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The set  $G^* \subset G$  is open and  $R^{**} = \{V_1, \dots, V_{l^*}\}$  an open covering of  $G^*$  such that  $V_l \subset \subset U_l$  for  $l \in \{1, \dots, l^*\}$ . We have:

*Cartan's Theorem.* There exists a constant  $K$  such that if  $\xi \in Z^l(R^*, q\mathcal{O})$  then  $\xi|_{R^{**}} = \delta\eta$  where  $\eta \in C^{l-1}(R^{**}, q\mathcal{O})$  and  $\|\eta\| \leq K\|\xi\|$  for  $l \geq 1$ .

This is a simple consequence of Theorem B and Banach's open mapping theorem.

Now we apply Cartan's theorem. We keep the notations as above. Let  $\hat{G} = G \times E^n(\rho)$  and put  $\hat{R}^* = \{U_l \times E^n(\rho)\}$ . Now  $\hat{R}^*$  is a Stein covering of  $\hat{G}$ . Let  $\hat{G}^* = G^* \times E^n(\rho)$  and  $\hat{R}^{**} = \{V_l \times E^n(\rho)\}$ . Let  $\hat{\xi} \in Z^l(\hat{R}^*, q\mathcal{O})$  and write  $\hat{\xi} = \sum \xi_{(v)} (t/\rho)^v$  with  $\xi_{(v)} \in Z^l(R^*, q\mathcal{O})$ . We assume  $\|\hat{\xi}\|_\rho = \sup_v \|\xi_{(v)}\| < \infty$ . Now Cartan's theorem gives  $\xi_{(v)}|_{R^{**}} = \delta\eta_v$  with  $\eta_v \in C^{l-1}(R^{**}, q\mathcal{O})$  and  $\|\eta_v\| \leq K\|\xi_{(v)}\| < \infty$ . It follows that  $\hat{\eta} = \sum \eta_v (t/\rho)^v$  is well defined in  $C^{l-1}(\hat{R}^{**}, q\mathcal{O})$  and by definition we have  $\|\hat{\eta}\|_\rho \leq K\|\hat{\xi}\|_\rho$ .

## SMOOTHING

We are given a sequence of admissible refinements of measure coverings in  $X(\rho_1)$ . Here  $\rho_1 < \rho_0 = \min \rho_l$  as usual. Let  $l$  be a fixed integer  $\geq 1$ . We are given  $\mathfrak{B}^* \ll \mathfrak{B}' = \mathfrak{B}_{3l} \ll \mathfrak{B}_{3l-1} \ll \dots \ll \mathfrak{B}_1 \ll \mathfrak{B}_0 \ll \mathfrak{B} \ll \mathfrak{U}^* \ll \mathfrak{U} = \mathfrak{U}_{3l} \ll \dots \ll \mathfrak{U}_0 \ll \mathfrak{U}'$ . Here it is also required that  $(\mathfrak{B}_{v+1}, \mathfrak{U}_{v+1}) \ll (\mathfrak{B}_v, \mathfrak{U}_v); (\mathfrak{B}^*, \mathfrak{U}^*) \ll \ll (\mathfrak{B}', \mathfrak{U})$  and  $(\mathfrak{B}_0, \mathfrak{U}_0) \ll (\mathfrak{B}, \mathfrak{U}')$ . These extra conditions mean: 1)  $\hat{U}_{i_0 \dots i_k}^{(v+1)} \cap \hat{V}_{i_0 \dots i_k}^{(v+1)} \subset (U_{i_0 \dots i_k}^{(v)} \cap V_{i_0 \dots i_k}^{(v)})_i$  for each  $i \in \{i_0, \dots, i_k\}$  and 2)  $(U_{i_0 \dots i_k}^{(v+1)} \cap V_{i_0 \dots i_k}^{(v+1)})_j \subset (U_{i_0 \dots i_k}^{(v)} \cap V_{i_0 \dots i_k}^{(v)})_i$  for all  $i, j \in \{i_0, \dots, i_k, i_0, \dots, i_l\}$ . Recall that all operations are done with respect to  $\rho_1$ . Let us put  $\hat{R}_{i_0 \dots i_k i_0 \dots i_k}^{(v)} = \hat{U}_{i_0 \dots i_k}^{(v)} \cap \hat{V}_{i_0 \dots i_k}^{(v)}$ . We consider elements  $\xi_{i_0 \dots i_k i_0 \dots i_k} \in \hat{\Gamma}(\hat{R}_{i_0 \dots i_k i_0 \dots i_k}^{(v)}, \mathbb{F})$ . Now we take a full collection  $\hat{\xi} = \{\xi_{i_0 \dots i_k i_0 \dots i_k}\}$  of such elements which is anticommutative in  $\{i_0, \dots, i_k\}$  and  $\{i_0, \dots, i_k\}$ . In this way we get a double complex  $C_v^{k, \kappa}$ . Here  $\delta : C_v^{k, \kappa} \rightarrow C_v^{k+1, \kappa}$  and  $\partial : C_v^{k, \kappa} \rightarrow C_v^{k, \kappa+1}$  are the usual coboundary operators.

