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The set $G^* \subset G$ is open and $R^{**} = \{V_1, \dots, V_{l^*}\}$ an open covering of G^* such that $V_l \subset \subset U_l$ for $l \in \{1, \dots, l^*\}$. We have:

Cartan's Theorem. There exists a constant K such that if $\xi \in Z^l(R^*, q\mathcal{O})$ then $\xi|_{R^{**}} = \delta\eta$ where $\eta \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\|\eta\| \leq K\|\xi\|$ for $l \geq 1$.

This is a simple consequence of Theorem B and Banach's open mapping theorem.

Now we apply Cartan's theorem. We keep the notations as above. Let $\hat{G} = G \times E^n(\rho)$ and put $\hat{R}^* = \{U_l \times E^n(\rho)\}$. Now \hat{R}^* is a Stein covering of \hat{G} . Let $\hat{G}^* = G^* \times E^n(\rho)$ and $\hat{R}^{**} = \{V_l \times E^n(\rho)\}$. Let $\hat{\xi} \in Z^l(\hat{R}^*, q\mathcal{O})$ and write $\hat{\xi} = \sum \xi_{(v)}(t/\rho)^v$ with $\xi_{(v)} \in Z^l(R^*, q\mathcal{O})$. We assume $\|\hat{\xi}\|_\rho = \sup_v \|\xi_{(v)}\| < \infty$. Now Cartan's theorem gives $\xi_{(v)}|_{R^{**}} = \delta\eta_v$ with $\eta_v \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\|\eta_v\| \leq K\|\xi_{(v)}\| < \infty$. It follows that $\hat{\eta} = \sum \eta_v(t/\rho)^v$ is well defined in $C^{l-1}(\hat{R}^{**}, q\mathcal{O})$ and by definition we have $\|\hat{\eta}\|_\rho \leq K\|\hat{\xi}\|_\rho$.

SMOOTHING

We are given a sequence of admissible refinements of measure coverings in $X(\rho_1)$. Here $\rho_1 < \rho_0 = \min \rho_l$ as usual. Let l be a fixed integer ≥ 1 . We are given $\mathfrak{B}^* \ll \mathfrak{B}' = \mathfrak{B}_{3l} \ll \mathfrak{B}_{3l-1} \ll \dots \ll \mathfrak{B}_1 \ll \mathfrak{B}_0 \ll \mathfrak{B} \ll \mathfrak{U}^* \ll \mathfrak{U} = \mathfrak{U}_{3l} \ll \dots \ll \mathfrak{U}_0 \ll \mathfrak{U}'$. Here it is also required that $(\mathfrak{B}_{v+1}, \mathfrak{U}_{v+1}) \ll (\mathfrak{B}_v, \mathfrak{U}_v); (\mathfrak{B}^*, \mathfrak{U}^*) \ll (\mathfrak{B}', \mathfrak{U})$ and $(\mathfrak{B}_0, \mathfrak{U}_0) \ll (\mathfrak{B}, \mathfrak{U}')$. These extra conditions mean: 1) $\hat{U}_{i_0 \dots i_\kappa}^{(v+1)} \cap \hat{V}_{l_0 \dots l_l}^{(v+1)} \subset (U_{i_0 \dots i_\kappa}^{(v)} \cap V_{l_0 \dots l_l}^{(v)})_i$ for each $i \in \{i_0, \dots, i_\kappa\}$ and 2) $(U_{i_0 \dots i_\kappa}^{(v+1)} \cap V_{l_0 \dots l_l}^{(v+1)})_j \subset (U_{i_0 \dots i_\kappa}^{(v)} \cap V_{l_0 \dots l_l}^{(v)})_i$ for all $i, j \in \{i_0, \dots, i_\kappa, l_0, \dots, l_l\}$.

Recall that all operations are done with respect to ρ_1 . Let us put $\hat{R}_{i_0 \dots i_\kappa l_0 \dots l_l}^{(v)} = \hat{U}_{i_0 \dots i_\kappa}^{(v)} \cap \hat{V}_{l_0 \dots l_l}^{(v)}$. We consider elements $\xi_{i_0 \dots i_\kappa l_0 \dots l_l} \in \hat{\Gamma}(\hat{R}_{i_0 \dots i_\kappa l_0 \dots l_l}^{(v)}, \mathbb{F})$.

