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# MULTIPLIERS OF UNIFORM CONVERGENCE

by Ronald DeVore

1. *Introduction.* If $A$ and $B$ are two classes of $2\pi$-periodic integrable functions we say that $(\lambda_k)$ is a multiplier sequence from $A$ into $B$ and we write $(\lambda_k) \in (A, B)$ if whenever

$$\sum_{0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a function in $A$

$$\sum_{0}^{\infty} \lambda_n (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a function in $B$. Let $C$ denote the class of $2\pi$-periodic continuous functions and $C_F$ the subclass of those functions in $C$ whose Fourier series converges uniformly. Karamata [1] has shown that $(\lambda_k) \in (C, C_F)$ if and only if

$$\int_{0}^{2\pi} | A_n(t) | \, dt = O(1) \quad (n \to \infty) \tag{1.1}$$

where

$$A_n(t) = \sum_{0}^{n} \lambda_k \cos kt.$$

This theorem contains as a special case an earlier result of Tomić [2] who showed that if $(\lambda_k)$ is monotone decreasing and convex (i.e. $\Delta^2 \lambda_k = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2} \geqq 0$) or more generally quasi-convex (i.e. $\sum_{0}^{\infty} (k+1) |\Delta^2 \lambda_k| < \infty$) then $(\lambda_k) \in (C, C_F)$ if and only if $\lambda_n \log n = O(1)$ $(n \to \infty)$.

It is interesting to see to what extent condition (1.1) can be relaxed if we restrict our attention to a sub-class of $C$ determined by some structural property. For example, let $\omega$ be a modulus of continuity and $C_\omega$ the subclass of $C$ consisting of those functions whose modulus of continuity $\omega(f, h)$ satisfies

$$\omega(f, h) = O(\omega(h)) \qquad (h \to 0).$$

Then Tomić [3] has shown that for a quasi-convex sequence $(\lambda_k)$ to be in $(C_\omega, C_F)$ it is sufficient that

$$(1.2)\qquad \omega\left(\frac{1}{n}\right)\lambda_n \log n = o(1) \qquad (n\to\infty).$$

Also Bojanic [4] has shown that sufficient conditions for $(\lambda_k)$ to be in $(C_\omega, C_F)$ are

$$(1.3)\qquad \int_0^{2\pi} \left|\sum_0^n \Lambda_k(t)\right| dt = O(n) \qquad (n\to\infty)$$

and

$$(1.4)\qquad \omega\left(\frac{1}{n}\right)\int_0^{2\pi} |\Lambda_n(t)|\, dt = o(1) \qquad (n\to\infty).$$

Of course, condition (1.3) is equivalent to $(\lambda_k)$ being a Fourier Stieljes sequence which in particular characterizes the class of multipliers $(C, C)$.

No necessary conditions have been given for $(\lambda_k)$ to be in $(C_\omega, C_F)$ and sufficient conditions have been restricted to quasi-convex and Fourier-Stieljes sequences. In order to obtain necessary and sufficient conditions for $(\lambda_k)$ to be in $(C_\omega, C_F)$, it is natural to attempt to make $C_\omega$ a Banach space in which trigonometric polynomials are dense and then invoke the Banach-Steinhaus theorem as Karamata did in characterizing $(C, C_F)$. The most natural norm is to define for $f \in C_\omega$

$$||f||_\omega = max\left(||f||_\infty, \sup_{h>0} \frac{\omega(f,h)}{\omega(h)}\right)$$

where $||f||_\infty$ is the usual supremum norm.

The normed space $(C_\omega, ||\cdot||_\omega)$ is a Banach space. However, trigonometric polynomials are not dense in $(C_\omega, ||\cdot||_\omega)$. For if $\omega(h) \neq O(h)$ $(h\to 0)$, then whenever $(T_n)$ is a sequence of trigonometric polynomials which converge in $||\cdot||_\omega$ to $f$, $f$ satisfies

$$\omega(f,h) = o(\omega(h)) \qquad (h\to 0).$$

In the case that $\omega(h) = O(h)$ $(h\to 0)$, then a sequence of trigonometric polynomials $(T_n)$ converge in $||\cdot||_\omega$ if and only both $T_n$ an $T_n'$ converge uniformly and therefore $f$ is the limit of the sequence $(T_n)$ only if $f$ is continuously differentiable. Accordingly, when $\omega(h) \neq O(h)$ $(h \to 0)$, we define $c_\omega$ as the class of those functions in $C_\omega$ for which