NORM IN  $C_v^{k, \kappa}$ : Let  $\hat{\xi} \in C_v^{k, \kappa}$ ; we put

$\|\hat{\xi}\|_\rho = \max_{i, (i_0, \dots, i_k, i_0, \dots, i_k)} \{ \|\hat{\xi}_{i_0 \dots i_k i_0 \dots i_k} \mid (R_{i_0 \dots i_k i_0 \dots i_k}^{(v+1)})_i(\rho) \|_i \mid i \in \{i_0, \dots, i_k\} \}$ . Here  $\rho \geq \rho_1$  and  $R_{i_0 \dots i_k i_0 \dots i_k}^{(v+1)} = U_{i_0 \dots i_k}^{(v+1)} \cap V_{i_0 \dots i_k}^{(v+1)}$  and  $\|\cdot\|_i$  is taken with respect to the chart  $\mathcal{W}_i$  as usual.

**SMOOTHING LEMMA:** Let  $\kappa > 0$ . There exists a constant  $K$  such that: If  $\hat{\xi} \in C_v^{k, \kappa}$  with  $\partial \hat{\xi} = 0$  and  $\|\hat{\xi}\|_\rho < \infty$  then we can find  $\hat{\eta} \in C_{v+3}^{k, \kappa-1}$  such that  $\hat{\xi} \mid C_{v+3}^{k, \kappa} = \partial \hat{\eta}$  and  $\|\hat{\eta}\|_\rho \leq K \|\hat{\xi}\|_\rho$ . Here  $\rho \leq \rho_2 = \gamma \rho_1$  with  $0 < \gamma < 1$  and  $K$  depends only on  $\rho_2$ .

*Proof.* Let us fix  $i_0, \dots, i_k$  in the following discussion. Let  $G = U_{i_0 \dots i_k}^{(v+1)}$  and put  $\hat{G} = (G)_i(\rho_1)$  for some  $i \in \{i_0, \dots, i_k\}$  which is also fixed now. Now  $G$  is Stein in  $X_0$  and  $\hat{G}$  is Stein in  $X$ . We put  $R^* = G \cap \mathfrak{B}_{v+1}$  which is a Stein covering of  $G$ . Also  $\hat{R}^* = \{(G \cap V_i^{(v+1)})_i(\rho_1)\}_{i=1, \dots, i^*}$  is a Stein covering of  $\hat{G}$ . Let  $\hat{\xi} = \{\hat{\xi}_{i_0 \dots i_k i_0 \dots i_k}\}$ . Now we look at the elements of  $\{\hat{\xi}_{i_0 \dots i_k i_0 \dots i_k}\} = \hat{\xi}_{i_0 \dots i_k} \in Z^\kappa(\hat{R}^*, \mathbb{F})$ . Here  $i_0, \dots, i_k$  is fixed as above. We get a cocycle because we have assumed that  $\partial \hat{\xi} = 0$ . More precisely we have considered the restriction of  $\hat{\xi}_{i_0 \dots i_k i_0 \dots i_k}$  to  $\hat{R}^*$ . We must verify that this restriction is possible.