Now we take a full collection $\hat{\xi} = \{\hat{\xi}_{i_0 \dots i_\kappa l_0 \dots l_l}\}$ of such elements which is anticommutative in $\{i_0, \dots, i_\kappa\}$ and $\{l_0, \dots, l_\kappa\}$. In this way we get a double complex $C_v^{k, \kappa}$. Here $\delta : C_v^{k, \kappa} \rightarrow C_v^{k+1, \kappa}$ and $\partial : C_v^{k, \kappa} \rightarrow C_v^{k, \kappa+1}$ are the usual coboundary operators.

NORM IN $C_v^{k, \kappa}$: Let $\hat{\xi} \in C_v^{k, \kappa}$; we put

$\|\hat{\xi}\|_\rho = \max_{i, (i_0, \dots, i_k, \iota_0, \dots, \iota_k)} \{ \|\hat{\xi}_{i_0 \dots i_k \iota_0 \dots \iota_k} \mid (R_{i_0 \dots i_k \iota_0 \dots \iota_k}^{(v+1)})_i(\rho) \|\}_i$ with $i \in \{i_0, \dots, i_k\}$. Here $\rho \geq \rho_1$ and $R_{i_0 \dots i_k \iota_0 \dots \iota_k}^{(v+1)} = U_{i_0 \dots i_k}^{(v+1)} \cap V_{\iota_0 \dots \iota_k}^{(v+1)}$ and $\|\cdot\|_i$ is taken with respect to the chart \mathscr{W}_i as usual.

SMOOTHING LEMMA: Let $\kappa > 0$. There exists a constant K such that: If $\hat{\xi} \in C_v^{k, \kappa}$ with $\partial \hat{\xi} = 0$ and $\|\hat{\xi}\|_\rho < \infty$ then we can find $\hat{\eta} \in C_{v+3}^{k, \kappa-1}$ such that $\hat{\xi} \mid C_{v+3}^{k, \kappa} = \partial \hat{\eta}$ and $\|\hat{\eta}\|_\rho \leq K \|\hat{\xi}\|_\rho$. Here $\rho \leq \rho_2 = \gamma \rho_1$ with $0 < \gamma < 1$ and K depends only on ρ_2 .

Proof. Let us fix i_0, \dots, i_k in the following discussion. Let $G = U_{i_0 \dots i_k}^{(v+1)}$ and put $\hat{G} = (G)_i(\rho_1)$ for some $i \in \{i_0, \dots, i_k\}$ which is also fixed now. Now G is Stein in X_0 and \hat{G} is Stein in X . We put $R^* = G \cap \mathfrak{B}_{v+1}$ which is a Stein covering of G . Also $\hat{R}^* = \{(G \cap V_{\iota}^{(v+1)})_i(\rho_1)\}_{\iota=1, \dots, \iota^*}$ is a Stein covering of \hat{G} . Let $\hat{\xi} = \{\hat{\xi}_{i_0 \dots i_k \iota_0 \dots \iota_k}\}$. Now we look at the elements of $\{\hat{\xi}_{i_0 \dots i_k \iota_0 \dots \iota_k}\} = \hat{\xi}_{i_0 \dots i_k} \in Z^\kappa(\hat{R}^*, \mathbf{F})$. Here i_0, \dots, i_k is fixed as above. We get a cocycle because we have assumed that $\partial \hat{\xi} = 0$. More precisely we have considered the restriction of $\hat{\xi}_{i_0 \dots i_k \iota_0 \dots \iota_k}$ to \hat{R}^* . We must verify that this restriction is possible.

