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# APPROXIMATION

We use positive  $n$ -tuples  $\rho, \dots$  with  $\rho \leq \rho_2 < \rho_3 < \rho_4 < \rho_1$  and  $\rho = \gamma'' \rho_1, \rho_2 = \gamma \rho_1, \rho_3 = \gamma' \rho_1, \rho_4 = \gamma''' \rho_1$ . The  $n$ -tuple  $\rho_1$  is defined as in the smoothing theorem.

*Definition:*  $H_*^l = \{ \xi \in H^l(X_0, \underline{F|X_0}) \text{ such that there exists } U = U(0) \text{ in } E^n \text{ with } \hat{\xi} \in H^l(\psi^{-1}(U), \mathbf{F}) \text{ and } \hat{\xi}|X_0 = \xi \}$ . Serre's theorem gives  $\dim_{\mathbf{C}} H_*^l \leq \dim_{\mathbf{C}} H^l(X_0, \underline{F|X_0}) < \infty$ . In the following discussion we are given  $\hat{b}_1, \dots, \hat{b}_r$  in  $Z^l(\hat{\mathcal{U}}'(\rho_4), \mathbf{F})$  such that  $\hat{b}_1|X_0, \dots, \hat{b}_r|X_0$  constitute a base of the complex vector space  $H_*^l$ . For this to be possible,  $\rho_4$  has to be chosen small enough. Here  $\hat{\mathcal{U}}'$  is a Stein covering of  $X(\rho_1)$  and defined as in the smoothing theorem. We also assume that we are given a sequence of measure coverings as there. Further we construct the sequence so that there are still sufficiently many measure coverings in between  $\mathfrak{B}$  and  $\mathcal{U}$ . These are denoted by  $\mathcal{U}_v^*$ . We have  $\mathcal{U} \gg \mathcal{U}_1^* \gg \mathcal{U}_2^* \gg \dots \gg \mathfrak{B}$ . The  $n$ -tuple  $\rho_3$  is also fixed from now on and  $K$  always denotes (possibly different) constants.

*Approximation Lemma:* Let  $\varepsilon > 0$ . Then we can find  $\rho_2$  such that: If  $\rho \leq \rho_2$  and  $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho), \mathbf{F})$  with  $\|\hat{\xi}\|_{\rho} < \infty$  (the norm is taken with respect to  $\hat{\mathcal{U}}_1^*(\rho)$ ), then there exist  $a_1, \dots, a_r \in I(E^n(\rho))$  and  $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$  such that  $\tilde{\xi} = \hat{\xi} - \sum_1^r a_i \hat{b}_i - \delta \eta$  on  $\mathfrak{B}(\rho)$ . Here  $\tilde{\xi} \in Z^l(\mathfrak{B}(\rho), \mathbf{F})$  and  $\|\tilde{\xi}\|_{\rho} \leq \varepsilon \|\hat{\xi}\|_{\rho}$  and  $\|a_v\|_{\rho}, \|\hat{\eta}\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$ .  $K$  is a fixed constant.

*Proof.* We shall first prove some results which are needed later on. Let  $S \in \Gamma(\hat{U}_{\iota_0 \dots \iota_l}(\rho), \mathbf{F})$ . Choose  $\iota \in \{\iota_0, \dots, \iota_l\}$ . Now  $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota} \subset \hat{U}_{\iota_0 \dots \iota_l}$  because  $\mathcal{U}_1^* \ll \mathcal{U}$ . The operations are always defined with respect to  $\rho_1$ . We can now restrict  $S$  to  $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}(\rho)$ . In the chart  $\mathcal{W}_{\iota}$  we can write  $S = \sum a_v (t/\rho)^v$ . Here  $a_v \in qI(U_{\iota_0 \dots \iota_l}^{(1)*})$ . Now the  $a_v$  are extended constantly and we get elements  $\hat{a}_v \in \Gamma((U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}, \mathbf{F})$ . Let us put  $S_v = \hat{a}_v| \hat{U}_{\iota_0 \dots \iota_l}^{(2)*}$ . We claim that  $\|S_v\|_{\rho_1} \leq K \|S\|_{\rho}$ . For obviously  $\|S\|_{\rho} \geq |a_v (U_{\iota_0 \dots \iota_l}^{(1)*})|$  and

