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3. An imbedding theorem

Lemma 3.1. If X is a compact complex manifold and S a coherent analytic sheaf over X, then $\Gamma(X, S)$ is a finite dimensional vector space (cf. remark concerning Theorem 2.2).

We will now prove an imbedding theorem (cf. [2], p. 343).

Theorem 3.2. If the complex manifold X is compact, connected, and carries a positive (negative) line bundle, then X can be imbedded biholomorphically in a complex projective space $\mathbf{P}^N$.

Proof: Suppose F is a line bundle on a compact complex manifold X with the property that for every $a \in X$ there exists a section $\sigma \in \Gamma(X, \underline{F})$ with $\sigma(a) \neq 0$. Then F defines a holomorphic mapping of X into a projective space $\mathbf{P}^k$ in the following way:

Since X is compact, $\Gamma(X, \underline{F})$ is finite-dimensional according to Lemma 3.1.

Let $\sigma_0, \ldots, \sigma_k$ be a basis of $\Gamma(X, \underline{F})$. Then the σ_j have no common zeros.

Since F is locally isomorphic to the product of an open subset of X and $\mathbf{C}$, the σ_j are locally given by holomorphic functions without common zeros.

We map X into $\mathbf{P}^k$ by $x \to (\sigma_0(x), \ldots, \sigma_k(x))$. The point in the projective space is independent of the isomorphism we are using, for if we use another isomorphism we get a point $(g(x)\,\sigma_0(x), \ldots, g(x)\,\sigma_k(x))$, where $g(x) \neq 0$ (cf. (1.1)).

We are now going to show that if F is positive, then there exists an integer γ such that the sections of $\Gamma(X, \underline{F}^\gamma)$ have no common zeros and such that the corresponding mapping is an imbedding.

For $a \in X$, let I be the sheaf of germs of holomorphic functions vanishing at a. Since I is coherent, we can apply the vanishing theorem of Kodaira. We conclude that there exists an integer $k(a)$ such that $H^1(X, I \otimes \underline{F}^{k \geqq k(a)}) = 0$

Since $\mathcal{O}_a / I_a \approx \mathbf{C}$, we have the following exact sequence

$$0 \to I \to \mathcal{O}(X) \to \mathbf{C}_a \to 0\,,$$

where $\mathbf{C}_a$ is a sheaf with stalk $\mathbf{C}$ at a and zero outside. From this it follows that the sequence

$$0 \to I \otimes \underline{F}^{k(a)} \to \underline{F}^{k(a)} \to \mathbf{C}_a \otimes \underline{F}^{k(a)} \to 0$$

is exact. We have $\mathbf{C}_a \otimes \underline{F}^{k(a)} \approx \tilde{F}_a^{k(a)}$, where $\tilde{F}_a^{k(a)}$ has stalk $F_a^{k(a)}$ at a and zero outside. Using the fact that $H^1(X, I \otimes \underline{F}^{k(a)}) = 0$, the exact cohomology sequence associated to the above sequence of sheaves gives us an exact sequence

$$\Gamma(X, \underline{F}^{k(a)}) \to \Gamma(X, \tilde{F}_a^{k(a)}) \to 0 .$$

This implies that given $e \in F_a^{k(a)}$ there exists $\sigma \in \Gamma(X, \underline{F}^{k(a)})$ such that $\sigma(a) = e$. Thus, for every $a \in X$ we can find an integer $k(a)$ and a neighbourhood V_a of a such that $\Gamma(X, \underline{F}^{k(a)})$ has a section not vanishing on V_a. Since X is compact, there are finitely many such neighbourhoods V_i $(i=1, ..., p)$ with corresponding sections of F^{k_i} such that $X = \bigcup_{i=1}^{p} V_i$. Letting $k = k_1 \cdot k_2 \cdot ... \cdot k_p$, we get p elements of $\Gamma(X, \underline{F}^k)$ without common zeros, for if $\sigma \in \Gamma(X, \underline{F})$ and $\sigma(x) \neq 0$, then $\sigma' = \underbrace{\sigma \otimes ... \otimes \sigma}_{l\text{ - times}} \in \Gamma(X, \underline{F}^l)$ and $\sigma'(x) \neq 0$.

Let $\underline{E} = \underline{F}^k$. Now, for $a \in X$, let $G = q_a^2$, where q_a is the ideal of germs of holomorphic functions vanishing at a. Using the above argument with $\underline{E}$ and G instead of $\underline{F}$ and I, we see that there exists an integer $s(a)$ such that the restriction mapping

$$\Gamma(X, \underline{E}^{s(a)}) \to \{ \mathcal{O}_a / q_a{}^2 \} \otimes \underline{E}_a{}^{s(a)}$$

