

So that

$$\left| \int_{0}^{2\pi} f(t) \Lambda_{n}(t-x) dt \right| = \left| \int_{0}^{2\pi} f'(t) \overline{\Lambda}_{n}(t-x) dt \right| \leq \int_{0}^{2\pi} \left| \overline{\Lambda}_{n}(t) \right| dt$$
with $\overline{\Lambda}_{n}(t) = \int_{0}^{t} \Lambda_{n}(u) du$.

Thus,

$$||| \Lambda_n |||_{\omega} \leq \int_0^{2\pi} |\overline{\Lambda}_n(t)| dt,$$

the function $g(x) = \frac{1}{2\pi} sgn \int_{0}^{x} \Lambda_{n}(t) dt$ is in C_{ω} and $||g||_{\omega} \le 1$. Also

$$\int_{0}^{2\pi} g(t) \Lambda_{n}(t) dt = |g(2\pi) \Lambda_{n}(2\pi) - \int_{0}^{2\pi} |\overline{\Lambda}_{n}(t)| dt| \ge \int_{0}^{2\pi} |\overline{\Lambda}_{n}(t)| dt - \lambda_{0}.$$

Thus,

$$\int_{0}^{2\pi} |\overline{\Lambda}_{n}(t)| dt - \lambda_{0} \leq |||\Lambda_{n}|||_{\omega} \leq \int_{0}^{2\pi} |\overline{\Lambda}_{n}(t)| dt \quad n = 1, 2, \dots$$

This shows that (4.2) is necessary and sufficient for (λ_k) to be in (c_{ω}, C_F) if $\omega(h) = h$.

Finally in the general case, the inequality

$$\left|\left|\int_{0}^{2\pi} f(t) \Lambda_{n}(t-x) dt\right|\right|_{\omega} \leq \omega(\mu_{n}) \int_{0}^{2\pi} \left|\Lambda_{n}(t)\right| dt$$

is a simple modification of Lemma 1 of [11] and we will not give its proof.

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