$$\omega(f, h) = o(\omega(h)) \qquad (h \to 0)$$

and when $\omega(h) = O(h)$ $(h \to 0)$ we define $c_\omega$ as the class of all continuously differentiable functions. $c_\omega$ is then a closed subspace of $C_\omega$ and it is easy to see that if $f \in c_\omega$, the Fejer sums of $f$

$$\sigma_n(f) = \int_0^{2\pi} f(t) F_n(t-x)\, dt$$

with

$$F_n(t) = \frac{1}{2\pi(n+1)} \left( \frac{\sin(n+1)\frac{1}{2}t}{\sin\frac{1}{2}t} \right)^2$$

converges in $||\cdot||_\omega$ to $f$. Thus, $c_\omega$ is precisely the closure of the class of trigonometric polynomials in $||\cdot||_\omega$. It therefore appears some what more natural to consider the class $c_\omega$ rather than the class $C_\omega$ in terms of problems involving multiplier sequences. For we then have

PROPOSITION 1. *The sequence* $(\lambda_k) \in (c_\omega, C_F)$ *if and only if*

$$|||\Lambda_n|||_\omega \equiv \sup_{\substack{f \varepsilon c_\omega \\ ||f||_\omega \leqq 1}} || \int_0^{2\pi} f(t) \Lambda_n(t-x)\, dt ||_\infty = O(1) \quad (n \to \infty).$$

This is an immediate application of the Banach-Steinhaus theorem [5, p. 60] and the fact that the operators

$$L_n(f)(x) = \int_0^{2\pi} f(t) \Lambda_n(t-x)\, dt$$

converge in $||\cdot||_\infty$ for each trigonometric polynomial $T$.

We shall find it convenient to use the following proposition which follows immediately from the fact that any function $f$ in $C_\omega$ with $||f||_\omega \leqq 1$ is the uniform limit of sequence of functions from the unit ball of $(c_\omega, ||\cdot||_\omega)$ (e.g. $\sigma_n(f)$ provides such a sequence of functions).

PROPOSITION 2. *If* $\Lambda$ (t) *is an integrable function then*

$$|||\Lambda|||_\omega = \sup_{\substack{f \varepsilon C_\omega \\ ||f||_\omega \leqq 1}} || \int_0^{2\pi} f(t) \Lambda_n(t-x)\, dt ||_\infty$$

In section 2, we shall consider quasi-convex sequences and show that in this case $(\lambda_k) \in (c_\omega, C_F)$ if and only if

$$\lambda_n \, \omega \left(\frac{1}{n}\right) \log n = O(1) \quad (n \to \infty).$$

In section 3, we shall give a necessary condition that $(\lambda_k)$ be in $(c_\omega, C_F)$ with no restrictions on $(\lambda_k)$. We shall show that $(\lambda_k) \in (c_\omega, C_F)$ only if

$$\omega\left(\frac{1}{n}\right) \int_0^{2\pi} | A_n(t) | \, dt = O(1) \quad (n \to \infty).$$

It is easy to see that this condition is in general not sufficient. For example, if $\omega(h) = h$, then simple integration by parts (see theorem 4.2) shows that

$$||| A_n |||_\omega = \int_0^{2\pi} | \int_0^t A_n(x) \, dx | \, dt + O(1) \quad (n \to \infty)$$

thus, if we let

$$\lambda_n = \begin{cases} n \, , n = 2^k \\ o, n \neq 2^k \qquad k = 0, 1, 2, \ldots \end{cases}$$

then

$$\int_0^{2\pi} | A_n(t) | \, dt = \int_0^{2\pi} | \sum_0^{[\log 2^n]} 2^k \cos 2^k t \, | \, dt = O(n) \quad (n \to \infty).$$

Whereas,

$$\int_0^{2\pi} | \int_0^t A_n(x) \, dx \, | \, dt = \int_0^{2\pi} | \sum_0^{[\log 2^n]} \sin 2^k t \, | \, dt$$

and it follows from a theorem of Helson [6] that

$$\int_0^{2\pi} | \int_0^t A_n(x) \, dx \, | \, dt \neq O(1) \quad (n \to \infty).$$

