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APPROXIMATION

We use positive *n*-tuples ρ , ... with $\rho \leqslant \rho_2 < \rho_3 < \rho_4 < \rho_1$ and $\rho = \gamma'' \rho_1$, $\rho_2 = \gamma \rho_1$, $\rho_3 = \gamma' \rho_1$, $\rho_4 = \gamma''' \rho_1$. The *n*-tuple ρ_1 is defined as in the smoothing theorem.

Definition: $H_*^l = \{ \xi \in H^l(X_0, \underline{F}|X_0) \text{ such that there exists } U = U(0) \text{ in } E^n \text{ with } \hat{\xi} \in H^l(\psi^{-1}(U), \overline{F}) \text{ and } \hat{\xi} \mid X_0 = \xi \}.$ Serre's theorem gives $\dim_{\mathbf{C}} H_*^l \leqslant \dim_{\mathbf{C}} H^l(X_0, \underline{F}|X_0) < \infty$. In the following discussion we are given $\hat{\mathfrak{b}}_1, \ldots, \hat{\mathfrak{b}}_r$ in $Z^l(\hat{\mathfrak{U}}'(\rho_4), \overline{F})$ such that $\hat{\mathfrak{b}}_1 \mid X_0, \ldots \hat{\mathfrak{b}}_r \mid X_0$ constitute a base of the complex vector space H_*^l . For this to be possible, ρ_4 has to be chosen small enough. Here $\hat{\mathfrak{U}}'$ is a Stein covering of $X(\rho_1)$ and defined as in the smoothing theorem. We also assume that we are given a sequence of measure coverings as there. Further we construct the sequence so that there are still sufficiently many measure coverings in between \mathfrak{B} and \mathfrak{U} . These are denoted by \mathfrak{U}_v^* . We have $\mathfrak{U} \gg \mathfrak{U}_1^* \gg \mathfrak{U}_2^* \gg \ldots \gg \mathfrak{B}$. The *n*-tupel ρ_3 is also fixed from now on and K always denotes (possibly different) constants.

Approximation Lemma: Let $\varepsilon > 0$. Then we can find ρ_2 such that: If $\rho \leqslant \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $||\hat{\xi}||_{\rho} < \infty$ (the norm is taken with respect to $\hat{\mathfrak{U}}_1^*(\rho)$), then there exist $a_1, \ldots a_r \in I(E^n(\rho))$ and $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ such that $\tilde{\xi} = \hat{\xi} - \sum_{1}^{r} a_i \hat{\mathfrak{b}}_i - \delta \hat{\eta}$ on $\hat{\mathfrak{B}}(\rho)$. Here $\tilde{\xi} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ and $||\tilde{\xi}||_{\rho} \leqslant \varepsilon$ $||\hat{\xi}||_{\rho}$ and $||a_v||_{\rho}, ||\hat{\eta}||_{\rho} \leqslant K ||\hat{\xi}||_{\rho}$. K is a fixed constant.

Proof. We shall first prove some results which are needed later on. Let $S \in \Gamma$ $(\hat{U}_{\iota_0 \dots \iota_l}(\rho), \mathbf{F})$. Choose $\iota \in \{\iota_0, \dots, \iota_l\}$. Now $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota} \subset \hat{U}_{\iota_0 \dots \iota_l}$ because $\mathfrak{U}_1^* \ll \mathfrak{U}$. The operations are always defined with respect to ρ_1 . We can now restrict S to $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}(\rho)$. In the chart \mathcal{W}_{ι} we can write $S = \sum a_{\nu} (t/\rho)^{\nu}$. Here $a_{\nu} \in qI(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}$. Now the a_{ν} are extended constantly and we get elements $\hat{a}_{\nu} \in \Gamma((U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}, \mathbf{F})$. Let us put $S_{\nu} = \hat{a}_{\nu} \mid \hat{U}_{\iota_0 \dots \iota_l}^{(2)*}$. We claim that $||S_{\nu}||_{\rho_1} \ll K ||S||_{\rho}$. For obviously $||S||_{\rho} \geqslant |a_{\nu}(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}|$ and

