Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: COMPACT ANALYTICAL VARIETIES
Autor: Narasimhan, Raghavan
Kapitel: 2. The vanishing theorem of Kodaira
DOI: https://doi.org/10.5169/seals-42344

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 19.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

$h_i \circ h_j^{-1}$ can be written $(x, \gamma'(x, c))$ where $\gamma'(x, c) \in \mathbf{C}$. According to the last property in (iii), for fixed $x \in U_i \cap U_j$ the mapping $c \to \gamma'(x, c)$ is a $\mathbf{C}$-isomorphism of $\mathbf{C}$ onto itself. Therefore

$$\gamma'(x, c) = g_{ij}(x) \cdot c \text{ , where } g_{ij}(x) \neq 0 \text{ ,} \tag{1.1}$$

and it is easily seen that g_{ij} is holomorphic in $U_i \cap U_j$.

The functions g_{ij} obviously satisfy the cocycle conditions

$$g_{ij}\, g_{jk}\, g_{ki} = 1 \quad \text{on} \quad U_i \cap U_j \cap U_k \text{ ,} \tag{1.2}$$

$$g_{ij}\, g_{ji} = 1 \quad \text{on} \quad U_i \cap U_j \text{ .} \tag{1.3}$$

The g_{ij} are called transition functions corresponding to the line bundle F.

Conversely, it is easy to prove (cf. [4], p. 135) that given an open covering $\{ U_i \}$ and functions g_{ij} without zeros in $U_i \cap U_j$ which satisfy the cocycle conditions, we can construct a line bundle which has g_{ij} as transition functions.

Now, let F be a line bundle over a complex manifold X, and let π be the corresponding projection. We denote $\pi^{-1}(a)$ by F_a. Let F_a^* be the $\mathbf{C}$-dual of F_a. Then

$$F^* = \bigcup_{a \in X} F_a$$

is in a natural way a holomorphic line bundle over X, which is called the dual bundle of F. If F has transition functions $\{ g_{ij} \}$, then F^* has transition functions $\{ g_{ij}^{-1} \}$.

Definition 1.6. Let F be a holomorphic line bundle over a compact complex manifold. Then F is *negative* if the zero cross section $\mathfrak{o}$ of F can be blown down to a point. F is *positive* if the dual bundle is negative.

In the sequel we let $\underline{F}$ denote the sheaf of germs of analytic sections of a line bundle F.

2. The vanishing theorem of Kodaira

This is the following theorem, which is our first main result:

Theorem 2.1. Let X be a compact connected complex manifold and F a positive line bundle on X and S a coherent analytic sheaf on X. Then there exists an integer $k(S, F)$ such that for $k > k(S, F)$ we have $H^q(X, S \otimes \underline{F}^k) = 0$ $(\forall q \geqslant 1)$.

The proof uses the following finiteness theorem:

Theorem 2.2. Let V be a complex manifold, S a coherent analytic sheaf on V, and $D \subset\subset V$ a strictly pseudoconvex subdomain of V. Then the cohomology groups $H^q(D, S)$ are finite-dimensional **C**-vector spaces if $q \geq 1$.

For a proof of Theorem 2.2 see Section 4.4 of the lectures by Malgrange in these notes.

Proof of Theorem 2.1.

Let E be the dual bundle of F. By hypothesis, E is negative. Thus, by Lemma 1.4, the zero cross section of E has a strictly pseudoconvex neighbourhood D.

By definition, we have a projection $\pi: E \to X$. We will now use π to " lift " S to a coherent analytic sheaf $\tilde{S}$ on E. To do this, we first consider the sheaf of abelian groups $\pi^{-1}(S)$ which to any point a of E assigns the stalk $S_{\pi(a)}$. Since $S_{\pi(a)}$ and the ring $\mathcal{O}_a(E)$ of germs of analytic functions at a both are modules over the ring $\mathcal{O}_{\pi(a)}(X)$, we can form the tensor product $\tilde{S}_a = S_a \otimes \mathcal{O}_a(E)$ over $\mathcal{O}_{\pi(a)}(X)$. Then $\tilde{S}_a$ is a module over $\mathcal{O}_a(E)$, and this defines $\tilde{S}$. Since S is coherent, $\tilde{S}$ is also coherent (cf [3], p. 401).

