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Now  $\omega$  takes the form

$$\omega - d\beta' = dz_1 \wedge \alpha'. \quad (6.23)$$

We distinguish the two cases  $k > 1$  and  $k = 1$ . In the first case we get from (6.23)

$$dz_1 \wedge \delta\alpha' = 0,$$

which implies that  $\delta\alpha' = 0$ . Since  $\alpha'$  is a form of type  $q - 1 \geq 1$ , we can apply once again Lemma 6.8 and get  $\alpha' = \delta\alpha''$ . Thus  $dz_1 \wedge \alpha' = d(dz_1 \wedge \alpha'')$ , and we get  $\omega = d(\beta' + dz_1 \wedge \alpha'')$ . This proves that the cohomology under consideration is trivial for  $k > 1$ .

Finally, in the case  $k = 1$ ,  $\alpha'$  is a meromorphic function, independent of  $z_2, \dots, z_n$ . Thus by (6.23),  $\omega = d\gamma$  for some  $\gamma$  if and only if in the Laurent expansion of  $\alpha'$  the coefficient of  $z_1^{-1}$  is zero. Thus the cohomology in dimension 1 is generated by  $z_1^{-1} dz_1$ , which completes the proof of Theorem 6.4.

## 7. LEFSCHETZ' THEOREM ON HYPERPLANE SECTIONS

The Lefschetz theorem in the slightly more general setting proved by Andreotti and Frankel [1], is the following:

*Theorem 7.1.* Let  $V$  be a submanifold of  $\mathbf{P}^n$  of complex dimension  $d$  and let  $D$  be a hyperplane section of  $V$  (not necessarily non-singular). Then there are natural isomorphisms

$$H^q(V, \mathbf{Z}) \simeq H^q(D, \mathbf{Z}), \quad (\forall q < d - 1),$$

and a natural injection

$$H^{d-1}(V, \mathbf{Z}) \rightarrow H^{d-1}(D, \mathbf{Z}).$$

*Proof.*  $X = V - D$  is a Stein manifold, since it is imbedded as a closed submanifold of  $\mathbf{C}^n$ . Now one knows that

$$H^q(V, D, \mathbf{Z}) \simeq H_c^q(X, \mathbf{Z}), \quad (7.1)$$

where the  $c$  indicates cohomology with compact support. On the other hand, since  $X$  is a topological manifold of dimension  $2d$ , Poincaré duality gives

$$H_c^q(X, \mathbf{Z}) \simeq H_{2d-q}(X, \mathbf{Z}). \quad (7.2)$$

Now we shall use the following theorem:

*Theorem 7.2.* Let  $X$  be a Stein manifold of dimension  $d$ . Then

$$H_r(X, \mathbf{Z}) = 0, \quad (\forall r > d). \quad (7.3)$$

Suppose this theorem is proved. Then (7.1) – (7.3) gives

$$H^q(V, D, \mathbf{Z}) = 0. \quad (\forall q < d). \quad (7.4)$$

Now we have the exact sequence

$$\dots \rightarrow H^q(V, D, \mathbf{Z}) \rightarrow H^q(V, \mathbf{Z}) \rightarrow H^q(D, \mathbf{Z}) \rightarrow H^{q+1}(V, D, \mathbf{Z}) \rightarrow \dots,$$

and using (7.4) we conclude that the mapping

$$H^q(V, \mathbf{Z}) \rightarrow H^q(D, \mathbf{Z})$$

is an isomorphism onto when  $q < d-1$  and an injection when  $q = d-1$ .

This proves Lefschetz' theorem.

The proof of Theorem 7.2 is based on *Morse theory*. Let  $X$  be a  $C^\infty$ -manifold with countable base. If  $f$  is a real-valued  $C^\infty$ -function on  $X$ , then a point  $a \in X$  is called *critical* for  $f$  if  $df(a) = 0$ . A critical point  $a$  is *non-degenerate*, if in local coordinates  $f(x) - f(a) = \sum a_{ij} (x_i - a_i)(x_j - a_j) + o(|x - a|^2)$ , where the symmetric matrix  $(a_{ij})$  is non-singular. It is non-degenerate of index  $r$  if  $(a_{ij})$  has  $r$  eigenvalues  $< 0$ . The non-degenerate critical points for  $f$  are necessarily isolated. We now quote some facts from Morse theory; for proofs, see [6].

*Lemma 7.3.* Suppose that  $f \in C^\infty(X)$ ,  $f \geq 0$ ,  $\alpha < \beta$ , and that  $X_\beta = \{x \in X; f(x) \leq \beta\}$  is compact.