we can use the Theorem I to prove that  $\|S_v\|_{\rho_1} \leq K \|\hat{a}_v\| (U_{\iota_0 \dots \iota_l}^{(1)*}(\rho_1))\|_{\iota} = K \|a_v(U_{\iota_0 \dots \iota_l}^{(1)*})\| \leq K \|S\|_{\rho}$ . Q.E.D.

Let  $S'_v$  be defined using some other  $\iota' \in \{\iota_0, \dots, \iota_l\}$ . Then  $S_v - S'_v \in \Gamma(\hat{U}_{\iota_0 \dots \iota_l}^{(2)*}, \mathbf{F})$ . We claim that  $\|S_v - S'_v\|_{\rho_4} \leq K \gamma''' \|S\|_{\rho}$ .

*Proof.* Define  $\alpha_s = \sum_{|\lambda|=s}^{\infty} a_{\lambda}(t/\rho)^{\lambda}$  and  $\beta_s = \sum_{|\lambda|=0}^{s-1} a_{\lambda}(t/\rho)^{\lambda}$  over  $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}(\rho)$ . We do the same for  $\iota'$  respectively and obtain  $\alpha'_s$  and  $\beta'_s$  over  $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota'}(\rho)$ . For the restrictions to  $\hat{U}_{\iota_0 \dots \iota_l}^{(2)*}$  we see that  $\alpha_s - \alpha'_s = -(\beta_s - \beta'_s)$ . Hence we get  $\|\alpha_s - \alpha'_s\|_{\rho_4} \leq K(\gamma''')^s \|\alpha_s - \alpha'_s\|_{\rho_1} = K(\gamma''')^s \|\beta_s - \beta'_s\|_{\rho_1} \leq K(\gamma''')^s \|\beta_s\|_{\rho_1} + K(\gamma''')^s \|\beta'_s\|_{\rho_1} \leq K(\gamma''')^s [\|\beta_s\|_{\rho_1}^* + \|\beta'_s\|_{\rho_1}^*] \leq K(\gamma''')^s (\gamma'')^{1-s} \|S\|_{\rho}$ . Here the norms are defined with respect to  $U_{\iota_0 \dots \iota_l}^{(3)*}$  except  $\|\cdot\|^*$  and  $\|S\|_{\rho}$  which are defined with respect to  $U_{\iota_0 \dots \iota_l}^{(1)*}$ . Now we look at the difference  $(S_v - S'_v) t^v / \rho^v$  on  $(U_{\iota_0 \dots \iota_l}^{(3)*})_{\mu}$  with  $|\nu| = s$ ,  $\mu \in \{\iota_0, \dots, \iota_l\}$ , and the power series development with respect to  $W_{\mu}$ . There is one term of order  $s$  which is equal to the corresponding term of  $\alpha_s - \alpha'_s$ . Therefore its norm is  $\leq K(\gamma''')^s \cdot (\gamma'')^{1-s} \|S\|_{\rho}$ . Moreover we have  $\|S_v(t/\rho)^v - S'_v(t/\rho)^v\|_{\rho_1} \leq (\gamma'')^{-s} \cdot K \|S\|_{\rho}$  where the first norm is defined with respect to  $U_{\iota_0 \dots \iota_l}^{(3)*}$ . For the sum  $\sum$  of terms of higher order than  $s$  in the power series of  $(S_v - S'_v) t^v / \rho^v$  we therefore get:  $\|\sum\|_{\rho_4} \leq (\gamma''')^{s+1} (\gamma'')^{-s} \cdot K \|S\|_{\rho}$ . Hence we get  $\|(S_v - S'_v)\|_{\rho_4} \leq \gamma''' \cdot K \|S\|_{\rho}$ . This proves our statement. We see that  $K$  is independent of  $\rho_4$  and  $S$ . The number  $\gamma'''$  depends on  $\rho_4$  only, so  $\gamma''' \cdot K$  gets very small if we make  $\rho_4$  very small.