In section 4, we shall examine sufficient conditions for $(\lambda_k)$ to be in $(c_\omega, C_F)$. First we shall obtain the result analogous to that of Bojanic. In particular, using the necessary condition given in Section 3, we shall prove that if $(\lambda_k)$ is a Stieltjes sequence then $(\lambda_k) \in (c_\omega, C_F)$ if and only if

$$\omega\left(\frac{1}{n}\right) \int_0^{2\pi} | A_n(t) | \, dt = O(1) \quad (N \to \infty)$$

Finally, we shall give a sufficient condition for $(\lambda_k)$ to be in $(c_\omega, C_F)$ with no restrictions on $(\lambda_k)$. We shall show that $(\lambda_k) \in (c_\omega, C_F)$ if

$$(1.5) \qquad \omega(\mu_n) \int_0^{2\pi} | \Lambda_n(t) | \, dt = O(1)$$

where

$$\mu_n = \frac{\int_0^{2\pi} | \int_0^t \Lambda_n(x) \, dx | \, dt}{\int_0^{2\pi} | \Lambda_n(t) | \, dt} .$$

This condition is also necessary in the case that $\omega(h) = O(h)$ $(h \to 0)$. However, it is generally not necessary. For example, if $F(x)$ is the classical Lebesgue function (see [7, p. 195]), then $F(x) - \frac{x}{2\pi}$ is continuous, of bounded variation, and its Fourier coefficients are not $o\left(\frac{1}{n}\right)$ $(n \to \infty)$. Thus, if $(\lambda_k)$ is the sequence of Fourier-Stieljes coefficients of $d\left(F(\mathrm{t}) - \frac{t}{2\pi}\right)$ we have using the theorem of Dirichlet-Jordan [7, p. 57] that

$$\lim_{n \to \infty} \int_0^{2\pi} | \sum_0^n \frac{\lambda_k}{k} \sin kt | \, dt = \int_0^{2\pi} | F(t) - \frac{t}{2\pi} | \, dt > 0 .$$

while by the result of Helson [6]

$$\int_0^{2\pi} | \sum_0^n \lambda_k \cos kt | \, dt \neq O(1) \quad (n \to \infty) .$$

Also,

$$\int_0^{2\pi} | \sum_0^n \lambda_k \cos kt | \, dt = O(\log n) \quad (n \to \infty)$$

since it is a Fourier-Stieljes series. So that, if we choose $\omega$ to satisfy the conditions

$$\omega\left(\frac{1}{n}\right) \int_0^{2\pi} | \sum_0^n \lambda_k \cos kt | \, dt = O(1) \quad (n \to \infty)$$

and

$$\omega(\mu_n) \int_0^{2\pi} \left| \sum_0^n \lambda_k \cos kt \right| dt \neq O(1) \qquad (n \to \infty)$$

with

$$\mu_n = \frac{\int_0^{2\pi} \left| \sum_0^{2n} \frac{\lambda_k}{k} \sin kt \right| dt}{\int_0^{2\pi} \left| \sum_0^n \lambda_k \cos kt \right| dt}$$

we see that (1.5) is in general not necessary.

Although, we give necessary and sufficient conditions for $(\lambda_k)$ to be in $(c_\omega, C_F)$ in the case that $(\lambda_k)$ is quasi-convex or a Stieljes sequence in general no conditions that are both necessary and sufficient are known.

2. *Quasi-convex sequences.* We consider first the simplest case of quasi convex sequences. If we apply Abel summation twice we find

$$\Lambda_n(t) = \sum_0^n (k+1) \Delta^2 \lambda_k F_k(t) + n \Delta \lambda_{n-1} F_n(t) + \lambda_n D_n(t)$$

where $D_n$ is the Dirichlet kernel

$$D_n(t) = \frac{1}{2\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin \frac{1}{2} t} .$$

From the quasi-convexity and the fact that $\int_0^{2\pi} |F_n(t)| dt = 1$, we have

$$||| \sum_0^n (k+1) \Delta^2 \lambda_k F_k |||_\omega \leqq \int_0^{2\pi} \left| \sum (k+1) \Delta^2 \lambda_k F_k(t) \right| dt = O(1) \qquad (n \to \infty)$$

for any modulus of continuity $\omega$. Thus

(2.1) $$||| \Lambda_n |||_\omega = O(1) + ||| n \Delta \lambda_{n-1} F_n + \lambda_n D_n |||_\omega \qquad (n \to \infty)$$

It follows from standard estimates that there exist positive constants $C_1, C_2$ such that

(2.2) $$C_1 \, \omega\left(\frac{1}{n}\right) \log n \leqq ||| D_n |||_\omega \leqq C_2 \, \omega\left(\frac{1}{n}\right) \log n .$$

This result is contained in theorems (3.1) and (4.1) so we shall not supply an independent proof.

