Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 14 (1968)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COMPACT ANALYTICAL VARIETIES

Autor: Narasimhan, Raghavan

Kapitel:4. Line bundle associated to a divisor **DOI:** https://doi.org/10.5169/seals-42344

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 30.10.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

easily seen that the sequence

$$0 \to H \to \mathcal{O}(X) \to \mathbf{C}_a \oplus \mathbf{C}_b \to 0$$

is exact. From this we conclude as above that there exists an integer s(a, b) such that the sequence

$$\Gamma\left(X, \underline{E}^{s(a,b)}\right) \to E_a^{s(a,b)} \oplus E_b^{s(a,b)} \to 0$$

is exact. Therefore there exists a neighbourhood W of (a,b) in $X \times X$ such that if $(a',b') \in W$, then the sections of $\Gamma(X,\underline{E}^{s(a,b)})$ separate a' and b'; that is, if $\sigma_0, ..., \sigma_k$ is a basis of $\Gamma(X,\underline{E}^{s(a,b)})$, then $(\sigma_0(a'), ..., \sigma_k(a'))$ and $(\sigma_0(b'), ..., \sigma_k(b'))$ are different points in \mathbf{P}^k . Let l be a positive integer, let $(a',b') \in W$, and let σ be a section of $\Gamma(X,\underline{E}^{s(a,b)})$ such that $\sigma(a') \neq 0$ and $\sigma(b') \neq 0$. Then $\sigma^{l-1} \otimes \sigma_0, ..., \sigma^{l-1} \otimes \sigma_k$ are sections of $\Gamma(X,\underline{E}^{ls(a,b)})$ such that $((\sigma^{l-1} \otimes \sigma_0)(a'), ..., (\sigma^{l-1} \otimes \sigma_k)(a'))$ and $((\sigma^{l-1} \otimes \sigma_0)(b'), ..., (\sigma^{l-1} \otimes \sigma_k)(b'))$ are different points in \mathbf{P}^k .

This means that for every positive integer l the sections of $\Gamma(X, E^{ls(a,b)})$ separate all point pairs in W. Thus, covering $X \times X - U$ by finitely many such neighbourhoods and taking s'' to be the product of the corresponding s(a, b), we find that the sections of $\Gamma(X, \underline{E}^{s''})$ separate all point pairs in $X \times X - U$.

Let $\alpha = s's''$ and let $\sigma_0, ..., \sigma_d$ be a basis of $\Gamma(X, \underline{E}^{\alpha})$. We claim that the mapping f from X into \mathbf{P}^d defined by $f(x) = (\sigma_0(x), ..., \sigma_d(x))$ is a biholomorphic imbedding of X into \mathbf{P}^d . That this mapping is regular follows from the fact that α is a multiple of s'. What remains to be proved is that the mapping is injective.

Suppose $a, b \in X$, $a \neq b$. If $(a, b) \in U$, then $a, b \in U_i$ for some i, and since α is a multiple of s', we have $f(a) \neq f(b)$. If $(a, b) \in X \times X - U$, then $f(a) \neq f(b)$ since α is a multiple of s''. This proves the theorem.

4. Line bundle associated to a divisor

Let X be a complex manifold and D an analytic subset of X of pure codimension 1 at every point. Such a set D is called a *divisor* of X. We shall construct a line bundle F on X, associated to D.

To do this, we observe that every point of X has a neighbourhood U in which there is a holomorphic function s such that $U \cap D = \{x \in U; s(x) = 0\}$, and s generates, at every point of U, the ideal of germs of holomorphic functions vanishing on D. Thus we get a covering of X by open sets U_j and

corresponding holomorphic functions s_j . The functions $g_{ij} = s_i/s_j$ are then holomorphic and $\neq 0$ on $U_i \cap U_j$ and $g_{ij} g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. The functions g_{ij} therefore define a line bundle F on X with transition functions g_{ij} (see sect. 1). This bundle F is determined by D uniquely up to isomorphism.