*Verification:* By definition of  $Z^\kappa(\hat{R}^*, \mathbb{F})$  we have to look at sets of the following type: (these are the sets where the cross-sections are defined)  $(G \cap V_{i_0}^{(v+1)})_i \cap \dots \cap (G \cap V_{i_k}^{(v+1)})_i = (G \cap V_{i_0 \dots i_k}^{(v+1)})_i = (R_{i_0 \dots i_k i_0 \dots i_k}^{(v+1)})_i$ . Now by 2) we have  $(R_{i_0 \dots i_k i_0 \dots i_k}^{(v+1)})_i \subset \bigcap_j (R_{i_0 \dots i_k i_0 \dots i_k}^{(v)})_j \subset (U_{i_0}^{(v)})_{i_0} \cap \dots \cap (V_{i_k}^{(v)})_{i_k} = \hat{R}_{i_0 \dots i_k i_0 \dots i_k}^{(v)}$ . Q.E.D.

Now we put  $G^* = U_{i_0 \dots i_k}^{(v+2)} \subset G$ . We let  $\hat{R}^{**} = \{(G^* \cap V_i^{(v+2)})_i\}_{i=1, \dots, i^*}$ . The system  $\hat{R}^{**}$  is a Stein covering of  $(G^*)_i$ . We are in a good position now. For we are given  $\hat{\xi}_{i_0 \dots i_k} \in Z^\kappa(\hat{R}^*, \mathbb{F})$ . Here  $\hat{R}^*$  is a Stein covering of  $\hat{G}$  and  $\hat{G}$  is a Stein manifold. We are working in the chart  $\mathcal{W}_i$  where the usual identifications are used. Hence we arrive at the following situation:  $G$  is a Stein manifold with a Stein covering  $R^* = \mathfrak{B}_{v+1} \cap G$ . Also  $G^* \subset G$  and  $R^{**} = \mathfrak{B}_{v+2} \cap G^*$  is a Stein covering of  $G^*$  such that  $R^{**} \subset R^*$ . The cocycle  $\hat{\xi}_{i_0 \dots i_k}$  is now considered as an element of  $Z^\kappa(\hat{R}^*, q\mathcal{O})$  which

we simply call  $\hat{\xi}_{i_0 \dots i_k}$  again. Now we apply the result after Cartan's theorem. Hence we can find a constant  $K$  such that for every  $\rho \leq \rho_2$  we get  $\eta \in C^{\kappa-1}(\hat{R}^{**}, q\mathcal{O})$  and  $\|\eta\|_\rho \leq K \|\hat{\xi}_{i_0 \dots i_k}\|_\rho$  with  $\partial\eta = \hat{\xi}_{i_0 \dots i_k}$ . But this means precisely that we can find  $\hat{\eta}_{i_0 \dots i_k} \in C^{\kappa-1}(\hat{R}^{**}(\rho), \mathbf{F})$  such that  $\|\hat{\eta}_{i_0 \dots i_k}\|_{i, \rho} \leq K \|\hat{\xi}_{i_0 \dots i_k}\|_{i, \rho}$  with  $\hat{\xi}_{i_0 \dots i_k} = \partial\hat{\eta}_{i_0 \dots i_k}$ . We have only constructed  $\hat{\eta}_{i_0 \dots i_k}$  using a fixed  $i \in \{i_0, \dots, i_k\}$ . Now we must let  $(i_0, \dots, i_k)$  vary. For each  $(i_0, \dots, i_k)$  we choose some  $i$  which only depends on the unordered  $(k+1)$ -tupel  $(i_0, \dots, i_k)$  and construct an element  $\hat{\eta}_{i_0 \dots i_k}$  as above. Now we can restrict everything to  $C_{v+3}^{k, \kappa-1}$ .

*Verification:* Consider a set where cross-sections over  $C_{v+3}^{k, \kappa-1}$  have to be defined, i.e. a set  $\hat{U}_{i_0 \dots i_k}^{(v+3)} \cap \hat{V}_{i_0 \dots i_k}^{(v+3)}$ . But by 1) follows  $\hat{U}_{i_0 \dots i_k}^{(v+3)} \cap \hat{V}_{i_0 \dots i_k}^{(v+3)} \subset (R_{i_0 \dots i_k, i_0 \dots i_k}^{(v+2)})_i$  for each  $i \in \{i_0, \dots, i_k\}$ . This inclusion shows that we get a well defined element  $\hat{\eta} \in C_{v+3}^{k, \kappa-1}$  by restricting the elements  $\hat{\eta}_{i_0 \dots i_k}$  to  $C_{v+3}^{k, \kappa-1}$ . We find that  $\hat{\xi} \mid C_{v+3}^{k, \kappa} = \partial\hat{\eta}$  now. The norm inequalities are not obvious, but recalling how  $\hat{\eta}$  is constructed here it is seen that we can apply Theorem I to obtain the required estimate.

