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Now $\omega$ takes the form

$$\omega - d\beta' = dz_1 \wedge \alpha' . \tag{6.23}$$

We distinguish the two cases $k > 1$ and $k = 1$. In the first case we get from (6.23)

$$dz_1 \wedge \delta\alpha' = 0 ,$$

which implies that $\delta\alpha' = 0$. Since $\alpha'$ is a form of type $q-1 \geqslant 1$, we can apply once again Lemma 6.8 and get $\alpha' = \delta\alpha''$. Thus $dz_1 \wedge \alpha' = d(dz_1 \wedge \alpha'')$, and we get $\omega = d(\beta' + dz_1 \wedge \alpha'')$. This proves that the cohomology under consideration is trivial for $k > 1$.

Finally, in the case $k = 1$, $\alpha'$ is a meromorphic function, independent of $z_2, \ldots, z_n$. Thus by (6.23), $\omega = d\gamma$ for some $\gamma$ if and only if in the Laurent expansion of $\alpha'$ the coefficient of ${z_1}^{-1}$ is zero. Thus the cohomology in dimension 1 is generated by ${z_1}^{-1}\, dz_1$, which completes the proof of Theorem 6.4.

## 7. Lefschetz' theorem on hyperplane sections

The Lefschetz theorem in the slightly more general setting proved by Andreotti and Frankel [1], is the following:

*Theorem 7.1.* Let $V$ be a submanifold of $\mathbf{P}^n$ of complex dimension $d$ and let $D$ be a hyperplane section of $V$ (not necessarily non-singular). Then there are natural isomorphisms

$$H^q(V, \mathbf{Z}) \simeq H^q(D, \mathbf{Z}) , \qquad (\forall q < d-1) ,$$

and a natural injection

$$H^{d-1}(V, \mathbf{Z}) \to H^{d-1}(D, \mathbf{Z}) .$$

*Proof.* $X = V - D$ is a Stein manifold, since it is imbedded as a closed submanifold of $\mathbf{C}^n$. Now one knows that

$$H^q(V, D, \mathbf{Z}) \simeq H^q_c(X, \mathbf{Z}) , \tag{7.1}$$

where the $c$ indicates cohomology with compact support. On the other hand, since $X$ is a topological manifold of dimension $2\,d$, Poincaré duality gives

$$H^q_c(X, \mathbf{Z}) \simeq H_{2d-q}(X, \mathbf{Z}) . \tag{7.2}$$

Now we shall use the following theorem:

*Theorem 7.2.* Let $X$ be a Stein manifold of dimension $d$. Then

$$H_r(X, \mathbf{Z}) = 0\,, \qquad (\forall r > d)\,. \tag{7.3}$$

Suppose this theorem is proved. Then (7.1) – (7.3) gives

$$H^q(V, D, \mathbf{Z}) = 0\,. \qquad (\forall q < d)\,. \tag{7.4}$$

Now we have the exact sequence

$$\ldots \to H^q(V, D, \mathbf{Z}) \to H^q(V, \mathbf{Z}) \to H^q(D, \mathbf{Z}) \to H^{q+1}(V, D, \mathbf{Z}) \to \ldots,$$

and using (7.4) we conclude that the mapping

$$H^q(V, \mathbf{Z}) \to H^q(D, \mathbf{Z})$$

is an isomorphism onto when $q < d-1$ and an injection when $q = d-1$.
This proves Lefschetz' theorem.

The proof of Theorem 7.2 is based on *Morse theory*. Let $X$ be a $C^\infty$-manifold with countable base. If $f$ is a real-valued $C^\infty$-function on $X$, then a point $a \in X$ is called *critical* for $f$ if $df(a) = 0$. A critical point $a$ is *non-degenerate*, if in local coordinates $f(x) - f(a) = \Sigma a_{ij}(x_i - a_i)\,(x_j - a_j) + o(|x-a|^2)$, where the symmetric matrix $(a_{ij})$ is non-singular. It is non-degenerate of index $r$ if $(a_{ij})$ has $r$ eigenvalues $< 0$. The non-degenerate critical points for $f$ are necessarily isolated. We now quote some facts from Morse theory; for proofs, see [6].

*Lemma 7.3.* Suppose that $f \in C^\infty(X)$, $f \geqslant 0$, $\alpha < \beta$, and that $X_\beta = \{x \in X; f(x) \leqslant \beta\}$ is compact.