Let  $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho), \mathbf{F})$  with  $\hat{\xi} = \{\hat{\xi}_{\iota_0 \dots \iota_l}\}$ . Choose  $\iota = \iota(\iota_0, \dots, \iota_l)$  as a function of the unordered  $(l+1)$ -tuple. We now fix  $\iota_0, \dots, \iota_l$  and write  $S = \hat{\xi}_{\iota_0 \dots \iota_l}$ . We apply to  $S$  the method described above and obtain  $\hat{\xi}_{\iota_0 \dots \iota_l}^{(v)} = S_v$ . We do this now for every  $\iota_0, \dots, \iota_l$  and consider  $\hat{\xi}_{(v)} = \{\hat{\xi}_{\iota_0 \dots \iota_l}^{(v)}\}$  as an element of  $C^l(\hat{\mathcal{U}}_2^*(\rho_4), \mathbf{F})$ . Of course  $\hat{\xi}_{(v)}$  depends on the choice of  $\iota = \iota(\iota_0 \dots \iota_l)$  here. Now we see that  $\|\hat{\xi}_{(v)}\|_{\rho_4} \leq \|\hat{\xi}_{(v)}\|_{\rho_1} \leq K \|\hat{\xi}\|_{\rho}$ . We also wish to estimate  $\delta \hat{\xi}_{(v)}$ . Because  $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho), \mathbf{F})$  we can use the preliminary result on  $\iota$  and  $\iota'$  to obtain  $\|\delta \hat{\xi}_{(v)}\|_{\rho_4} \leq K \gamma''' \|\hat{\xi}\|_{\rho}$ .

We shall also need another result:

*Induction Lemma:* There exists  $\hat{\eta}_v \in C^l(\hat{\mathcal{U}}_4^*(\rho_3), \mathbf{F})$  such that  $\delta\hat{\eta}_v = \delta\hat{\xi}_{(v)}$  on  $\hat{\mathcal{U}}_4^*(\rho_3)$  and  $\|\hat{\eta}_v\|_{\rho_3} \leq K \|\delta\hat{\xi}_{(v)}\|_{\rho_4}$ .