The main result of this section is

THEOREM 2.1. *If* $(\lambda_k)$ *is a quasi-convex sequence then* $(\lambda_k) \in (c_\omega, C_F)$ *if and only if*

$$\lambda_n\, \omega\left(\frac{1}{n}\right) \log n \;=\; O(1) \quad (n \to \infty). \tag{2.1}$$

Proof: We first consider the case when $(\lambda_n)$ is a bounded sequence. Then by a result of Tomic [3]

$$n\, \Delta\, \lambda_{n-1} \;=\; o(1)\,.$$

Thus from (2.1) we have

$$|||\, A_n\, |||_\omega \;=\; O(1) + |||\, \lambda_n D_n\, |||_\omega$$

and the theorem follows immediately from the inequalities (2.2).

We shall now show that the case $(\lambda_k)$ unbounded does not arise. Tomić [3] has shown that if $(\lambda_k)$ is quasi convex and unbounded then

$$\lambda_n \;=\; An + B + o(1) \qquad (n \to \infty) \tag{2.3}$$

and

$$n\, \Delta\, \lambda_{n-1} \;=\; -\; An + o\left(\frac{1}{n}\right). \qquad (n \to \infty) \tag{2.4}$$

thus if

$$\lambda_n\, \omega\left(\frac{1}{n}\right) \log n \;=\; O(1) \qquad (n \to \infty)$$

we must have

$$\frac{\lambda_n}{n} \log n \;=\; O(1) \qquad (n \to \infty)$$

and therefor $(\lambda_n)$ cannot satisfy (2.3) and the conditions (2.1) and $(\lambda_k)$ unbounded are not compatible. Secondly, if $(\lambda_k)$ is unbounded then by virtue of (2.1)

$$|||\, A_n\, |||_\omega \;=\; O(1) + |||\, n\, \Delta\, \lambda_{n-1} F_n + \lambda_n D_n\, |||_\omega$$

and thus by (2.2) (2.3), and (2.4) we must have

$$|||\, A_n\, |||_\omega \geqq An - A\, C_2\, n\, \omega\left(\frac{1}{n}\right) \log n\,. \tag{2.5}$$

For $\omega(h) = h$, (2.5) fails and thus $(\lambda_k) \notin (c_\omega, C_F)$ for any $\omega$. Thus, $(\lambda_k)$ unbounded and $(\lambda_k) \in (c_\omega, C_F)$ are also incompatible.

3. *A necessary condition for* $(\lambda_k)$ *to be in* $(c_\omega, C_F)$. In this section, we shall give a necessary condition for $(\lambda_k)$ to be in $(c_\omega, C_F)$. Our main result is the following theorem.

THEOREM 3.1. *There exists an absolute constant* $C>0$ *such that for any trigonometric polynomial* $T$ *of degree* n we have

$$|||T|||_\omega \geqq C\omega\left(\frac{1}{n}\right)\int_0^{2\pi} |T|\,dt \qquad n = 1, 2, \ldots$$

An immediate corollary of this theorem and Proposition 1 is

COROLLARY 3.1. *A necessary condition for the sequence* $(\lambda_k)$ *to be in* $(c_\omega, C_F)$ *is that*

$$\omega\left(\frac{1}{n}\right)\int_0^{2\pi} |A_n|\,dt = O(1), \qquad (n\to\infty)$$

We shall need some preliminary results concerning representations of trigonometric polynomials. Let $x_k = \dfrac{2k\pi}{3n}$, $k = 0, 1, 2, \ldots, 3n-1$. Then if $T$ is a trigonometric polynomial of degree $n$, we have (see [8, p. 33])

$$T(x) = \frac{2}{3n}\sum_0^{3n-1} T(x_k)\,K_n(x-x_k) \tag{3.1}$$

where

$$K_n(t) = \frac{1}{\pi}\,\frac{\sin\left(\frac{3n}{2}t\right)\sin\left(\frac{n}{2}t\right)}{2n\left(\sin\frac{t}{2}\right)^2}\,. \tag{3.2}$$