If $f \in \Gamma(X, F)$, then the isomorphism $F \mid U_j \simeq U_j \times \mathbb{C}$ gives a holomorphic function f_j on U_j corresponding to f. The functions f_j are related by $f_i = g_{ij} f_j$ on $U_i \cap U_j$. Conversely, if f_j are holomorphic functions on U_j , satisfying this condition, then there is a section f of F on X, which corresponds to f_j on U_j . In particular, the s_j define a section s_D of F on X, and we have $D = \{x \in X; s_D(x) = 0\}$.

Example. Let $X = \mathbf{P}^n$, and let H be the hyperplane defined in the homogeneous coordinates $z_0, ..., z_n$ by $z_0 = 0$. Then the process above associates to H a line bundle F on \mathbf{P}^n . As defining functions we can use $s_j(z_0, ..., z_n) = z_0/z_j$ on the set U_j where $z_j \neq 0$, (j=0, ..., n). We shall prove that F is positive.

Each homogeneous coordinate z_k defines a section $s^{(k)}$ of F, which on each U_j corresponds to the holomorphic function z_k/z_j , for the transition functions are $g_{ij} = s_i/s_j = z_j/z_i$ and we have $z_k/z_i = (z_k/z_j) g_{ij}$. Now any section of F can be regarded as a holomorphic function on $E = F^*$, which is linear on the fibres of E. In particular, $s^{(0)}$, ..., $s^{(k)}$ give a holomorphic mapping $\varphi \colon E \to \mathbb{C}^{n+1}$. It is clear that the zero section in E is equal to $\varphi^{-1}(0)$. It is seen by direct verification that φ maps E onto \mathbb{C}^{n+1} and $E - \varphi^{-1}(0)$ biholomorphically onto $\mathbb{C}^{n+1} - \{0\}$. Hence E is negative and F is positive (see sect. 1).

If V is a submanifold of \mathbf{P}^n , then the restriction of F to V is a positive line bundle associated to the hyperplane section $D = V \cap H$. In fact, the dual of the restriction is the restriction $E \mid V$ of E to V, and we can use the restriction of φ to $E \mid V$ as "blowing down mapping".

Let again X be a complex manifold, D a divisor of X, and F the line bundle on X, associated to D. What are the sections of F^k ?

If $U \in \Gamma(X, F^k)$, then s is represented in local coordinates on U_j by a holomorphic function f_j . The f_j are connected by $f_i = g_{ij}^k f_j$ on $U_i \cap U_j$, because the functions g_{ij}^k are transition functions for F^k . Now $s_i^k = g_{ij}^k s_j^k$ on $U_i \cap U_j$, the s_i being local equations for the set D as above, and thus $f_i/s_i^k = f_j/s_j^k$ on $U_i \cap U_j$. Hence there exists a meromorphic function f on X such that $f_i = s_i^k f$ on U_i .

This means that f is meromorphic with poles only on D and of order $\leq k$. Conversely, if f is such a meromorphic function, then $f_j = s_j^k f$ are holomorphic on U_j and satisfy $f_i = g_{ij}^k f_j$ on $U_i \cap U_j$. Therefore they give a section s of F^k . This correspondence is obtained simply by associating to the section u of F^k , the meromorphic function $u \otimes s_D^{-k}$.

Let us consider again the space \mathbf{P}^n and the bundle F associated to a hyperplane section. Let $(z_0,...,z_n)$ denote homogeneous coordinates for \mathbf{P}^n . If $u \in \Gamma(\mathbf{P}^n, F^k)$, u defines, for $z \in \mathbf{P}^n$, an element of $F_z = (E_z^*)^k$, E being the dual bundle to F, hence a map of E_z into \mathbf{C} which is homogeneous of degree k. Thus, u defines a map \hat{u} of $E \to \mathbf{C}$, homogeneous of degree k on each fibre. If φ denotes the map of E into \mathbf{C}^{n+1} defined above, $\hat{u}: E \to \mathbf{C}$ is holomorphic, and vanishes on $\varphi^{-1}(0)$, and so defines a holomorphic function v on \mathbf{C}^{n+1} which is homogeneous of degree k (v is holomorphic also at 0 since a continuous function holomorphic outside a point in \mathbf{C}^{n+1} , $n \geqslant 1$, is holomorphic also at this point). The Taylor expansion of v about 0 shows that v is a homogeneous polynomial of degree k. Thus, any $u \in \Gamma(\mathbf{P}^n, F^k)$ can be identified with a homogeneous polynomial of degree k in the homogeneous coordinates $(z_0, ..., z_n)$ [i.e. the sections $s^{(0)}, ..., s^{(n)}$ of F defined above].