(a) If $f$ has no critical points in $\{x \in X\colon \alpha \leqslant f(x) \leqslant \beta\}$, then $X_\alpha$ is a deformation retract of $X_\beta$, and hence

$$H_r(X_\beta, X_\alpha, \mathbf{Z}) = 0\,, \qquad (\forall r \geqq 0)\,.$$

(b) If all critical points of $f$ in $\{x \in X;\ \alpha \leqslant f(x) \leqslant \beta\}$ are non-degenerate of index $\leqslant d$, then

$$H_r(X_\beta, X_\alpha, \mathbf{Z}) = 0\,, \qquad (\forall r > d)\,.$$

In particular, if all critical points of $f$ in $X_\beta$ are non-degenerate of index $\leqslant d$, then

$$H_r(X_\beta, \mathbf{Z}) = 0\,, \qquad (\forall r > d)\,.$$

In the proof of Theorem 7.2 we shall also use the following lemma of Morse:

*Lemma 7.4.* Let $X$ be a $C^\infty$-manifold with countable base. Then every real function $g \in C^\infty(X)$ can be approximated in the topology of $C^\infty(X)$ by real functions $f \in C^\infty(X)$, whose critical points are all non-degenerate.

The topology of $C^\infty(X)$ is the topology of uniform convergence of all derivatives on compact sets. Therefore the lemma explicitly means the following:

Let $\varepsilon > 0$, an integer $r \geqslant 0$ and a compact set $K \subset X$ be given, and let $K = K_1 \cup \ldots \cup K_k$, where each $K_j$ is compact and contained in an open set $U_j$, where we have a coordinate system. Then there is a function $f$ of the prescribed type such that

$$\sup_j \ \sup_{|\alpha| \leq r} \ \sup_{x \in K_j} |D^\alpha f(x) - D^\alpha g(x)| < \varepsilon .$$

(Here $D^\alpha$ means a derivative of order $|\alpha|$ in the coordinates on $U_j$.)

To prove Lemma 7.4 we shall use a Lemma of Sard (see [8, Ch. I,§3, Th. 4]):

*Lemma 7.5.* Let $\Omega$ be an open subset of $\boldsymbol{R}^n$ and $f: \Omega \to \boldsymbol{R}^n$ a $C^1$-mapping. Let $A$ be the critical set of $f$, i.e. the set of $a \in \Omega$ where $\det(\partial f_i(a)/\partial x_j) = 0$. Then $f(A)$ has Lebesgue measure 0 in $\boldsymbol{R}^n$. In particular, $f(A)$ is nowhere dense in $\mathbf{R}^n$.

*Proof of Lemma 7.4.* Suppose first that $X$ is an open subset $\Omega$ of $\mathbf{R}^n$. If $g \in C^\infty(\Omega)$ is realvalued, consider the mapping

$$\varphi : \Omega \ni x \to (\partial g/\partial x_1, \ldots, \partial g/\partial x_n) \in \mathbf{R}^n .$$

The critical set $A$ of $\varphi$ is the set in $\Omega$ where

$$\det(\partial^2 g/\partial x_i \, \partial x_j) = 0 .$$

The lemma of Sard, applied to $\varphi$, shows that there are arbitrarily small $\varepsilon_1, \ldots, \varepsilon_n \in \mathbf{R}$ such that $(\varepsilon_1, \ldots, \varepsilon_n) \notin \varphi(A)$. Put

$$f(x) = g(x) - \varepsilon_1 x_1 - \ldots - \varepsilon_n x_n .$$

A point $x \in \Omega$ is a critical point of $f$ if and only if $\partial g/\partial x_j = \varepsilon_j$, $(j=1, \ldots, n)$.

At such points $\varphi(x) = (\varepsilon_1, \ldots, \varepsilon_n) \in \varphi(A)$ and hence $\det(\partial^2 g/\partial x_i \, \partial x_j) \neq 0$. Hence all critical points of $f$ are non-degenerate.

Since $\varepsilon_1, \ldots, \varepsilon_n$ can be chosen arbitrarily small, the lemma is proved in the case $X = \Omega$.

The general case now follows by a category argument. From the special case we conclude that we can cover $X$ by denumerably many relatively compact open subsets $U_j$ of $X$, such that $\mathcal{U}_j$ is dense in the space of real $C^\infty$-functions, where $\mathcal{U}_j$ denotes the set of real $C^\infty$-functions, whose critical points in $\overline{U}_j$ are all non-degenerate. It is also easy to see that every $\mathcal{U}_j$ is open in the space of real $C^\infty$-functions. Since this space is a real Fréchet space, we can therefore use Baire's theorem to conclude that the set of all real $C^\infty$-functions, whose critical points in $X$ are all non-degenerate, i.e. $\cap\, \mathcal{U}_j$, is dense. This proves the lemma of Morse.