is surjective. Since the residue classes in $\mathcal{O}_a / q_a{}^2$ are sets of germs f of holomorphic functions at a with fixed values of $f(a)$ and $df(a)$, this implies that we can find a neighbourhood U_a of a and sections $\sigma_1, ..., \sigma_t \in \Gamma(X, \underline{E}^{s(a)})$ which are nowhere zero in U_a such that the mapping given by $\sigma_1, ..., \sigma_t$ is regular and injective in U_a. We observe that for every positive integer I we can find sections $\sigma_1{}^{(1)}, ..., \sigma_t{}^{(1)} \in \Gamma(X, E^{ls(a)})$ which have the same properties in U_a. In fact, if σ is a section of $\underline{E}^{s(a)}$ which has no zeros on a set $M \subset X$, we set

$$\sigma' = \sigma \otimes ... \otimes \sigma, (l-1) \text{ times}.$$

Then $\sigma' \otimes \sigma_1, ..., \sigma' \otimes \sigma_t$ are sections of $\underline{E}^{ls(a)}$, and define the same mapping (at least on M) as $\sigma_1, ..., \sigma_t$.

We can cover X by finitely many such neighbourhoods $U_1, ..., U_r$. If $s' = s_1 \cdot ... \cdot s_r$, then there are elements of $\Gamma(X, \underline{E}^{s'})$ which give a regular, injective mapping in each U_i $(1 \leqslant i \leqslant r)$.

We are now going to show that we can separate points in X by sections of a suitable $\underline{E}^{\alpha}$. Let $U = \bigcup_{i=1}^{r} (U_i \times U_i)$. For $(a, b) \in X \times X - U$, let H be the sheaf of germs of holomorphic functions vanishing at a and b. It is easily seen that the sequence

$$0 \to H \to \mathcal{O}(X) \to \mathbf{C}_a \oplus \mathbf{C}_b \to 0$$

is exact. From this we conclude as above that there exists an integer $s\,(a, b)$ such that the sequence

$$\Gamma\,(X, \underline{E}^{s(a,b)}) \to E_a{}^{s(a,b)} \oplus E_b{}^{s(a,b)} \to 0$$

is exact. Therefore there exists a neighbourhood W of (a, b) in $X \times X$ such that if $(a', b') \in W$, then the sections of $\Gamma\,(X, \underline{E}^{s(a,b)})$ separate a' and b'; that is, if $\sigma_0, \ldots, \sigma_k$ is a basis of $\Gamma\,(X, \underline{E}^{s(a,b)})$, then $(\sigma_0(a'), \ldots, \sigma_k(a'))$ and $(\sigma_0(b'), \ldots, \sigma_k(b'))$ are different points in $\mathbf{P}^k$. Let l be a positive integer, let $(a', b') \in W$, and let σ be a section of $\Gamma\,(X, \underline{E}^{s(a,b)})$ such that $\sigma\,(a') \neq 0$ and $\sigma\,(b') \neq 0$. Then $\sigma^{l-1} \otimes \sigma_0, \ldots, \sigma^{l-1} \otimes \sigma_k$ are sections of $\Gamma\,(X, \underline{E}^{l\,s(a,b)})$ such that $((\sigma^{l-1} \otimes \sigma_0)\,(a'), \ldots, (\sigma^{l-1} \otimes \sigma_k)\,(a'))$ and $((\sigma^{l-1} \otimes \sigma_0)\,(b'), \ldots, (\sigma^{l-1} \otimes \sigma_k)\,(b'))$ are different points in $\mathbf{P}^k$.

This means that for every positive integer l the sections of $\Gamma\,(X, E^{ls(a,b)})$ separate all point pairs in W. Thus, covering $X \times X - U$ by finitely many such neighbourhoods and taking s'' to be the product of the corresponding $s\,(a, b)$, we find that the sections of $\Gamma\,(X, \underline{E}^{s''})$ separate all point pairs in $X \times X - U$.

Let $\alpha = s's''$ and let $\sigma_0, \ldots, \sigma_d$ be a basis of $\Gamma\,(X, \underline{E}^{\alpha})$. We claim that the mapping f from X into $\mathbf{P}^d$ defined by $f\,(x) = (\sigma_0(x), \ldots, \sigma_d(x))$ is a biholomorphic imbedding of X into $\mathbf{P}^d$. That this mapping is regular follows from the fact that α is a multiple of s'. What remains to be proved is that the mapping is injective.

Suppose $a, b \in X$, $a \neq b$. If $(a, b) \in U$, then $a, b \in U_i$ for some i, and since α is a multiple of s', we have $f\,(a) \neq f\,(b)$. If $(a, b) \in X \times X - U$, then $f\,(a) \neq f\,(b)$ since α is a multiple of s''. This proves the theorem.

4. Line bundle associated to a divisor

Let X be a complex manifold and D an analytic subset of X of pure codimension 1 at every point. Such a set D is called a *divisor* of X. We shall construct a line bundle F on X, associated to D.

To do this, we observe that every point of X has a neighbourhood U in which there is a holomorphic function s such that $U \cap D = \{\, x \in U; s(x) = 0 \,\}$, and s generates, at every point of U, the ideal of germs of holomorphic functions vanishing on D. Thus we get a covering of X by open sets U_j and