Also [8, p. 33]

$$\int_0^{2\pi} |T(x)|\,dx \leqq \frac{1}{n}{\sum_0^{3n-1}}' |T(x_k)|\,. \tag{3.3}$$

Now to the proof of theorem (3.1). Let $0<\delta<\frac{1}{4}$. We wish to estimate

$$\int_{-\frac{\pi\delta}{3n}}^{\frac{\pi\delta}{3n}} K_n(t)\,dt$$

from below. We have for $|t| \leqq \frac{\pi\delta}{3n}$

$$K_n(t) \geqq \frac{1}{\pi}\left(\frac{\left(\frac{2}{\pi}\right)\left(\frac{3nt}{2}\right)\left(\frac{2}{\pi}\right)\left(\frac{nt}{2}\right)}{2n\left(\frac{t}{2}\right)^2}\right) = \frac{6}{\pi^3}n.$$

So that,

$$(3.4) \qquad \int_{\frac{-\pi\delta}{3n}}^{\frac{\delta\pi}{3n}} K_n(t)\,dt \geqq \frac{6n}{\pi^3}\cdot\frac{2\pi\delta}{3n} = \frac{4}{\pi^2}\delta.$$

Secondly, for $k \neq 0$ we estimate $\int_{x_k - \frac{\pi}{3n}}^{x_k + \frac{2\pi\delta}{3n}} K_n\,(t)\,dt$ from above. For $|t - x_k| \leqq \frac{2\pi\delta}{3n}$, we have

$$K_n(t) \leqq \frac{\sin\frac{\delta\pi}{2}}{2n\left(\frac{2\pi}{3n}(k-\frac{1}{2})\right)^2} \leqq \frac{\delta\pi}{4n}\,\frac{1}{\left(\frac{2\pi}{3n}(k-\frac{1}{2})\right)^2} = \frac{9\delta}{8\pi}\,\frac{n}{(k-\frac{1}{2})^2}.$$

Thus

$$(3.5) \qquad \int_{x_k - \frac{2\pi\delta}{3n}}^{x_k + \frac{2\pi}{3n}} |\,K_n(t)\,|\,dt \leqq \frac{4\,\delta\pi}{3n}\cdot\frac{9\delta}{8\pi}\,\frac{n}{(k-\frac{1}{2})^2} = \frac{3}{2}\,\frac{\delta^2}{(k-\frac{1}{2})^2}.$$

Let $g_\delta(x)$ be the $2\pi$-periodic continuous function which has the value one on the interval $\left[\frac{-\pi\delta}{3n}, \frac{\pi\delta}{3n}\right]$ has the value zero on $[-\pi, \pi] - \left[\frac{-2\pi\delta}{3n}, \frac{2\pi\delta}{3n}\right]$ and is linear on the intervals $\left[\frac{-\pi\delta}{3n}, \frac{-\pi\delta}{3n}\right]$ and $\left[\frac{\pi\delta}{3n}, \frac{2\pi\delta}{3n}\right]$.

The function

$$\bar{g}_\delta(x) = \omega\left(\frac{\delta\pi}{3n}\right)\sum_{k=0}^{3n-1} Sgn\,(T(x_k))\,g_\delta(x - x_k)$$

is in $C_\omega$ and $\|\bar{g}_\delta\|_\omega \leqq 1$. Also,

$$T(x_k)\int_0^{2\pi} \bar{g}_\delta(x)\, K_n(x-x_k)\, dx \geqq \omega\left(\frac{\delta\pi}{3n}\right) |\, T(x_k)\,| \int_{x_k-\frac{\pi\delta}{3n}}^{x_k+\frac{\pi\delta}{3n}} |\, K_n(x-x_k)\,|\, dx$$

$$-\ \omega\left(\frac{\delta\pi}{3n}\right) |\, T(x_k)\,| \sum_{\substack{j=0\\ j\neq k}}^{3n-1} \int_{x_j-\frac{2\pi\delta}{3n}}^{x_j+\frac{2\pi\delta}{3n}} |\, K_n(x-x_k)\,|\, dx$$