(a) If  $f$  has no critical points in  $\{x \in X; \alpha \leq f(x) \leq \beta\}$ , then  $X_\alpha$  is a deformation retract of  $X_\beta$ , and hence

$$H_r(X_\beta, X_\alpha, \mathbf{Z}) = 0, \quad (\forall r \geq 0).$$

(b) If all critical points of  $f$  in  $\{x \in X; \alpha \leq f(x) \leq \beta\}$  are non-degenerate of index  $\leq d$ , then

$$H_r(X_\beta, X_\alpha, \mathbf{Z}) = 0, \quad (\forall r > d).$$

In particular, if all critical points of  $f$  in  $X_\beta$  are non-degenerate of index  $\leq d$ , then

$$H_r(X_\beta, \mathbf{Z}) = 0, \quad (\forall r > d).$$

In the proof of Theorem 7.2 we shall also use the following lemma of Morse:

*Lemma 7.4.* Let  $X$  be a  $C^\infty$ -manifold with countable base. Then every real function  $g \in C^\infty(X)$  can be approximated in the topology of  $C^\infty(X)$  by real functions  $f \in C^\infty(X)$ , whose critical points are all non-degenerate.

The topology of  $C^\infty(X)$  is the topology of uniform convergence of all derivatives on compact sets. Therefore the lemma explicitly means the following:

Let  $\varepsilon > 0$ , an integer  $r \geq 0$  and a compact set  $K \subset X$  be given, and let  $K = K_1 \cup \dots \cup K_k$ , where each  $K_j$  is compact and contained in an open set  $U_j$ , where we have a coordinate system. Then there is a function  $f$  of the prescribed type such that

$$\sup_j \sup_{|\alpha| \leq r} \sup_{x \in K_j} |D^\alpha f(x) - D^\alpha g(x)| < \varepsilon.$$

(Here  $D^\alpha$  means a derivative of order  $|\alpha|$  in the coordinates on  $U_j$ .)

To prove Lemma 7.4 we shall use a Lemma of Sard (see [8, Ch. I, §3, Th. 4]):

*Lemma 7.5.* Let  $\Omega$  be an open subset of  $\mathbf{R}^n$  and  $f: \Omega \rightarrow \mathbf{R}^n$  a  $C^1$ -mapping. Let  $A$  be the critical set of  $f$ , i.e. the set of  $a \in \Omega$  where  $\det(\partial f_i(a)/\partial x_j) = 0$ . Then  $f(A)$  has Lebesgue measure 0 in  $\mathbf{R}^n$ . In particular,  $f(A)$  is nowhere dense in  $\mathbf{R}^n$ .

*Proof of Lemma 7.4.* Suppose first that  $X$  is an open subset  $\Omega$  of  $\mathbf{R}^n$ . If  $g \in C^\infty(\Omega)$  is realvalued, consider the mapping

$$\varphi: \Omega \ni x \rightarrow (\partial g/\partial x_1, \dots, \partial g/\partial x_n) \in \mathbf{R}^n.$$

The critical set  $A$  of  $\varphi$  is the set in  $\Omega$  where

$$\det(\partial^2 g/\partial x_i \partial x_j) = 0.$$

The lemma of Sard, applied to  $\varphi$ , shows that there are arbitrarily small  $\varepsilon_1, \dots, \varepsilon_n \in \mathbf{R}$  such that  $(\varepsilon_1, \dots, \varepsilon_n) \notin \varphi(A)$ . Put

$$f(x) = g(x) - \varepsilon_1 x_1 - \dots - \varepsilon_n x_n.$$

A point  $x \in \Omega$  is a critical point of  $f$  if and only if  $\partial g/\partial x_j = \varepsilon_j$ , ( $j=1, \dots, n$ ).

At such points  $\varphi(x) = (\varepsilon_1, \dots, \varepsilon_n) \in \varphi(A)$  and hence  $\det(\partial^2 g/\partial x_i \partial x_j) \neq 0$ . Hence all critical points of  $f$  are non-degenerate.

Since  $\varepsilon_1, \dots, \varepsilon_n$  can be chosen arbitrarily small, the lemma is proved in the case  $X = \Omega$ .

The general case now follows by a category argument. From the special case we conclude that we can cover  $X$  by denumerably many relatively

compact open subsets  $U_j$  of  $X$ , such that  $\mathcal{U}_j$  is dense in the space of real  $C^\infty$ -functions, where  $\mathcal{U}_j$  denotes the set of real  $C^\infty$ -functions, whose critical points in  $\bar{U}_j$  are all non-degenerate. It is also easy to see that every  $\mathcal{U}_j$  is open in the space of real  $C^\infty$ -functions. Since this space is a real Fréchet space, we can therefore use Baire's theorem to conclude that the set of all real  $C^\infty$ -functions, whose critical points in  $X$  are all non-degenerate, i.e.  $\cap \mathcal{U}_j$ , is dense. This proves the lemma of Morse.