As an application of the vanishing theorem of Kodaira, we now prove the following result due to Chow (cf. [3], p. 170).

Theorem 4.1. Let A be a subvariety of \mathbf{P}^n . Then there exist homogeneous polynomials $f_1, ..., f_k$ such that $A = \{ a \in \mathbf{P}_n; f_1(a) = ... = f_k(a) = 0 \}$.

Proof. We first prove that if $b \notin A$, then there exists a homogeneous polynomial f vanishing on A with $f(b) \neq 0$. Let S be the sheaf of germs of holomorphic functions vanishing on A and let I be the sheaf of germs of holomorphic functions vanishing at b. Let F be the line bundle associated to a hyperplane section of A. Then F is positive. We get an exact sequence

$$0 \to I \otimes S \otimes F^m \to S \otimes F^m \to S_b \otimes F_b^m \to 0 \; .$$

By the vanishing theorem of Kodaira, part of the corresponding cohomology sequence will be

$$H^{o}(\mathbf{P}^{n}, S \otimes F^{m}) \to H^{o}(\mathbf{P}^{n}, S_{b} \otimes F_{b}^{m}) \to 0$$
,

if m is sufficiently large. Thus there exists $f \in H^0(P^n, S \otimes F^m)$ which is not zero at b. Since $S \subset \mathcal{O}$, we may look upon $H^0(S \otimes F^m)$ as a subspace of $H^0(F^m)$. It is then the subspace of those sections of $H^0(F^m)$ which vanish

on A. Since $f \in H^0(\mathbf{P}^n, F^m)$, this gives the desired homogeneous polynomial.

To prove the theorem, it now suffices to consider all homogeneous polynomials which vanish on A without being identically zero and apply the Hilbert basis theorem.

5. MEROMORPHIC FORMS

Let X be a complex manifold. A holomorphic differential form is a form which in local coordinates can be written as a finite sum

$$\omega = \sum a_{i_1 \dots i_k} dz_{i_1} \wedge \dots \wedge dz_{i_k}$$
 (5.1)

with holomorphic coefficients $a_{i_1...i_k}$.

A form is called meromorphic if it has locally the form (5.1) with coefficients that are meromorphic functions. Every meromorphic function can be written locally as $f\omega$ where f is a meromorphic function and ω a holomorphic form. The exterior differentiation d, satisfying $d^2 = 0$, extends naturally to meromorphic forms.

Let D be a divisor of X and let $\Omega^p(k, D) = \Omega^p(X, k, D)$ be the sheaf of germs of meromorphic p-forms on X with poles only on D and of order $\leq k$, and let $\Omega^p = \Omega^p(X)$ be the sheaf of germs of holomorphic p-forms on X.

Lemma 5.1. There is a natural isomorphism

$$\Omega^p(k,D) \simeq \Omega^p \otimes F^k$$
.

Proof. A germ in $\Omega^p(k, D)$ at $a \in X$ is represented by a form $f\omega$, where f is a meromorphic function in a neighbourhood U of a, with poles only on D and of order $\leq k$, and ω is a holomorphic form on U. Now to f corresponds biuniquely a section $s \in \Gamma(U, F^k)$ (see Sect. 4), which gives a germ $s_a \in \underline{F}_a^k$. Also ω defines a germ $\omega_a \in \Omega_a^p$.

The desired mapping $\Omega^p(K, D) \to \Omega^p \otimes F^k$ is now uniquely defined by

$$f\omega \to \omega_a \otimes s_a$$
.

To see that it is an isomorphism, it is sufficient to observe that the inverse mapping of $\Omega^p \otimes \underline{F}^k$ into $\Omega^p(k, D)$ is induced by the bilinear mapping $\Omega^p \oplus \underline{F}^k \to \Omega^p(k, D)$, which is given by

$$(\omega_a, s_a) \to (f\omega)_a$$
, $(a \in X)$.