

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 14 (1968)  
**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** REPRESENTATIONS OF COMPACT GROUPS AND SPHERICAL HARMONICS  
**Autor:** Coifman, R. R. / Weiss, Guido  
**Kapitel:** §6. The Fourier transform of functions on  $\mathbb{R}^n$   
**DOI:** <https://doi.org/10.5169/seals-42346>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 27.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

where

$$t = \frac{\chi_2 \bar{x}_2}{|x|^2} \cdot 1)$$

### § 6. THE FOURIER TRANSFORM OF FUNCTIONS ON $\mathbf{R}^n$

We have shown that  $L^2(\Sigma_{n-1})$  can be decomposed into a direct sum of mutually orthogonal subspaces (the spaces  $\mathcal{H}_n^{(k)}$ ) that are invariant and irreducible under the action of rotations. There exists a corresponding decomposition of  $L^2(\mathbf{R}^n)$  and the spaces making up this decomposition are intimately connected with the Fourier transform of functions of  $n$  real variables. In this section we shall construct these spaces and study the action of the Fourier transform restricted to them. We shall see that also in this situation the rotation group  $SO(n)$  and its representations play a central role.

If  $f$  belongs to  $L^1(\mathbf{R}^n)$  its *Fourier transform*  $\hat{f}$  is defined by letting

$$(\mathcal{F}f)(y) = \hat{f}(y) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x \cdot y} dx$$

for  $y \in \mathbf{R}^n$ .<sup>1)</sup>

Perhaps the simplest class of functions that is invariant under the action of the Fourier transform is the collection of *radial functions*. We recall that these are the functions on  $\mathbf{R}^n$  that depend only on  $|x|$ ; equivalently,  $f$  is radial if  $\rho_v f = f$  for all  $v \in SO(n)$ , where the operator  $\rho_v$  is defined by

$$(\rho_v f)(x) = f(v^{-1}x)$$

for all  $x \in \mathbf{R}^n$ . Since Lebesgue measure is invariant under the action of rotations and  $v = v^*$  when  $v \in SO(n)$ ,

$$\int_{\mathbf{R}^n} f(x) e^{-2\pi i x \cdot v^{-1}y} dx = \int_{\mathbf{R}^n} f(x) e^{-2\pi i v x \cdot y} dx = \int_{\mathbf{R}^n} f(v^{-1}x) e^{-2\pi i x \cdot y} dx.$$

That is,

$$(6.1) \quad (\mathcal{F}\rho_v)f = (\rho_v \mathcal{F})f$$

<sup>1)</sup> It is not hard to use these results in order to obtain analogous results for  $SO(3)$ . We refer the reader to VILENKIN [11] for complete details.

<sup>1)</sup> When  $f \in L^2(\mathbf{R}^n)$  the integral defining  $\hat{f}$  is not defined in the Lebesgue sense. In this case,  $\hat{f}$  is usually defined as the limit in the  $L^2$  mean of the sequence  $\hat{f}^k(y) = \int_{|x| \leq k} f(x) e^{-2\pi i x \cdot y} dx$ . In order to avoid technical difficulties that arise from this definition we shall restrict our attention to integrable functions

for all  $f \in L^1(\mathbf{R}^n)$ . This basic property, that Fourier transformation commutes with the action of rotations clearly implies.

**THEOREM (6.2).** *If  $f \in L^1(\mathbf{R}^n)$  is radial then  $\hat{f}$  is also a radial function.*

In order to extend this invariance property we introduce, for  $k = 0, 1, 2, \dots$ , the class of functions  $\mathfrak{h}^{(k)} = \mathfrak{h}_n^{(k)}$  mapping  $\mathbf{R}^n$  into  $\mathbf{C}^{d_k}$  having the form

$$F(x) = f(|x|)(Y_1(\xi), \dots, Y_{d_k}(\xi)) = (F_1(x), \dots, F_{d_k}(x)),$$

where

$$x = |x| \xi, \int_0^\infty f(r) r^{n-1} dr < \infty \text{ }^1) \text{ and } \{ Y_1, Y_2, \dots, Y_{d_k} \}$$

is an orthonormal basis of  $\mathcal{H}_n^{(k)}$  such that  $Y_1 = a_k^{-1} Z_1$  (that is, orthonormality is to be taken with respect to the inner product (2.6)). Such a basis was considered, for example, in theorem (2.16). When  $k = 0$  this class is precisely the set of radial functions. It will be convenient if we choose the  $Y_1, \dots, Y_{d_k}$  to be real-valued.