which by virtue of (3.4) and (3.5) is

$$\geqq \omega\left(\frac{\delta\pi}{3n}\right) |\, T(x_k)\,| \left(\frac{4}{\pi^2}\,\delta - \frac{3}{2}\,\delta^2 \sum_{\substack{j=0\\ j\neq k}}^{3n-1} \frac{1}{(j-k-\frac{1}{2})^2}\right)$$

$$\geqq \omega\left(\frac{\delta\pi}{3n}\right) |\, T(x_k)\,| \left(\frac{4}{\pi^2}\,\delta - \frac{3}{2}\,\delta^2 \sum_{j=0}^{\infty} \frac{1}{(j-\frac{1}{2})^2}\right)$$

Thus if we choose $\delta_0 > 0$ such that

$$\left(\frac{4}{\pi^2}\,\delta_0 - \frac{3}{2}\,\delta_0^2 \sum_{j=0}^{\infty} \frac{1}{(j-\frac{1}{2})^2}\right) = C_0 > 0$$

We have, using the elementary properties of a modulus of continuity that

$$T\,(x_k)\int_0^{2\pi} \bar{g}_{\delta_o}(x)\, K_n(x-x_k)\, dx \geqq C\omega\left(\frac{1}{n}\right) |\, T(x_k)\,| \quad k = 0, 1, 2, \ldots, 3n-1$$

where $C$ is an absolute positive constant. Finally,

$$\int_0^{2\pi} \bar{g}_{\delta_o}(x)\, T(x)\, dx = \frac{2}{3n} \sum_{k=0}^{3n-1} T(x_k) \int_0^{2\pi} \bar{g}_{\delta_o}(x)\, K_n(x-x_k)\, dx \geqq$$

$$\geqq C\,\omega\left(\frac{1}{3n}\right) \cdot \frac{2}{3n} \sum_{k=0}^{3n-1} |\, T(x_k)\,|$$

which by virtue of (3.3.) is

$$\geqq \frac{2}{3}\, C\,\omega\left(\frac{1}{n}\right) \int_0^{2\pi} |\, T(x)\,|\, dx\, .$$

Thus, using Proposition 2,

$$||| T_n |||_\omega \geqq \int_0^{2\pi} \bar{g}_{\delta_0}(x)\, T(x)\, dx \geqq \frac{2}{3}\, C\, \omega\left(\frac{1}{n}\right) \int_0^{2\pi} | T(x) |\, dx$$

and the theorem is proved.

4. *Sufficient conditions for* $(\lambda_k)$ *to be in* $(c_\omega, C_F)$. We first establish the result analogous to that of Bojanic (1.3) and (1.4). The proof is essentially that of Haršiladze [9].

THEOREM 4. 1. *If* $(\lambda_k)$ *is a Stieljes sequence and if*

$$\omega\left(\frac{1}{n}\right) \int_0^{2\pi} | \Lambda_n(x) |\, dx = O(1) \qquad (n \to \infty)$$

*then* $(\lambda_k) \in (c_\omega, C_F)$.

Proof: Let $V_n(f)$ be the de la Vallée Poussin sums of $f$

$$V_n(f) = \int_0^{2\pi} f(t) \left(2F_{2n}(t-x) - F_n(t-x)\right) dt\,.$$

It is well known [10, p. 92] that

$$||f - V_n(f)||_\infty \leqq C\, \omega\left(f, \frac{1}{n}\right) \tag{4.1}$$

where $C$ is a constant independent of $f$ and $n$. Also if $T$ is a trigonometric polynomial of degree $n$ then

$$V_n(T) = T\,.$$

Thus if $f \in C_\omega$, $||f||_\omega \leqq 1$

$$\int_0^{2\pi} f(t)\, \Lambda_n(t-x)\, dt = \int_0^{2\pi} \left(f(t) - V_n(f)(t)\right) \Lambda_n(t-x)\, dt + $$

$$+ \int_0^{2\pi} V_n(f)(t) \Lambda_n(t-x)\, dt.$$

We have

$$\int_0^{2\pi} \left| \int_0^{2\pi} \left(2F_{2n}(t) - F_n(t)\right) \Lambda_n(t-x)\, dt \right| dx = O(1) \quad (n \to \infty).$$