*Proof.* The proof uses the assumption that  $\psi_{(l+1)}(\mathbf{F})$  is coherent. Because the coherence of direct images is proved by downward induction on  $l$ , this assumption can be made. Moreover it is assumed that the main theorem is proved for dimension  $l+1$  already. Let us now put  $\alpha = \delta\hat{\xi}_{(v)} \in B^{l+1}(\hat{\mathcal{U}}_2^*(\rho_4), \mathbf{F})$  and  $\hat{\eta}_v = \beta \in C^l(\hat{\mathcal{U}}_4^*(\rho_3), \mathbf{F})$ . We have to prove the existence of  $\beta$ . We may assume that  $\rho_4$  is so small that the main theorem is valid for  $\rho \leq \rho_4$  in the case of dimension  $l+1$ . So there are cocycles  $\omega_1, \dots, \omega_r \in Z^{l+1}(\hat{\mathcal{U}}(\rho_4), \mathbf{F})$  such that  $\alpha = \sum C_\lambda \omega_\lambda + \delta\eta$ , where  $C_\lambda \in I(E^n(\rho_4))$  and  $\eta \in C^l(\hat{\mathcal{U}}_4^*(\rho_4), \mathbf{F})$ . We have to assume that between  $\hat{\mathcal{U}}_4^*$  and  $\mathcal{U}_2^*$  there are very many measure coverings. The cross-sections  $\psi_{(l+1)}(\omega_\lambda)$  give a homomorphism  $r\mathcal{O} \rightarrow \psi_{(l+1)}(\mathbf{F})$  over  $E^n(\rho_4)$ . Because  $\psi_{(l+1)}(\mathbf{F})$  is coherent the kernel  $\mathcal{N}$  is coherent again. Over  $E^n(\rho')$  with  $\rho_3 < \rho' < \rho_4$  we find an epimorphism  $p\mathcal{O} \rightarrow \mathcal{N}$ . Denote by  $n_1, \dots, n_p$  the images of the unit cross-sections in  $p\mathcal{O}$ . Write  $n_\lambda = (e_{\lambda 1}, \dots, e_{\lambda r})$  as an  $r$ -tupel of holomorphic functions. The image of  $n_\lambda$  in  $\Gamma(E^n(\rho'), \psi_{(l+1)}(\mathbf{F}))$  is  $\psi_{(l+1)}(\sum_{\mu=1}^r e_{\lambda\mu} \omega_\mu)$  and zero. We may choose  $\rho_2$  and then  $\rho_3$  and  $\rho'$  very small. Then it follows that  $\hat{n}_\lambda = \sum e_{\lambda\mu} \omega_\mu$  is a coboundary. If  $\rho_3 < \rho'' < \rho'$  there are cochains  $\eta_\lambda \in C^l(\hat{\mathcal{U}}_4^*(\rho''), \mathbf{F})$  such that  $\delta\eta_\lambda = \hat{n}_\lambda$ . Now  $(C_1, \dots, C_r) \in \Gamma(E^n(\rho_4), \mathcal{N})$ . By the methods of sheaf theory we can lift this cross-section to  $p\mathcal{O}$ . Using a "Banach open mapping theorem" we see that the map  $\Gamma(E^n(\rho'), p\mathcal{O}) \rightarrow \Gamma(E^n(\rho'), \mathcal{N})$  is open. This means here that we can find holomorphic functions  $a_\lambda$  over  $E^n(\rho_3)$  such that  $C_\mu = \sum a_\lambda e_{\lambda\mu}$  and  $\|a_\lambda\|_{\rho_3} \leq K \max_\mu \|C_\mu\|_{\rho'} \leq K \max_\mu \|C_\mu\|_{\rho_4}$ . We get  $\sum C_\mu \omega_\mu = \sum a_\lambda e_{\lambda\mu} \omega_\mu = \sum a_\lambda \hat{n}_\lambda = \delta(\sum a_\lambda \eta_\lambda)$ . This leads to  $\alpha \in C^{l+1}(\hat{\mathcal{U}}_4^*(\rho_3)) = \delta(\eta + \sum a_\lambda \eta_\lambda)$ . The estimates required obviously hold. Q.E.D.

Let us now put  $\hat{\xi}_{(v)}^* = \hat{\xi}_{(v)} - \hat{\eta}_v \in Z^l(\hat{\mathcal{U}}_4^*(\rho_3), \mathbf{F})$ . We can write  $\hat{\xi}_{(v)}^*|X_0 = \sum a_{v\lambda} \hat{b}_\lambda|X_0 + \delta\gamma_v$  over  $\mathcal{U}_6^*$ . Here  $a_{v\lambda}$  are complex numbers and  $\gamma_v \in C^{l-1}(\mathcal{U}_6^*, F|X_0)$ . Cartan's theorem and the result after that give the estimates  $|a_{v\lambda}| \leq K \|\hat{\xi}_{(v)}^*\|_{\rho_3} \leq K \|\hat{\xi}\|_\rho$  and  $\|\gamma_v\|_{\rho_3} \leq K \|\hat{\xi}_{(v)}^*\|_{\rho_3} \leq K \|\hat{\xi}\|_\rho$ . Here  $\hat{\gamma}_v \in C^{l-1}(\hat{\mathcal{U}}_7^*(\rho_3), \mathbf{F})$  has been obtained by a constant

extension of  $\gamma_v$ . Let us now put  $\hat{\xi}_{(v)}^{(1)} = \hat{\xi}_{(v)}^* - \sum a_{v\lambda} \hat{b}_\lambda - \delta \hat{\gamma}_v$ . Here  $\hat{\xi}_{(v)}^{(1)} \in C^l(\hat{\mathfrak{U}}_7^*(\rho_3), \mathbb{F})$ . Using the previous estimates and the fact that the  $\hat{b}_\lambda$  are finite we find that  $\|\hat{\xi}_{(v)}^{(1)}\|_{\rho_3} \leq K \|\hat{\xi}_{(v)}\|_{\rho_4} \leq K \|\hat{\xi}\|_\rho$ .