Let  $T^{(k)} = (t_{lj}^{(k)})$  be the matrix of the representation  $S^{k,n}$  with respect to the basis  $\{ Y_1, Y_2, \dots, Y_{d_k} \}$ ; that is, the functions  $t_{lj}^{(k)} = t_{lj}$  satisfy

$$(S_v^{k,n} Y_j)(\xi) = Y_j(v^{-1}\xi) = \sum_{l=1}^{d_k} t_{lj}(v) Y_l(\xi)$$

for  $j = 1, 2, \dots, d_k$ . If we let

$$\rho_v F = (\rho_v F_1, \dots, \rho_v F_{d_k})$$

we then have

$$\begin{aligned} (\rho_v F)(x) &= f(|x|)(Y_1(v^{-1}\xi), \dots, Y_{d_k}(v^{-1}\xi)) = \\ &= f(|x|)(Y_1(\xi), \dots, Y_{d_k}(\xi)) \begin{pmatrix} t_{11}(v) & \dots & t_{1d_k}(v) \\ t_{21}(v) & \dots & t_{2d_k}(v) \\ \dots & \dots & \dots \\ t_{d_k1}(v) & \dots & t_{d_kd_k}(v) \end{pmatrix} = (T_v^{(k)} F)(x), \end{aligned}$$

The last equality being the definition of the operator  $T_v^{(k)}$  acting on  $F$ . That is,

$$(6.3) \quad \rho_v F = T_v^{(k)} F$$

<sup>1)</sup> This condition merely assures us that the radial function  $g(x) = f(|x|)$  is integrable on  $\mathbf{R}^n$ .

for all  $v \in SO(n)$ . If we now apply the Fourier transform to each component of  $\rho_v F$ , it follows from (6.2) and (6.3) that

$$(6.4) \quad \rho_v \hat{F} = \rho_v (\hat{F}_1, \dots, \hat{F}_{d_k}) = T_v^{(k)} \hat{F}.$$

The following, together with relation (6.4), shows that  $\hat{F}$  must have the same form as  $F$ ; that is,

$$(6.5) \quad \hat{F}(y) = \tilde{f}(|y|)(Y_1(\eta), \dots, Y_{d_k}(\eta))$$

for all  $y = |y| \eta \in \mathbf{R}^n$ .

**THEOREM (6.6).** *Suppose  $G = (G_1, \dots, G_{d_k})$  is a continuous function mapping  $\mathbf{R}^n$  into  $\mathbf{C}^{d_k}$  such that*

$$(6.7) \quad \rho_v G = T_v^{(k)} G$$

for all  $v \in SO(n)$ , then

$$G(y) = a_k^{-1} G_1(|y|\mathbf{1})(Y_1(\eta), \dots, Y_{d_k}(\eta))$$

for all  $y = |y| \eta$  in  $\mathbf{R}^n$ .

*Proof.* Let  $v \in SO(n)$  be such that  $y = |y| v' \mathbf{1} = |y| v^{-1} \mathbf{1}$ . Then, by (6.7)

$$G(y) = G(v^{-1}|y|\mathbf{1}) = (T_v^{(k)} G)(|y|\mathbf{1}).$$

Consequently,

$$(6.8) \quad G_j(y) = \sum_{l=1}^{d_k} t_{lj}(v) G_l(|y|\mathbf{1})$$

for  $j = 1, 2, \dots, d_k$ . If  $u \in SO(n-1)$  then  $y = |y| v^{-1} \mathbf{1} = |y| v^{-1} u^{-1} \mathbf{1} = |y| (uv)^{-1} \mathbf{1}$ ; thus, if we replace  $v$  by  $uv$  in (6.8) we obtain

$$G_j(y) = \sum_{l=1}^{d_k} t_{lj}(uv) G_l(|y|\mathbf{1}).$$

Integrating over  $SO(n-1)$ , therefore,

$$G_j(y) = \sum_{l=1}^{d_k} G_l(|y|\mathbf{1}) \int_{SO(n-1)} t_{lj}(uv) du.$$

But, by (3.2) and theorem (3.5) (or (3.15))

$$\int_{S_{0(n-1)}} t_{lj}(uv) du = \begin{cases} t_{lj}(v) & \text{when } l = 1 \\ 0 & \text{when } l > 1 \end{cases}.$$