Verification: By definition of $Z^\kappa(\hat{R}^*, \mathbf{F})$ we have to look at sets of the following type: (these are the sets where the cross-sections are defined) $(G \cap V_{\iota_0}^{(v+1)})_i \cap \dots \cap (G \cap V_{\iota_k}^{(v+1)})_i = (G \cap V_{\iota_0 \dots \iota_k}^{(v+1)})_i = (R_{i_0 \dots i_k \iota_0 \dots \iota_k}^{(v+1)})_i$. Now by 2) we have $(R_{i_0 \dots i_k \iota_0 \dots \iota_k}^{(v+1)})_i \subset \bigcap_j (R_{i_0 \dots i_k \iota_0 \dots \iota_k}^{(v)})_j \subset (U_{i_0}^{(v)})_{i_0} \cap \dots \cap (V_{\iota_k}^{(v)})_{\iota_k} = \hat{R}_{i_0 \dots i_k \iota_0 \dots \iota_k}^{(v)}$. Q.E.D.

Now we put $G^* = U_{i_0 \dots i_k}^{(v+2)} \subset \subset G$. We let $\hat{R}^{**} = \{(G^* \cap V_{\iota}^{(v+2)})_i\}_{\iota=1, \dots, \iota^*}$. The system \hat{R}^{**} is a Stein covering of $(G^*)_i$. We are in a good position now. For we are given $\hat{\xi}_{i_0 \dots i_k} \in Z^\kappa(\hat{R}^*, \mathbf{F})$. Here \hat{R}^* is a Stein covering of \hat{G} and \hat{G} is a Stein manifold. We are working in the chart \mathscr{W}_i where the usual identifications are used. Hence we arrive at the following situation: G is a Stein manifold with a Stein covering $R^* = \mathfrak{B}_{v+1} \cap G$. Also $G^* \subset \subset G$ and $R^{**} = \mathfrak{B}_{v+2} \cap G^*$ is a Stein covering of G^* such that $R^{**} \subset \subset R^*$. The cocycle $\hat{\xi}_{i_0 \dots i_k}$ is now considered as an element of $Z^\kappa(\hat{R}^*, q\mathcal{O})$ which

we simply call $\hat{\xi}_{i_0 \dots i_k}$ again. Now we apply the result after Cartan's theorem. Hence we can find a constant K such that for every $\rho \leq \rho_2$ we get $\eta \in C^{\kappa-1}(\hat{R}^{**}, q\mathcal{O})$ and $\|\eta\|_\rho \leq K \|\hat{\xi}_{i_0 \dots i_k}\|_\rho$ with $\partial\eta = \hat{\xi}_{i_0 \dots i_k}$. But this means precisely that we can find $\hat{\eta}_{i_0 \dots i_k} \in C^{\kappa-1}(\hat{R}^{**}(\rho), \mathbb{F})$ such that $\|\hat{\eta}_{i_0 \dots i_k}\|_{i, \rho} \leq K \|\hat{\xi}_{i_0 \dots i_k}\|_{i, \rho}$ with $\hat{\xi}_{i_0 \dots i_k} = \partial\hat{\eta}_{i_0 \dots i_k}$. We have only constructed $\hat{\eta}_{i_0 \dots i_k}$ using a fixed $i \in \{i_0, \dots, i_k\}$. Now we must let (i_0, \dots, i_k) vary. For each (i_0, \dots, i_k) we choose some i which only depends on the unordered $(k+1)$ -tupel (i_0, \dots, i_k) and construct an element $\hat{\eta}_{i_0 \dots i_k}$ as above. Now we can restrict everything to $C_{v+3}^{k, \kappa-1}$.

Verification: Consider a set where cross-sections over $C_{v+3}^{k, \kappa-1}$ have to be defined, i.e. a set $\hat{U}_{i_0 \dots i_k}^{(v+3)} \cap \hat{V}_{i_0 \dots i_k}^{(v+3)}$. But by 1) follows $\hat{U}_{i_0 \dots i_k}^{(v+3)} \cap \hat{V}_{i_0 \dots i_k}^{(v+3)} \subset (R_{i_0 \dots i_k, i_0 \dots i_k}^{(v+2)})_i$ for each $i \in \{i_0, \dots, i_k\}$. This inclusion shows that we get a well defined element $\hat{\eta} \in C_{v+3}^{k, \kappa-1}$ by restricting the elements $\hat{\eta}_{i_0 \dots i_k}$ to $C_{v+3}^{k, \kappa-1}$. We find that $\hat{\xi} \mid C_{v+3}^{k, \kappa} = \partial\hat{\eta}$ now. The norm inequalities are not obvious, but recalling how $\hat{\eta}$ is constructed here it is seen that we can apply Theorem I to obtain the required estimate.