we can use the Theorem I to prove that $||S_{\nu}||_{\rho_{1}} \leqslant K ||\hat{a}_{\nu}||_{(U_{\iota_{0}}^{(1)^{*}}...\iota_{l})_{\iota}}(\rho_{1})||_{\iota} = K |a_{\nu}(U_{\iota_{0}}^{(1)^{*}}...\iota_{l})| \leqslant K ||S||_{\rho}.$ Q.E.D.

Let S_{ν}' be defined using some other $\iota' \in \{\iota_0, \dots \iota_l\}$. Then $S_{\nu} - S_{\nu}' \in \Gamma(\hat{U}_{\iota_0}^{(2)*}, \dots, \iota_l)$. We claim that $||S_{\nu} - S_{\nu}'||_{\rho_4} \leqslant K\gamma''' ||S||_{\rho}$.

Proof. Define $\alpha_s = \sum_{|\lambda|=s}^{\infty} a_{\lambda}(t/\rho)^{\lambda}$ and $\beta_s = \sum_{|\lambda|=0}^{s-1} a_{\lambda}(t/\rho)^{\lambda}$ $(U_{\iota_0}^{(1)*}, (\rho))$. We do the same for ι' respectively and obtain α'_s and β'_s over $(U_{\iota_0 \ldots \iota_l}^{(1)^*})_{\iota'}(\rho)$. For the restrictions to $\hat{U}_{\iota_0 \ldots \iota_l}^{(2)^*}$ we see that $\alpha_s - \alpha_s' =$ $-(\beta_s - \beta_s)$. Hence we get $\|\alpha_s - \alpha_s'\|_{\rho_4} \leq K(\gamma''')^s \|\alpha_s - \alpha_s'\|_{\rho_1} = K(\gamma''')^s \|\beta_s - \alpha_s'\|_{\rho_1}$ $-\beta_{s} \|_{\rho_{1}} \leqslant K(\gamma^{\prime\prime\prime})^{s} \|\beta_{s}\|_{\rho_{1}} + K(\gamma^{\prime\prime\prime})^{s} \|\beta_{s}^{'}\|_{\rho_{1}} \leqslant K(\gamma^{\prime\prime\prime})^{s} [\|\beta_{s}\|_{\rho_{1}}^{*} + \|\beta_{s}^{'}\|_{\rho_{1}}^{*}] \leqslant$ $\leq K(\gamma''')^s (\gamma'')^{1-s} ||S||_{\rho}$. Here the norms are defined with respect to $U_{\iota_0 \dots \iota_l}^{(3)*}$ except $\|\cdot\|^*$ and $\|\cdot\|_{\rho}$ which are defined with respect to $U_{\iota_0 \ldots \iota_l}^{(1)^*}$. Now we look at the difference $(S_{\nu} - S_{\nu}') t^{\nu}/\rho^{\nu}$ on $(U_{\iota_0}^{(3)*})_{\mu}$ with $|\nu| = s, \mu \in {\iota_0, ... \iota_l}$, and the power series development with respect to W_{μ} . There is one term of order s which is equal to the corresponding term of $\alpha_s - \alpha_s$. Therefore its norm is $\leqslant K(\gamma''')^s$. $(\gamma'')^{1-s} ||S||_{\rho}$. Moreover we have $||S_{\nu}(t/\rho)^{\nu} - S_{\nu}'(t/\rho)^{\nu}||_{\rho_1} \leqslant$ $\leq (\gamma'')^{-s} \cdot K ||S||_{\rho}$ where the first norm is defined with respect to $U^{(3)*}_{\iota_0 \ldots \iota_l}$. For the sum \sum of terms of higher order than s in the power series of $(S_v -S_{v}^{'}$) t^{v}/ρ^{v} we therefore get: $||\sum||_{\rho_{4}} \leqslant (\gamma''')^{s+1} (\gamma'')^{-s} \cdot K||S||_{\rho}$. Hence we get $||(S_v - S_v)||_{\rho_4} \leqslant \gamma''' \cdot K ||S||_{\rho}$. This proves our statement. We see that K is independent of ρ_4 and S. The number $\gamma^{\prime\prime\prime}$ depends on ρ_4 only, so $\gamma^{\prime\prime\prime} \cdot K$ gets very small if we make ρ_4 very small.