This equality and (2.17) show that

$$G_j(y) = G_j(v^{-1}|y|\mathbf{1}) = G_1(|y|\mathbf{1}) \overline{Y_j(v^{-1}\mathbf{1})} a_k^{-1}.$$

(Since

$$\overline{t_{lj}(v)} = t_{jl}(v^{-1}) = \overline{Y_j(v^{-1}\mathbf{1})}.)$$

Writing  $y = |y|\eta$ , where  $\eta = v^{-1}\mathbf{1}$ , and using the fact that  $Y_j$  is real-valued, we obtain the desired result

$$G_j(y) = a_k^{-1} G_1(|y|\mathbf{1}) \hat{Y}_j(\eta),$$

$j = 1, 2, \dots, d_k.$

**THEOREM (6.9).** *Let  $Y$  be a spherical harmonic of degree  $k$  and  $f$  a function on  $(-\infty, \infty)$  satisfying*

$$(i) \quad \int_0^\infty |f(r)| r^{n-1} dr < \infty.$$

*If  $h(x) = f(|x|) Y(\xi)$ , when  $x = |x|\xi \in \mathbf{R}^n$ , then  $h \in L^1(\mathbf{R}^n)$  and*

$$\hat{h}(y) = \tilde{f}(|y|) Y(\eta)$$

*for all  $y = |y|\eta \in \mathbf{R}^n$ . The transformation  $f \rightarrow \tilde{f}$  depends only on  $k$  and  $n$  and, in particular, is independent of  $Y \in \mathcal{H}_n^{(k)}$ .*

*Proof.* Let  $\{Y_1, \dots, Y_{d_k}\}$  be the basis of  $\mathcal{H}_n^{(k)}$  that was used in the previous theorem and  $F(x) = (f(|x|) Y_1(\xi), \dots, f(|x|) Y_{d_k}(\xi)) = (F_1(x), \dots, F_{d_k}(x))$ . Condition (i) guarantees that each of the functions  $F_j, j = 1, \dots, d_k$ , is integrable.<sup>1)</sup> Thus,  $\hat{F} = (\hat{F}_1, \dots, \hat{F}_{d_k})$  is well defined, continuous (as can be very easily shown), and satisfies relation (6.4). By theorem (6.6), therefore,

---

1) Using polar coordinates  $x = |x|\xi$ , with  $\xi \in \Sigma_{n-1}$ , we have  $\int_{\mathbf{R}^n} |F_j(x)| dx = \int_{\Sigma_{n-1}} \omega_{n-1} \{ \int_0^\infty |f(r)| r^{n-1} dr \} |Y_j(\xi)| d\xi < \infty$ , where  $\omega_{n-1}$  is the "area" of  $\Sigma_{n-1}$ .

$$\hat{F}(y) = a_k^{-1} \hat{F}_1(|y|\mathbf{1})(Y_1(\eta), \dots, Y_{d_k}(\eta)) \text{ for } y = |y|\eta \in \mathbf{R}^n.$$

Putting  $\tilde{f}(|y|) = a_k^{-1} \hat{F}_1(|y|\mathbf{1})$  we obtain equality (6.5). Since  $\{Y_1, \dots, Y_{d_k}\}$  is a basis of  $\mathcal{H}_n^{(k)}$  we can find coefficients  $b_1, \dots, b_{d_k}$  such that

$$Y = \sum_{l=1}^{d_k} b_l Y_l.$$

Thus,

$$h(x) = \sum_{l=1}^{d_k} f(|x|) b_l Y_l(\xi).$$

We have just shown that the Fourier transform of  $F_l(x) = f(|x|) Y_l(\xi)$  has the values  $\tilde{f}(|y|) Y_l(\eta)$ . Thus,

$$\hat{h}(y) = \sum_{l=1}^{d_k} b_l \tilde{f}(|y|) Y_l(\eta) = \tilde{f}(|y|) \sum_{l=1}^{d_k} b_l Y_l(\eta) = \tilde{f}(|y|) Y(\eta).$$

This proves the theorem.