SMOOTHING THEOREM. There exists a constant K such that: If $\hat{\xi} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbb{F})$ with $\|\hat{\xi}\|_\rho < \infty$ then we can find $\hat{\xi}^* \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbb{F})$ and $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}'(\rho), \mathbb{F})$ for which $\hat{\xi}^* \mid \hat{\mathfrak{B}}'(\rho) = \hat{\xi} \mid \hat{\mathfrak{B}}'(\rho) + \delta\hat{\eta}$ and $\|\hat{\xi}^*\|_\rho$ and $\|\hat{\eta}\|_\rho \leq K \|\hat{\xi}\|_\rho$. Here $\rho \leq \rho_2 < \rho_1$ and K only depends on ρ_2 .

Proof. Before we can use the double complex $\{C_v^{k, \kappa}\}$ we must introduce two "ε-maps". To define the ε_1 -map, let $Z_v^{k, \kappa} \subset C_v^{k, \kappa}$ consist of all $\hat{\xi} \in C_v^{k, \kappa}$ such that $\delta\hat{\xi} = \partial\hat{\xi} = 0$. Now we shall define the ε_1 -map: $\varepsilon_1: Z^l(\hat{\mathfrak{B}}, \mathbb{F}) \rightarrow Z_0^{0, l}$. A section belonging to an element of $C_0^{0, l}$ is defined on some set $\hat{U}_{i_0}^{(0)} \cap \hat{V}_{i_0, \dots, i_l}^{(0)} \subset \hat{V}_{i_0 \dots i_l}$ where sections of elements of $Z^l(\hat{\mathfrak{B}}, \mathbb{F})$ are defined. Hence we get a natural restriction map ε_1 which also maps cocycles into cocycles. It is easy to verify that $\|\varepsilon_1(\hat{\xi})\|_\rho \leq K \|\hat{\xi}\|_\rho$. Theorem I can be used because $(U_i^{(1)} \cap V_{i_0 \dots i_l}^{(1)})_i \subset (V_{i_0 \dots i_l}^{(0)})_i$ for every i and every $i \in \{i_0, \dots, i_l\}$. Recall that the norm in $Z^l(\hat{\mathfrak{B}}, \mathbb{F})$ is defined with respect to

$\hat{\mathfrak{B}}_0$ here. The “ ε_2 -map” : we shall construct a map $\varepsilon_2: Z_{3l}^{l,0} \rightarrow Z^l(\hat{\mathcal{U}}, \mathbf{F})$. Let $\hat{\xi} = \{ \hat{\xi}_{i_0, \dots, i_l, \iota_0} \} \in Z_{3l}^{l,0}$. Here $\hat{\xi}_{i_0, \dots, i_l, \iota_0}$ is defined on $\hat{R}_{i_0 \dots i_l, \iota_0}^{(3l)}$. Because $\hat{\delta}\hat{\xi} = 0$ we see that the elements $\hat{\xi}_{i_0 \dots i_l, \iota_0}$ are independent of ι_0 . Now $\bigcup_{\iota=1}^{\iota^*} \hat{V}_{\iota}^{(3l)}$ covers $X(\rho_1)$. If we put $\varepsilon_2(\hat{\xi})_{i_0 \dots i_l} = \hat{\xi}_{i_0 \dots i_l, \iota_0}$ in $\hat{U}_{i_0 \dots i_l}^{(3l)} \cap \hat{V}_{\iota_0}^{(3l)}$ then we see that $\varepsilon_2(\hat{\xi})_{i_0 \dots i_l}$ is a well defined section on $\hat{U}_{i_0 \dots i_l}^{(3l)}$. In this way we obtain $\varepsilon_2(\hat{\xi}) \in Z^l(\hat{\mathcal{U}}, \mathbf{F})$. Here $\varepsilon_2(\hat{\xi})$ is a cocycle because $\hat{\delta}\hat{\xi} = 0$. Now we prove that $\| \varepsilon_2(\hat{\xi}) \|_{\rho} \leq K \| \hat{\xi} \|_{\rho}$.