Let $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $\hat{\xi} = \{\hat{\xi}_{\iota_0 \dots \iota_l}\}$. Choose $\iota = \iota(\iota_0, \dots, \iota_l)$ as a function of the unordered (l+1)-tuple. We now fix ι_0, \dots, ι_l and write $S = \hat{\xi}_{\iota_0 \dots \iota_l}$. We apply to S the method described above and obtain $\hat{\xi}_{\iota_0 \dots \iota_l}^{(\nu)} = \{\hat{\xi}_{\iota_0 \dots \iota_l}^{(\nu)}\}$ as an element of $C^l(\hat{\mathfrak{U}}_2^*(\rho_4), \mathbf{F})$. Of course $\hat{\xi}_{(\nu)}$ depends on the choice of $\iota = \iota(\iota_0 \dots \iota_l)$ here. Now we see that $||\hat{\xi}_{(\nu)}||_{\rho_4} \leq ||\hat{\xi}_{(\nu)}||_{\rho_1} \leq K||\hat{\xi}||_{\rho}$. We also wish to estimate $\delta\hat{\xi}_{(\nu)}$. Because $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ we can use the preliminary result on ι and ι' to obtain $||\delta\hat{\xi}_{(\nu)}||_{\rho_4} \leq K\gamma''' ||\hat{\xi}||_{\rho}$.

We shall also need another result:

Induction Lemma: There exists $\hat{\eta}_v \in C^l(\hat{\mathfrak{U}}_4^*(\rho_3), \mathbf{F})$ such that $\hat{\delta \eta}_v = \hat{\delta \xi}_{(v)}$ on $\hat{\mathfrak{U}}_4^*(\rho_3)$ and $\|\hat{\eta}_v\|_{\rho_3} \leqslant K \|\hat{\delta \xi}_{(v)}\|_{\rho_4}$.