Now we also have  $\hat{\xi}_{(v)}^{(1)}|_{X_0} = 0$ . It follows that

$$\|\hat{\xi}_{(v)}^{(1)}\|_\rho \leq \gamma/\gamma' \|\hat{\xi}_{(v)}^{(1)}\|_{\rho_3} \leq \gamma/\gamma' \cdot K \|\hat{\xi}\|_\rho.$$

Finally we put in  $\hat{\mathfrak{U}}_9^*(\rho)$ :

$$\begin{aligned} \hat{\xi}^{(1)} &= \Sigma \hat{\xi}_{(v)}^{(1)} (t/\rho)^v = \\ &= \Sigma \hat{\xi}_{(v)} (t/\rho)^v - \Sigma \hat{\eta}_v (t/\rho)^v - \Sigma a_{v\lambda} (t/\rho)^v \hat{b}_\lambda - \delta (\Sigma \hat{\gamma}_v (t/\rho)^v) \\ &= \hat{\xi} - \hat{\eta} - \Sigma a_\lambda \hat{b}_\lambda - \delta \hat{\gamma}. \end{aligned}$$

Using the fact that the sum of the absolute values of the coefficients in the power series expansion of  $\hat{\xi}_{(v)}^{(1)}$  by  $(t/\rho)$  is smaller than  $\gamma/\gamma' \cdot K \|\hat{\xi}\|_\rho$  and that with respect to  $\hat{\eta}_v$  is smaller than  $\gamma''' \cdot K \|\hat{\xi}\|_\rho$  we find:  $\|\hat{\xi}^{(1)}\|_\rho \leq \gamma/\gamma' \cdot K \|\hat{\xi}\|_\rho$  and  $\|\hat{\eta}\|_\rho \leq \gamma''' \cdot K \|\hat{\xi}\|_\rho$  and  $\|a_\lambda\|_\rho \leq K \|\hat{\xi}\|_\rho$ . We take the restriction to  $\hat{\mathfrak{B}}(\rho)$  and now  $\tilde{\xi} = \hat{\xi}^{(1)} - \hat{\eta} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbb{F})$  is the desired element. Of course we have to choose  $\rho_4$  and then  $\rho_2$  small enough, for example let  $\gamma''' < \varepsilon/2 K$  and  $\gamma \leq \varepsilon\gamma'/2 K$ .

### MAIN THEOREM

There exists  $\rho_2$  and a constant  $K$  such that if  $\rho \leq \rho_2$  and  $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbb{F})$  with  $\|\hat{\xi}\|_\rho < \infty$  then we can find  $a_1, \dots, a_r \in I(E^n(\rho))$  and  $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbb{F})$  such that  $\hat{\xi} = \sum a_\lambda \hat{b}_\lambda + \delta \hat{\eta}$  on  $\hat{\mathfrak{B}}(\rho)$  with  $\|\hat{\eta}\|_\rho$  and  $\|a_v\|_\rho \leq K \|\hat{\xi}\|_\rho$ .

*Proof.* We have one constant  $K$  from the smoothing theorem. Now we find  $\rho_2$  with an  $\varepsilon$  in the Approximation Lemma such that  $\varepsilon \cdot K < 1/2$ . We shall use this  $\rho_2$  and prove the theorem here. We are given  $\hat{\xi}_0 = \hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbb{F})$  with  $\|\hat{\xi}\|_\rho < \infty$ . The Approximation Lemma gives  $\tilde{\xi}_1 =$