*Proof of Theorem 7.2.* Let $X$ be a Stein manifold of dimension $d$, and let $K$ be a compact subset of $X$ such that

$$K = \{x \in X;\ |f(x)| \leq ||f||_K, \quad \forall f \text{ holomorphic on } X\}.$$

(Since $X$ is a Stein manifold, every compact subset of $X$ is contained in some $K$ of this kind.) Choose an open set $U$ such that $K \subset U \subset\subset X$. For every $a \in \partial U$ we can find a holomorphic function $f$ on $X$ such that $|f(x)| \geqslant 1$ in a neighbourhood of $a$ and $||f||_K < 1$. Since $\partial U$ is compact, we can therefore choose holomorphic functions $f_1, \dots, f_k$ on $X$ such that

$$\max |f_j(a)| \geq 1, \qquad (\forall a \in \partial U),$$

and

$$||f_j||_K < 1, \qquad (\forall j).$$

By replacing each $f_j$ by a sufficiently high power, we can also arrange that the function

$$p(x) = \Sigma\, |f_j(x)|^2$$

satisfies $p(x) < 1$ on $K$ and $p(x) \geqslant 1$ on $\partial U$. We can also assume that the rank of $(f_1, \dots, f_k)$ is maximal at all points of $U$.

Now $p \in C^\infty(X)$, $p \geqslant 0$, and $U_\beta = \{x \in U; p(x) \leqslant \beta\}$ is compact and contains $K$ if $\beta < 1$ is chosen so that $p(x) < \beta$ in $K$. By calculating the Levi form and using the maximality of the rank of $(f_1, \dots, f_k)$, we see that $p$ is strongly plurisubharmonic.

Because of Morse's lemma we can also assume that all critical points of $p$ in $U_\beta$ are non-degenerate. We shall prove that they are all of index $\leqslant d$.

We expand $p$ at a critical point $a \in U_\beta$ in a local coordinate system:

$$\begin{aligned} p(x) &= p(a) + 2\mathrm{Re} \sum \frac{\partial^2 p(a)}{\partial z_i\, \partial z_j}(z_i - a_i)(z_j - a_j) \\ &\quad + \sum \frac{\partial^2 p(a)}{\partial z_i\, \partial \bar{z}_j}(z_i - a_i)(\bar{z}_j - \bar{a}_j) + \dots \\ &= p(a) + \mathrm{Re}\, Q(z - a) + L(z - a) + \dots . \end{aligned}$$

Here $L(z-a)$ is the Levi form of $p$ at the point $a$. Now, since $p$ is strongly plurisubharmonic, we can choose the coordinates so that $L(z-a) = |z-a|^2$. Then we see that if $\zeta$ is an eigenvector corresponding to an eigenvalue $< 0$ of the symmetric matrix of the real quadratic form Re $Q(z) + L(z)$, then $i\zeta$ is an eigenvector corresponding to an eigenvalue $> 0$. Hence the number of negative eigenvalues is $\leqslant d$, since the real dimension of $X$ is $2d$. Thus the index of the critical point $a$ is $\leqslant d$.

Now using Lemma 7.3 (b), we see that

$$H_r(U_\beta, \mathbf{Z}) = 0\,, \qquad (\forall r > d)\,.$$

From this it follows that

$$H_r(X, \mathbf{Z}) = 0\,, \qquad (\forall r > d)\,,$$

because the singular cycles defining the homology groups $H_r(X, \mathbf{Z})$ have compact supports, and any compact subset of $X$ is contained in some compact set $K$ with a corresponding $U_\beta \supset K$.

A refinement of the above argument leads to the stronger (homotopy) statement:

*Any Stein manifold of (complex) dimension d has the same homotopy type as a CW complex of (real) dimension* $\leqslant d$. (See [6]).

Moreover, the Lefschetz theorem has an analogue in homology and in homotopy [6]. The latter, for example, asserts that, if $V$, $D$ are as in Th. 7.1, then the relative homotopy groups $\pi_q(V, D) = 0$ for $q < d$.

Th. 7.2 has been generalised in various directions. It has a relative analogue (relative to a Runge domain). Further, Th. 7.2 remains true if $X$ is any Stein space (with singularities) of complex dimension $d$, but the corresponding cohomology statement is proved only for some other coefficient groups [5, 7]. Note that in view of the use of Poincaré duality, this does not lead to a Lefschetz theorem for algebraic varieties with singularities.

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