Proof. The proof uses the assumption that $\psi_{(l+1)}(\mathbf{F})$ is coherent. Because the coherence of direct images is proved by downward induction on l, this assumption can be made. Moreover it is assumed that the main theorem is proved for dimension l+1 already. Let us now put $\alpha = \delta \xi_{(v)} \in$ $\in B^{l+1}(\hat{\mathfrak{U}}_{2}^{*}(\rho_{4}), \mathbf{F})$ and $\hat{\eta}_{v} = \beta \in C^{l}(\hat{\mathfrak{U}}_{4}^{*}(\rho_{3}), \mathbf{F})$. We have to prove the existence of β . We may assume that ρ_4 is so small that the main theorem is valid for $\rho \leqslant \rho_4$ in the case of dimension l+1. So there are cocycles $\omega_1, ..., \omega_r \in Z^{l+1}(\widehat{\mathfrak{U}}(\rho_4), \mathbf{F})$ such that $\alpha = \sum C_{\lambda} \omega_{\lambda} + \delta \eta$, where $C_{\lambda} \in$ $\in I(E^n(\rho_4))$ and $\eta \in C^l(\mathfrak{U}_4^*(\rho_4), \mathbf{F})$. We have to assume that between \mathfrak{U}_4^* and \mathfrak{U}_{2}^{*} there are very many measure coverings. The cross-sections $\psi_{(l+1)}(\omega_{\lambda})$ give a homomorphism $r\mathcal{O} \to \psi_{(l+1)}(\mathbf{F})$ over $E^n(\rho_4)$. Because $\psi_{(l+1)}(\mathbf{F})$ is coherent the kernel \mathcal{N} is coherent again. Over $E^{n}(\rho')$ with $\rho_{3} < \rho' < \rho_{4}$ we find an epimorphism $p\mathcal{O} \to \mathcal{N}$. Denote by $n_1, ..., n_p$ the images of the unit cross-sections in p0. Write $n_{\lambda} = (e_{\lambda 1}, \dots, e_{\lambda r})$ as an r-tupel of holomorphic functions. The image of n_{λ} in $\Gamma\left(E^{n}\left(\rho'\right),\psi_{(l+1)}\left(F\right)\right)$ is $\psi_{(l+1)}\left(\sum_{i}e_{\lambda\mu}\omega_{\mu}\right)$ and zero. We may choose ρ_2 and then ρ_3 and ρ' very small. Then it follows that $n_{\lambda} = \sum e_{\lambda\mu} \omega_{\mu}$ is a coboundary. If $\rho_3 < \rho'' < \rho'$ there are cochains $\eta_{\lambda} \in C^{l}\left(\mathfrak{U}_{4}^{*}(\rho''), \mathbf{F}\right)$ such that $\delta \eta_{\lambda} = n_{\lambda}$. Now $(C_{1}, ..., C_{r}) \in$ $\in \Gamma(E^n(\rho_4), \mathcal{N})$. By the methods of sheaf theory we can lift this crosssection to p0. Using a "Banach open mapping theorem" we see that the map $\Gamma(E^n(\rho'), p\emptyset) \to \Gamma(E^n(\rho'), \mathcal{N})$ is open. This means here that we can find holomorphic functions a_{λ} over $E^{n}(\rho_{3})$ such that $C_{\mu} = \sum a_{\lambda} e_{\lambda\mu}$ and $||a_{\lambda}||_{\rho_3} \leqslant K \max ||C_{\mu}||_{\rho'} \leqslant K \max ||C_{\mu}||_{\rho_4}$. We get $\sum C_{\mu} \omega_{\mu} = \sum a_{\lambda} e_{\lambda\mu} \omega_{\mu}$ $= \sum a_{\lambda} \hat{n}_{\lambda} = \delta \left(\sum a_{\lambda} \eta_{\lambda} \right). \text{ This leads to } \alpha \mid C^{l+1} \left(\hat{\mathfrak{U}}_{4}^{*} (\rho_{3}) \right) = \delta \left(\eta + \sum a_{\lambda} \eta_{\lambda} \right).$ The estimates required obviously hold. Q.E.D.

Let us now put $\hat{\xi}_{(v)}^* = \hat{\xi}_{(v)} - \hat{\eta}_v \in Z^l(\hat{\mathfrak{U}}_4(\rho_3), \mathbf{F})$. We can write $\hat{\xi}_{(v)}^* \mid X_0 = \sum_{i=1}^n a_{v\lambda} \hat{\mathfrak{b}}_{\lambda} \mid X_0 + \delta \gamma_v$ over \mathfrak{U}_6^* . Here $a_{v\lambda}$ are complex numbers and $\gamma_v \in C^{l-1}(\mathfrak{U}_6^*, F \mid X_0)$. Cartan's theorem and the result after that give the estimates $|a_{v\lambda}| \leq K ||\hat{\xi}_{(v)}^*||_{\rho_3} \leq K |||_{\rho_3} \leq K ||_$

extension of γ_{ν} . Let us now put $\hat{\xi}_{(\nu)}^{(1)} = \hat{\xi}_{(\nu)}^* - \sum a_{\nu\lambda} \hat{\mathfrak{b}}_{\lambda} - \hat{\delta\gamma}_{\nu}$. Here $\hat{\xi}_{(\nu)}^{(1)} \in C^l(\hat{\mathfrak{U}}_7^*(\rho_3), \mathbf{F})$. Using the previous estimates and the fact that the $\hat{\mathfrak{b}}_{\lambda}$ are finite we find that $||\hat{\xi}_{(\nu)}^{(1)}||_{\rho_3} \leqslant K ||\hat{\xi}_{(\nu)}||_{\rho_4} \leqslant K ||\hat{\xi}||_{\rho}$.