**SMOOTHING THEOREM.** There exists a constant  $K$  such that: If  $\hat{\xi} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbf{F})$  with  $\|\hat{\xi}\|_\rho < \infty$  then we can find  $\hat{\xi}^* \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$  and  $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}'(\rho), \mathbf{F})$  for which  $\hat{\xi}^* \mid \hat{\mathfrak{B}}'(\rho) = \hat{\xi} \mid \hat{\mathfrak{B}}'(\rho) + \partial\hat{\eta}$  and  $\|\hat{\xi}^*\|_\rho$  and  $\|\hat{\eta}\|_\rho \leq K \|\hat{\xi}\|_\rho$ . Here  $\rho \leq \rho_2 < \rho_1$  and  $K$  only depends on  $\rho_2$ .

*Proof.* Before we can use the double complex  $\{C_v^{k, \kappa}\}$  we must introduce two “ $\varepsilon$ -maps”. To define the  $\varepsilon_1$ -map, let  $Z_v^{k, \kappa} \subset C_v^{k, \kappa}$  consist of all  $\hat{\xi} \in C_v^{k, \kappa}$  such that  $\delta\hat{\xi} = \partial\hat{\xi} = 0$ . Now we shall define the  $\varepsilon_1$ -map:  $\varepsilon_1: Z^l(\hat{\mathfrak{B}}, \mathbf{F}) \rightarrow Z_0^{0, l}$ . A section belonging to an element of  $C_0^{0, l}$  is defined on some set  $\hat{U}_{i_0}^{(0)} \cap \hat{V}_{i_0, \dots, i_l}^{(0)} \subset \hat{V}_{i_0 \dots i_l}$  where sections of elements of  $Z^l(\hat{\mathfrak{B}}, \mathbf{F})$  are defined. Hence we get a natural restriction map  $\varepsilon_1$  which also maps cocycles into cocycles. It is easy to verify that  $\|\varepsilon_1(\hat{\xi})\|_\rho \leq K \|\hat{\xi}\|_\rho$ . Theorem I can be used because  $(U_i^{(1)} \cap V_{i_0 \dots i_l}^{(1)})_i \subset (V_{i_0 \dots i_l}^{(0)})_i$  for every  $i$  and every  $i \in \{i_0, \dots, i_l\}$ . Recall that the norm in  $Z^l(\hat{\mathfrak{B}}, \mathbf{F})$  is defined with respect to

$\hat{\mathfrak{B}}_0$  here. The “ $\varepsilon_2$ -map” : we shall construct a map  $\varepsilon_2: Z_{3l}^{l,0} \rightarrow Z^l(\hat{\mathfrak{U}}, \mathbf{F})$ . Let  $\hat{\xi} = \{\hat{\xi}_{i_0, \dots, i_l, \iota_0}\} \in Z_{3l}^{l,0}$ . Here  $\hat{\xi}_{i_0, \dots, i_l, \iota_0}$  is defined on  $\hat{R}_{i_0 \dots i_l, \iota_0}^{(3l)}$ . Because  $\hat{\partial}\hat{\xi} = 0$  we see that the elements  $\hat{\xi}_{i_0 \dots i_l, \iota_0}$  are independent of  $\iota_0$ . Now  $\bigcup_{\iota=1}^* \hat{V}_{\iota}^{(3l)}$  covers  $X(\rho_1)$ . If we put  $\varepsilon_2(\hat{\xi})_{i_0 \dots i_l} = \hat{\xi}_{i_0 \dots i_l, \iota_0}$  in  $\hat{U}_{i_0 \dots i_l}^{(3l)} \cap \hat{V}_{\iota_0}^{(3l)}$  then we see that  $\varepsilon_2(\hat{\xi})_{i_0 \dots i_l}$  is a well defined section on  $\hat{U}_{i_0 \dots i_l}^{(3l)}$ . In this way we obtain  $\varepsilon_2(\hat{\xi}) \in Z^l(\hat{\mathfrak{U}}, \mathbf{F})$ . Here  $\varepsilon_2(\hat{\xi})$  is a cocycle because  $\hat{\partial}\hat{\xi} = 0$ . Now we prove that  $\|\varepsilon_2(\hat{\xi})\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$ .