From Theorem 2.2 it now follows that $H^q(D, \tilde{S})$ are finite-dimensional **C**-vector spaces for $q \geq 1$. We complete the proof of Theorem 2.1 by constructing for every N a natural injection

$$\sum_{k=0}^{N} H^q(X, S \otimes \underline{F}^k) \to H^q(D, \tilde{S}),$$

where the sum is the direct sum as vector spaces. In fact, since $\dim \sum_{k=0}^{N} H^q = \sum_{k=0}^{N} \dim H^q$, the existence of such injections would imply the existence of the desired integer $k(S, F)$.

Let a be a point of the zero cross section $\mathfrak{o}$ in the negative bundle E, and let U be a neighbourhood of a such that $E_U \approx U \times \mathbf{C}$. Identifying $a \in \mathfrak{o} \subset E$ with the point $\pi(a) \in X$, we denote by $\mathcal{O}_a(E)$ and $\mathcal{O}_a(X)$ the rings of germs of analytic functions on E at a and on X at a, respectively.

To a germ $f \in \mathcal{O}_a(E)$ corresponds a Taylor series $\sum_{\nu=0}^{\infty} f_\nu(x) z^\nu$, converging in some neighbourhood $U' \times D_r$, where $U' \subset U$ and $D_r = \{z; |z| < r\}$.

For $x \in U$, let $e'(x) \in E_x$ correspond to $(x, 1)$ under the isomorphism $E_x \approx U \times \mathbf{C}$, and let $e(x) \in F_x$ be defined by $< e(x), e'(x) > = 1$. Then

$e(x)$ is a holomorphic section of F over U, and every germ $p \in \underline{F}_a^k$ is represented by $p(x)\, e(x) \otimes e(x) \otimes \ldots \otimes e(x)$, ($k$ factors $e(x)$), where $p(x)$ is holomorphic in a neighbourhood of a. But $p(x)\, e(x) \otimes e(x) \otimes \ldots \otimes e(x) \in \underline{F}_x^k$ can be identified with the multilinear functional

$$(z_1, \ldots, z_k) \to p(x)\, z_1 \cdot \ldots \cdot z_k$$

and therefore also with the polynomial $p(x)\, z^k$.

Hence, for every N we obtain an injection

$$i_N \colon \sum_{k=0}^{N} \underline{F}_a^k \to \mathcal{O}_a(E)$$

by mapping $(p_0, p_1, \ldots, p_N) \in \sum_0^N \underline{F}_a^k$ onto the germ at a of $\sum_{k=0}^{N} f_k(x)\, z^k$, where $f_k(x)$ is holomorphic in a neighbourhood of a and $f_k(x)\, z^k$ corresponds to $p_k \in \underline{F}_a^k$ in the way described above. Further the map $q_N \colon \sum_0^\infty f_\nu(x) z^\nu \to f_k(x) z^k$ gives rise to a homomorphism $\mathcal{O}_a(E) \to \underline{F}_a^k$ such that $q_N \circ i_N = \mathrm{id}$. It is obvious that this mapping i_N is injective.

From i_N we also obtain a homomorphism

$$j_N : S \otimes_{\mathcal{O}(X)} \sum_0^N \underline{F}^k \to S \otimes_{\mathcal{O}(X)} \mathcal{O}(E) = \tilde{S},$$

and the corresponding homomorphism

$$j_N^* : H^q(X, S \otimes \sum_0^N \underline{F}^k) \to H^q(\mathfrak{d}, \tilde{S}).$$

Further, the map q_N defined above gives rise to a homomorphism

$$\tilde{S} \to S \otimes_{\mathcal{O}(X)} \sum_0^N \underline{F}^k,$$

and hence a map

$$\eta_N \colon H^q(\mathcal{O}, \tilde{S}) \to H^q(X, S \otimes \sum_0^N \underline{F}^k)$$

such that $\eta_N \circ j_N^* = \mathrm{id}$. Hence j_N^* is injective.

This mapping can be factored as follows

$$H^q(\mathcal{O}, S \otimes \sum_0^N \underline{F}^k) = \sum_0^N H^q(S \otimes \underline{F}^k) \xrightarrow{\alpha} H^q(D, \tilde{S}) \xrightarrow{\beta} H^q(\mathfrak{d}, \tilde{S}),$$

and as $\beta \circ \alpha$ is an injection, α also is an injection, which proves the theorem.