Now we also have $\hat{\xi}_{(\nu)}^{(1)} \mid X_0 = 0$. It follows that

$$||\hat{\xi}_{(\nu)}^{(1)}||_{\rho} \leqslant \gamma/\gamma' ||\hat{\xi}_{(\nu)}^{(1)}||_{\rho_{3}} \leqslant \gamma/\gamma' \cdot K ||\hat{\xi}||_{\rho}.$$

Finally we put in $\widehat{\mathfrak{U}}_{9}^{*}(\rho)$:

$$\hat{\xi}^{(1)} = \Sigma \hat{\xi}^{(1)}_{(\nu)} (t/\rho)^{\nu} =$$

$$= \Sigma \hat{\xi}_{(\nu)} (t/\rho)^{\nu} - \Sigma \hat{\eta}_{\nu} (t/\rho)^{\nu} - \Sigma a_{\nu\lambda} (t/\rho)^{\nu} \hat{b}_{\lambda} - \delta (\Sigma \hat{\gamma}_{\nu} (t/\rho)^{\nu})$$

$$= \hat{\xi} - \hat{\eta} - \Sigma a_{\lambda} \hat{b}_{\lambda} - \delta \hat{\gamma}.$$

Using the fact that the sum of the absolute values of the coefficients in the power series expansion of $\hat{\xi}_{(\nu)}^{(1)}$ by (t/ρ) is smaller than $\gamma/\gamma' \cdot K || \hat{\xi} ||_{\rho}$ and that with respect to $\hat{\eta}_{\nu}$ is smaller than $\gamma''' \cdot K || \hat{\xi} ||_{\rho}$ we find: $|| \hat{\xi}^{(1)} ||_{\rho} \leq \sqrt{\gamma'} \cdot K || \hat{\xi} ||_{\rho}$ and $|| \hat{\eta} ||_{\rho} \leq \gamma''' \cdot K || \hat{\xi} ||_{\rho}$ and $|| a_{\lambda} ||_{\rho} \leq K || \hat{\xi} ||_{\rho}$. We take the restriction to $\hat{\mathfrak{B}}(\rho)$ and now $\tilde{\xi} = \hat{\xi}^{(1)} - \hat{\eta} \in Z^{l}(\hat{\mathfrak{B}}(\rho), \mathbb{F})$ is the desired element. Of course we have to choose ρ_{4} and then ρ_{2} small enough, for example let $\gamma''' < \varepsilon/2 K$ and $\gamma \leq \varepsilon \gamma'/2 K$.

MAIN THEOREM

There exists ρ_2 and a constant K such that if $\rho \leqslant \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $||\hat{\xi}||_{\rho} < \infty$ then we can find $a_1, ..., a_r \in I(E^n(\rho))$ and $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{V}}(\rho), \mathbf{F})$ such that $\hat{\xi} = \sum a_{\lambda} \hat{\mathfrak{b}}_{\lambda} + \delta \hat{\eta}$ on $\hat{\mathfrak{V}}(\rho)$ with $||\hat{\eta}||_{\rho}$ and $||a_{\nu}||_{\rho} \leqslant ||\hat{\xi}||_{\rho}$.

Proof. We have one constant K from the smoothing theorem. Now we find ρ_2 with an ε in the Approximation Lemma such that $\varepsilon \cdot K < 1/2$. We shall use this ρ_2 and prove the theorem here. We are given $\hat{\xi}_0 = \hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $||\hat{\xi}||_{\rho} < \infty$. The Approximation Lemma gives $\tilde{\xi}_1 = \mathbb{E}[\hat{\mathfrak{U}}(\rho), \mathbf{F})$ with $||\hat{\xi}||_{\rho} < \infty$.