*Verification.* A computation of  $\|\varepsilon_2(\hat{\xi})\|_{\rho}$  involves the following:  $\varepsilon_2(\hat{\xi}) = \{\xi_{i_0 \dots i_l}^{(2)}\}$ . Look at some  $\xi_{i_0 \dots i_l}^{(2)}$  in the chart  $\mathcal{W}_i$  with  $i \in \{i_0, \dots, i_l\}$ . We write  $\hat{\xi}_{i_0 \dots i_l}^{(2)} = \sum a_v (t/\rho)^v$  over  $(U_{i_0 \dots i_l}^*)_i$  and compute  $\sup_v |a_v(U_{i_0 \dots i_l}^*)|$ . A computation of  $\|\hat{\xi}\|_{\rho}$  involves the following: Look at  $\hat{\xi}_{i_0 \dots i_l}$  over  $(U_{i_0 \dots i_l}^* \cap V_{\iota}^*)_i$  in a chart  $W_i$ . Here  $\iota$  is fixed. We write  $\hat{\xi}_{i_0 \dots i_l, \iota} = \sum a_v^{(\iota)} (t/\rho)^v$  and compute  $\sup_v |a_v^{(\iota)}(U_{i_0 \dots i_l}^* \cap V_{\iota}^*)|$ . Now  $\bigcup_{\iota=1}^* V_{\iota}^*$  covers  $X_0$ . Hence we would have  $\sup_{v, \iota} |a_v^{(\iota)}(U_{i_0 \dots i_l}^* \cap V_{\iota}^*)| = \sup_v |a_v(U_{i_0 \dots i_l}^*)|$  if  $a_v = a_v^{(\iota)}$  in  $U_{i_0 \dots i_l}^* \cap V_{\iota}^*$ . But this is obvious since  $\xi_{i_0 \dots i_l}^{(2)} = \hat{\xi}_{i_0 \dots i_l, \iota}$  in  $(U_{i_0 \dots i_l}^* \cap V_{\iota}^*)_i$ . Hence we have  $\|\varepsilon_2(\hat{\xi})\|_{\rho} \leq \|\hat{\xi}\|_{\rho}$ .

Now we are ready to start the proof of the smoothing theorem. We let  $K$  denote a constant, which may be different at different occurrences.

We also introduce a double complex  $\{\tilde{C}_v^{k, \kappa}\}$  using  $(\mathfrak{B}, \mathfrak{B})$ , i.e. it is defined just as the previous double complex was, using  $\mathfrak{B}$ -sets instead of  $\mathfrak{U}$ -sets. We shall inductively construct the following elements:

$$\hat{\xi}_v = \{\hat{\xi}_{i_0 \dots i_v, \iota_0 \dots \iota_{l-v}}\} \in Z_{3v}^{v, l-v}$$

$$\tilde{\xi}_v = \{\tilde{\xi}_{i_0 \dots i_v, \iota_0 \dots \iota_{l-v}}\} \in \tilde{Z}_{3v}^{v, l-v}; \quad v = 0, \dots, l$$

$$\hat{\eta}_v = \{\hat{\eta}_{i_0 \dots i_{v-1}, \iota_0 \dots \iota_{l-v}}\} \in C_{3v}^{v-1, l-v}$$

$$\tilde{\eta}_v = \{\tilde{\eta}_{i_0 \dots i_{v-1}, \iota_0 \dots \iota_{l-v}}\} \in \tilde{C}_{3v}^{v-1, v-1}; \quad v = 1, \dots, l$$

$$\tilde{\gamma}_v = \{ \tilde{\gamma}_{i_0 \dots i_{v-1}, i_0 \dots i_{l-v-1}} \} \in \tilde{C}_{3v-3}^{v-1, l-v-1}; \quad v = 1, \dots, (l-1)$$

$$\text{and } \tilde{\gamma}_l = \{ \tilde{\gamma}_{i_0 \dots i_{l-1}} \} \in C^{l-1}(\mathfrak{B}_{3l}).$$