Since $(\lambda_k)$ is a Stieltjes sequence. Thus

$$\|\int_0^{2\pi} f(t)\, \Lambda_n(t-x)\, dt\,\|_\infty \leqq \|\int_0^{2\pi} (f(t) - V_n(f)(t)( \Lambda_n(t-x)\, dt\,\|_\infty +$$

$$+ \|f\|_\infty \int_0^{2\pi} |\int_0^{2\pi} (2F_{2n}(t) - F_n(t)( \Lambda_n(t-x)\, dt\,|\, dx \leqq$$

$$\leqq \|\int_0^{2\pi} (f(t) - V_n(f)(t))\, \Lambda_n(t-x)\, dt\,\|_\infty + O(1) \quad (n \to \infty)$$

which by virtue of (4.1) is

$$\leqq C\,\omega\left(\frac{1}{n}\right) \int_0^{2\pi} |\Lambda_n(t)|\, dt + O(1) \quad (n \to \infty)\,.$$

As a corollary of theorem 4.1 and theorem 3.1, we have

COROLLARY 4.1. *A Stieljes Sequence* $(\lambda_k)$ *is in* $(c_\omega, C_F)$ *if and only if*

$$\omega\left(\frac{1}{n}\right) \int_0^{2\pi} |\Lambda_n(t)|\, dt = O(1) \quad (n \to \infty).$$

We shall now give a sufficient condition for $(\lambda_k)$ to be in $(c_\omega, C_F)$ which requires no special restriction on $(\lambda_k)$.

THEOREM 4.2. *A sufficient condition for* $(\lambda_k)$ *to be in* $(c_\omega, C_F)$ *is that*

$$\omega(\mu_n) \int_0^{2\pi} |\Lambda_n(t)|\, dt = O(1) \quad (n \to \infty) \tag{4.2}$$

where

$$\mu_n = \frac{\int_0^{2\pi} |\int_0^{x} \Lambda_n(t)\, dt\,|\, dx}{\int_0^{2\pi} |\Lambda_n(t)|\, dt} \quad n = 0, 1, 2, \dots .$$

*If* $\omega(h) = h$ *then* (4.2) *is also necessary.*

Proof: We consider first the case when $\omega(h) = h$.
If $f \in C_\omega$ with $\|f\|_\omega \leqq 1$ then

$$|f'(x)| \leqq 1 \; a.\ e.$$

So that

$$\left| \int_0^{2\pi} f(t)\, \Lambda_n(t-x)\, dt \right| = \left| \int_0^{2\pi} f'(t)\, \overline{\Lambda}_n(t-x)\, dt \right| \leqq \int_0^{2\pi} |\overline{\Lambda}_n(t)|\, dt$$

$$\text{with } \overline{\Lambda}_n(t) = \int_0^t \Lambda_n(u)\, du.$$

Thus,

$$|||\Lambda_n|||_\omega \leqq \int_0^{2\pi} |\overline{\Lambda}_n(t)|\, dt,$$

the function $g(x) = \frac{1}{2\pi} sgn \int_0^x \Lambda_n(t)\, dt$ is in $C_\omega$ and $||g||_\omega \leqq 1$. Also

$$\int_0^{2\pi} g(t)\, \Lambda_n(t)\, dt = \left| g(2\pi)\, \Lambda_n(2\pi) - \int_0^{2\pi} |\overline{\Lambda}_n(t)|\, dt \right| \geqq \int_0^{2\pi} |\overline{\Lambda}_n(t)|\, dt - \lambda_0.$$

Thus,

$$\int_0^{2\pi} |\overline{\Lambda}_n(t)|\, dt - \lambda_0 \leqq |||\Lambda_n|||_\omega \leqq \int_0^{2\pi} |\overline{\Lambda}_n(t)|\, dt \qquad n = 1, 2, \ldots$$

This shows that (4.2) is necessary and sufficient for $(\lambda_k)$ to be in $(c_\omega, C_F)$ if $\omega(h) = h$.

Finally in the general case, the inequality

$$\left|\left| \int_0^{2\pi} f(t)\, \Lambda_n(t-x)\, dt \right|\right|_\omega \leqq \omega(\mu_n) \int_0^{2\pi} |\Lambda_n(t)|\, dt$$

is a simple modification of Lemma 1 of [11] and we will not give its proof.

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