It is not hard to give an explicit form for the mapping  $f \rightarrow \tilde{f}$  in terms of the *Bessel functions*

$$J_\lambda(t) = \frac{(t/2)^\lambda}{\Gamma\left(\frac{2\lambda+1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 e^{its} (1-s^2)^{\frac{2\lambda-1}{2}} ds.$$

We shall show, in fact, that

$$(6.10) \quad \tilde{f}(t) = \gamma_{k,n} t^{\frac{2-n}{2}} \int_0^\infty f(r) J_{k+\frac{n-2}{2}}(2\pi tr) r^{n/2} dr. \quad ^1)$$

Since  $\tilde{f}$  is independent of  $Y \in \mathcal{H}_n^{(k)}$  let us choose  $h(x) = f(|x|) Z_1(\xi) = f(|x|) P^{(k)}(\xi \cdot \mathbf{1})$ . Then

$$\hat{f}(y) = \int_{\mathbf{R}} e^{-2\pi i y \cdot x} f(|x|) P^{(k)}(\xi \cdot \mathbf{1}) dx =$$

<sup>1)</sup> We shall not calculate  $\gamma_{k,n}$ . The fact that this constant equals  $2\pi i^{-k}$  can be shown by evaluating the integral in (6.10) when  $f(r) = e^{-r^2}$  (see STEIN and WEISS [10], Chapter IV, section 3) or by using the constants obtained below.

$$= \omega_{n-1} \int_0^{\infty} r^{n-1} f(r) \left\{ \int_{\Sigma_{n-1}} e^{-2\pi i |y| r (\eta \cdot \xi)} P^{(k)}(\xi, \mathbf{1}) d\xi \right\} dr$$

Writing  $y = t \eta$ , this means that we have to compute

$$\int_{\Sigma_{n-1}} e^{-2\pi i r t (\eta \cdot \xi)} P^{(k)}(\xi, \mathbf{1}) d\xi.$$

But, by the Funk-Hecke theorem (4.16) this integrál is equal to

$$P^{(k)}(\eta, \mathbf{1}) a_k^{-2} c_n \int_{-1}^1 e^{-2\pi i r t s} P^{(k)}(s) (1-s^2)^{\frac{n-3}{2}} ds.$$

On the other hand, by (4.4), and, then integrating by parts  $k$  times we have

$$\begin{aligned} \int_{-1}^1 e^{-2\pi i r t s} P^{(k)}(s) (1-s^2)^{\frac{n-3}{2}} ds &= \alpha_{k,n} \int_{-1}^1 e^{-2\pi i r t s} \left[ \frac{d^k}{dt^k} (1-s^2)^{k+\frac{n-3}{2}} \right] ds \\ &= \beta_{k,n} \int_{-1}^1 (rt)^k e^{2\pi i r t s} (1-s^2)^{k+\frac{n-3}{2}} ds. \end{aligned}$$

The last integral, however, is the one involved in the definition of  $J_\lambda$  when  $\lambda = (2k+n-2)/2$ . Equality (6.10) now follows immediately.<sup>1)</sup>

### BIBLIOGRAPHY

- [1] BATEMAN, H., *Bateman Manuscript Project*, Vol. 1 and 2, N. Y. (1953).
- [2] CALDERÓN, A. P., *Integrales Singulares y sus Aplicaciones a Ecuaciones Diferenciales Hiperbólicas*. Fasc. 3, Cursos y Seminarios de Matematica, Univ. de Buenos Aires (1959).
- [3] — and A. ZYGMUND, Singular Integral Operators and Differential Equations. *Am. J. of Math.*, Vol. LXXIX, No. 4 (1957), pp. 901-921.
- [4] CARTAN, E., *Œuvres Complètes*. Gauthier-Villars, Paris (1939).
- [5] GODEMONT, R., A Theory of Spherical Functions, I. *Trans. Am. Math. Soc.*, 73 (1952), pp. 496-556.
- [6] DIEUDONNÉ, J. *Representacion de Grupos Compactos y Funciones Esfericas*. Fasc. 14, Cursos y Seminarios de Matematica, Univ. de Buenos Aires (1964).

<sup>1)</sup> The Bessel functions we have encountered here arise in much the same way as did the ultraspherical Polynomials. Instead of the group  $SO(n)$ , however, one must study the group of all rigid motions on  $\mathbf{R}^{(n)}$  (see VILENKIN [11] for details).