Verification. A computation of $\| \varepsilon_2(\hat{\xi}) \|_{\rho}$ involves the following: $\varepsilon_2(\hat{\xi}) = \{ \xi_{i_0 \dots i_l}^{(2)} \}$. Look at some $\xi_{i_0 \dots i_l}^{(2)}$ in the chart \mathcal{W}_i with $i \in \{ i_0, \dots, i_l \}$. We write $\hat{\xi}_{i_0 \dots i_l}^{(2)} = \sum a_v (t/\rho)^v$ over $(U_{i_0 \dots i_l}^*)_i$ and compute $\sup_v | a_v (U_{i_0 \dots i_l}^*)_i |$. A computation of $\| \hat{\xi} \|_{\rho}$ involves the following: Look at $\hat{\xi}_{i_0 \dots i_l}$ over $(U_{i_0 \dots i_l}^* \cap V_{\iota}^*)_i$ in a chart W_i . Here ι is fixed. We write $\hat{\xi}_{i_0 \dots i_l, \iota} = \sum a_v^{(\iota)} (t/\rho)^v$ and compute $\sup_{\iota^*} | a_v^{(\iota)} (U_{i_0 \dots i_l}^* \cap V_{\iota}^*)_i |$. Now $\bigcup_{\iota} V_{\iota}^*$ covers X_0 . Hence we would have $\sup_{v, \iota} | a_v^{(\iota)} (U_{i_0 \dots i_l}^* \cap V_{\iota}^*)_i | = \sup_v | a_v (U_{i_0 \dots i_l}^*)_i |$ if $a_v = a_v^{(\iota)}$ in $U_{i_0 \dots i_l}^* \cap V_{\iota}^*$. But this is obvious since $\xi_{i_0 \dots i_l}^{(2)} = \hat{\xi}_{i_0 \dots i_l, \iota}$ in $(U_{i_0 \dots i_l}^* \cap V_{\iota}^*)_i$. Hence we have $\| \varepsilon_2(\hat{\xi}) \|_{\rho} \leq \| \hat{\xi} \|_{\rho}$.

Now we are ready to start the proof of the smoothing theorem. We let K denote a constant, which may be different at different occurrences.

We also introduce a double complex $\{ \tilde{C}_v^{k, \kappa} \}$ using $(\mathfrak{B}, \mathfrak{B})$, i.e. it is defined just as the previous double complex was, using \mathfrak{B} -sets instead of \mathcal{U} -sets.

We shall inductively construct the following elements:

$$\hat{\xi}_v = \{ \hat{\xi}_{i_0 \dots i_v, \iota_0 \dots \iota_{l-v}} \} \in Z_{3v}^{v, l-v}$$

$$\tilde{\xi}_v = \{ \tilde{\xi}_{i_0 \dots i_v, \iota_0 \dots \iota_{l-v}} \} \in \tilde{Z}_{3v}^{v, l-v}; \quad v = 0, \dots, l$$

$$\hat{\eta}_v = \{ \hat{\eta}_{i_0 \dots i_{v-1}, \iota_0 \dots \iota_{l-v}} \} \in C_{3v}^{v-1, l-v}$$

$$\tilde{\eta}_v = \{ \tilde{\eta}_{i_0 \dots i_{v-1}, \iota_0 \dots \iota_{l-v}} \} \in \tilde{C}_{3v}^{v-1, v-1}; \quad v = 1, \dots, l$$

$$\tilde{\gamma}_v = \{ \tilde{\gamma}_{i_0 \dots i_{v-1}, i_0 \dots i_{l-v-1}} \} \in \tilde{C}_{3v-3}^{v-1, l-v-1}; \quad v = 1, \dots, (l-1)$$

and $\tilde{\gamma}_l = \{ \tilde{\gamma}_{i_0 \dots i_{l-1}} \} \in C^{l-1}(\mathfrak{B}_{3l})$.