*The construction:*  $\hat{\xi} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbb{F})$  is given. The whole construction is done using  $\rho$  instead of  $\rho_1$  and we omit  $\rho$  to simplify the notation. We put  $\varepsilon_1(\hat{\xi}) = \hat{\xi}_0 \in Z_0^{0,l}$ . Now we apply the Smoothing Lemma and get  $\hat{\eta}_1$  such that  $\partial \hat{\eta}_1 = \hat{\xi}_0$  with  $\|\hat{\eta}_1\|_\rho \leq K \|\hat{\xi}_0\|_\rho \leq K \|\hat{\xi}\|_\rho$ . Put  $\hat{\xi}_1 = \delta \hat{\eta}_1$ . Obviously  $\|\hat{\xi}_1\|_\rho \leq K \|\hat{\eta}_1\|_\rho$ . Inductively we find  $\delta \hat{\eta}_v = \hat{\xi}_{v-1}$  and we put  $\hat{\xi}_v = \delta \hat{\eta}_v$  where  $\hat{\eta}_v$  are found from the Smoothing Lemma. Finally we get  $\hat{\xi}_l$  and we have  $\|\hat{\xi}_l\|_\rho \leq K \|\hat{\xi}\|_\rho$ . Now we define  $\tilde{\xi}_v$  and  $\tilde{\eta}_v$  as follows. Put  $\tilde{\xi}_0 = \hat{\xi}_0$  where  $\tilde{\xi}_0 \in \tilde{Z}_0^{0,l}$  is obtained by natural restriction of  $\hat{\xi}_0$ . Put  $\tilde{\eta}_v = (-1)^v \{ \tilde{\xi}_{i_0 \dots i_{v-1}, i_0 \dots i_{l-v-1}} \}$  which is well defined with respect to  $(\mathfrak{B}_{3v}, \mathfrak{B}_{3v})$  by taking natural restrictions. Put  $\tilde{\xi}_v = \delta \tilde{\eta}_v$  for  $v = 1, \dots, l$ . A computation shows that  $\tilde{\xi}_{v-1} = \partial \tilde{\eta}_v$  when  $v = 1, \dots, l$ . Notice that this is trivial when  $v = 1$ . In the following discussion each  $\hat{\eta}_v$  is restricted to  $(\mathfrak{B}_{3v}, \mathfrak{B}_{3v})$ . We have  $\partial(\tilde{\eta}_1 - \hat{\eta}_1) = 0$ . Hence we find  $\tilde{\eta}_1 - \hat{\eta}_1 = \delta \tilde{\gamma}_1$  by the Smoothing Lemma. Now we define  $\tilde{\gamma}_v$  such that  $\partial \tilde{\gamma}_v = \tilde{\eta}_v - \hat{\eta}_v - \delta \tilde{\gamma}_{v-1}$  inductively. This is possible because  $\partial(\tilde{\eta}_v - \hat{\eta}_v - \delta \tilde{\gamma}_{v-1}) = 0$ , for we have  $\partial(\tilde{\eta}_v - \hat{\eta}_v - \delta \tilde{\gamma}_{v-1}) = \tilde{\xi}_{v-1} - \hat{\xi}_{v-1} - \delta \partial \tilde{\gamma}_{v-1} = \delta \tilde{\eta}_{v-1} - \delta \hat{\eta}_{v-1} - \delta(\tilde{\eta}_{v-1} - \hat{\eta}_{v-1}) = 0$ . We get finally  $\tilde{\gamma}_{l-1} \in \tilde{C}_{3l}^{l-2,0}$  and then  $\delta \tilde{\gamma}_{l-1} \in \tilde{C}_{3l}^{l-1,0}$ . We have  $\partial(\tilde{\eta}_l - \hat{\eta}_l - \delta \tilde{\gamma}_{l-1}) = 0$ . Therefore we can put  $\tilde{\gamma}_l = \varepsilon_2(\tilde{\eta}_l - \hat{\eta}_l - \delta \tilde{\gamma}_{l-1})$ . It follows that  $\tilde{\gamma}_l \in C^{l-1}(\mathfrak{B}_{3l})$  and  $\delta \tilde{\gamma}_l = \varepsilon_2(\tilde{\xi}_l - \hat{\xi}_l)$ . We have  $\varepsilon_2(\tilde{\xi}_l) = -\hat{\xi}|_{\mathfrak{B}'}$  and for  $\varepsilon_2(\tilde{\xi}_l) = -\hat{\xi}^*$  and  $\hat{\eta} = \tilde{\eta}$  the required equation  $\hat{\xi}^* = \hat{\xi} + \delta \hat{\eta}$ . The estimates follow immediately from the construction and the Smoothing Lemma.