*Proof of Theorem 7.2.* Let  $X$  be a Stein manifold of dimension  $d$ , and let  $K$  be a compact subset of  $X$  such that

$$K = \{x \in X; |f(x)| \leq \|f\|_K, \quad \forall f \text{ holomorphic on } X\}.$$

(Since  $X$  is a Stein manifold, every compact subset of  $X$  is contained in some  $K$  of this kind.) Choose an open set  $U$  such that  $K \subset U \subset \subset X$ . For every  $a \in \partial U$  we can find a holomorphic function  $f$  on  $X$  such that  $|f(x)| \geq 1$  in a neighbourhood of  $a$  and  $\|f\|_K < 1$ . Since  $\partial U$  is compact, we can therefore choose holomorphic functions  $f_1, \dots, f_k$  on  $X$  such that

$$\max |f_j(a)| \geq 1, \quad (\forall a \in \partial U),$$

and

$$\|f_j\|_K < 1, \quad (\forall j).$$

By replacing each  $f_j$  by a sufficiently high power, we can also arrange that the function

$$p(x) = \sum |f_j(x)|^2$$

satisfies  $p(x) < 1$  on  $K$  and  $p(x) \geq 1$  on  $\partial U$ . We can also assume that the rank of  $(f_1, \dots, f_k)$  is maximal at all points of  $U$ .

Now  $p \in C^\infty(X)$ ,  $p \geq 0$ , and  $U_\beta = \{x \in U; p(x) \leq \beta\}$  is compact and contains  $K$  if  $\beta < 1$  is chosen so that  $p(x) < \beta$  in  $K$ . By calculating the Levi form and using the maximality of the rank of  $(f_1, \dots, f_k)$ , we see that  $p$  is strongly plurisubharmonic.

Because of Morse's lemma we can also assume that all critical points of  $p$  in  $U_\beta$  are non-degenerate. We shall prove that they are all of index  $\leq d$ .

We expand  $p$  at a critical point  $a \in U_\beta$  in a local coordinate system:

$$\begin{aligned} p(x) &= p(a) + 2\operatorname{Re} \sum \frac{\partial^2 p(a)}{\partial z_i \partial z_j} (z_i - a_i)(z_j - a_j) \\ &\quad + \sum \frac{\partial^2 p(a)}{\partial z_i \partial \bar{z}_j} (z_i - a_i)(\bar{z}_j - \bar{a}_j) + \dots \\ &= p(a) + \operatorname{Re} Q(z - a) + L(z - a) + \dots \end{aligned}$$

Here  $L(z-a)$  is the Levi form of  $p$  at the point  $a$ . Now, since  $p$  is strongly plurisubharmonic, we can choose the coordinates so that  $L(z-a) = |z-a|^2$ . Then we see that if  $\zeta$  is an eigenvector corresponding to an eigenvalue  $< 0$  of the symmetric matrix of the real quadratic form  $\operatorname{Re} Q(z) + L(z)$ , then  $i\zeta$  is an eigenvector corresponding to an eigenvalue  $> 0$ . Hence the number of negative eigenvalues is  $\leq d$ , since the real dimension of  $X$  is  $2d$ . Thus the index of the critical point  $a$  is  $\leq d$ .

Now using Lemma 7.3 (b), we see that

$$H_r(U_\beta, \mathbf{Z}) = 0, \quad (\forall r > d).$$

From this it follows that

$$H_r(X, \mathbf{Z}) = 0, \quad (\forall r > d),$$

because the singular cycles defining the homology groups  $H_r(X, \mathbf{Z})$  have compact supports, and any compact subset of  $X$  is contained in some compact set  $K$  with a corresponding  $U_\beta \supset K$ .

A refinement of the above argument leads to the stronger (homotopy) statement:

*Any Stein manifold of (complex) dimension  $d$  has the same homotopy type as a CW complex of (real) dimension  $\leq d$ . (See [6]).*

Moreover, the Lefschetz theorem has an analogue in homology and in homotopy [6]. The latter, for example, asserts that, if  $V, D$  are as in Th. 7.1, then the relative homotopy groups  $\pi_q(V, D) = 0$  for  $q < d$ .

Th. 7.2 has been generalised in various directions. It has a relative analogue (relative to a Runge domain). Further, Th. 7.2 remains true if  $X$  is any Stein space (with singularities) of complex dimension  $d$ , but the corresponding cohomology statement is proved only for some other coefficient groups [5, 7]. Note that in view of the use of Poincaré duality, this does not lead to a Lefschetz theorem for algebraic varieties with singularities.

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