The construction: $\hat{\xi} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbb{F})$ is given. The whole construction is done using ρ instead of ρ_1 and we omit ρ to simplify the notation. We put $\varepsilon_1(\hat{\xi}) = \hat{\xi}_0 \in Z_0^{0,l}$. Now we apply the Smoothing Lemma and get $\hat{\eta}_1$ such that $\partial \hat{\eta}_1 = \hat{\xi}_0$ with $\| \hat{\eta}_1 \|_\rho \leq K \| \hat{\xi}_0 \|_\rho \leq K \| \hat{\xi} \|_\rho$. Put $\hat{\xi}_1 = \delta \hat{\eta}_1$. Obviously $\| \hat{\xi}_1 \|_\rho \leq K \| \hat{\eta}_1 \|_\rho$. Inductively we find $\delta \hat{\eta}_v = \hat{\xi}_{v-1}$ and we put $\hat{\xi}_v = \delta \hat{\eta}_v$ where $\hat{\eta}_v$ are found from the Smoothing Lemma. Finally we get $\hat{\xi}_l$ and we have $\| \hat{\xi}_l \|_\rho \leq K \| \hat{\xi} \|_\rho$. Now we define $\tilde{\xi}_v$ and $\tilde{\eta}_v$ as follows. Put $\tilde{\xi}_0 = \hat{\xi}_0$ where $\tilde{\xi}_0 \in \tilde{Z}_0^{0,l}$ is obtained by natural restriction of $\hat{\xi}_0$. Put $\tilde{\eta}_v = (-1)^v \{ \tilde{\xi}_{i_0 \dots i_{v-1}, i_0 \dots i_{l-v}} \}$ which is well defined with respect to $(\mathfrak{B}_{3v}, \mathfrak{B}_{3v})$ by taking natural restrictions. Put $\tilde{\xi}_v = \delta \tilde{\eta}_v$ for $v = 1, \dots, l$. A computation shows that $\tilde{\xi}_{v-1} = \partial \tilde{\eta}_v$ when $v = 1, \dots, l$. Notice that this is trivial when $v = 1$. In the following discussion each $\hat{\eta}_v$ is restricted to $(\mathfrak{B}_{3v}, \mathfrak{B}_{3v})$. We have $\partial(\tilde{\eta}_1 - \hat{\eta}_1) = 0$. Hence we find $\tilde{\eta}_1 - \hat{\eta}_1 = \delta \tilde{\gamma}_1$ by the Smoothing Lemma. Now we define $\tilde{\gamma}_v$ such that $\partial \tilde{\gamma}_v = \tilde{\eta}_v - \hat{\eta}_v - \delta \tilde{\gamma}_{v-1}$ inductively. This is possible because $\partial(\tilde{\eta}_v - \hat{\eta}_v - \delta \tilde{\gamma}_{v-1}) = 0$, for we have $\partial(\tilde{\eta}_v - \hat{\eta}_v - \delta \tilde{\gamma}_{v-1}) = \tilde{\xi}_{v-1} - \hat{\xi}_{v-1} - \delta \partial \tilde{\gamma}_{v-1} = \delta \tilde{\eta}_{v-1} - \delta \hat{\eta}_{v-1} - \delta(\tilde{\eta}_{v-1} - \hat{\eta}_{v-1}) = 0$. We get finally $\tilde{\gamma}_{l-1} \in \tilde{C}_{3l}^{l-2,0}$ and then $\delta \tilde{\gamma}_{l-1} \in \tilde{C}_{3l}^{l-1,0}$. We have $\partial(\tilde{\eta}_l - \hat{\eta}_l - \delta \tilde{\gamma}_{l-1}) = 0$. Therefore we can put $\tilde{\gamma}_l = \varepsilon_2(\tilde{\eta}_l - \hat{\eta}_l - \delta \tilde{\gamma}_{l-1})$. It follows that $\tilde{\gamma}_l \in C^{l-1}(\mathfrak{B}_{3l})$ and $\delta \tilde{\gamma}_l = \varepsilon_2(\tilde{\xi}_l - \hat{\xi}_l)$. We have $\varepsilon_2(\tilde{\xi}_l) = -\hat{\xi} | \mathfrak{B}'$ and for $\varepsilon_2(\tilde{\xi}_l) = -\hat{\xi}^*$ and $\hat{\eta} = \tilde{\eta}$ the required equation $\hat{\xi}^* = \hat{\xi} + \delta \hat{\eta}$. The estimates follow immediately from the construction and the Smoothing Lemma.