1. Preliminaries

COMPACT ANALYTICAL VARIETIES

by Raghavan NARASIMHAN

CONTENTS

INTRODUCTION

These lectures deal with the vanishing theorem of Kodaira (cf. e.g. [2], p. 344) and some of its consequences, and with Lefschetz' theorem on hyperplane sections (cf. [1]). Only complex manifolds (and not complex spaces) are considered, but most of the results in the first part could be carried over to the more general case (with similar proofs).

1. PRELIMINARIES

We first give some definitions:

Definition 1.1. Let V be a complex manifold and D a relatively compact, open subset of V. Then D is *strongly pseudoconvex* if for every $x_0 \in \partial D$ there exist a neighbourhood U of x_0 and a real-valued C^2-function φ defined in U such that

(1) $$d\varphi(x_0) \neq 0,$$

(2) $$H(\varphi)(x_0) > 0 \text{ for all } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{C}^n - \{0\}.$$

(Here $H(\varphi)$ is the complex Hessian form

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \alpha_i \bar{\alpha}_j$$

with respect to some system of local coordinates),

(3) $$D \cap U = \{ x \in U; \quad \varphi(x) < 0 \} .$$

It can be shown that strong pseudoconvexity of D is equivalent to the following property: For every $x_0 \in \partial D$ there exist a neighbourhood U of x_0 and a biholomorphic mapping f: $U \to \Omega \subset \mathbf{C}^n$ such that $f(U \cap D)$ has a strictly convex boundary (in the Euclidean sense).

Definition 1.2. Let V be a complex manifold and A a subset of V. We say that A " can be blown down to a point " if there exist an analytic space X, a point $x_0 \in X$, and a mapping f: $V \to X$ such that $f(A) = x_0$ and f: $V - A \to X - \{ x_0 \}$ is an analytic isomorphism.

To give an example of sets which can be blown down to a point, we mention the following theorem (for a proof see [2], pp. 338 and 340):

Theorem 1.3. If D is strongly pseudoconvex, then D has a maximal compact analytic subset A whose dimension at any point is > 0 and each component of A can be blown down to a point.

Lemma 1.4. If A can be blown down to a point, then A has a fundamental system of strongly pseudoconvex neighbourhoods.

Proof. Let X, x_0, and f be as in Definition 1.2. The lemma follows from the fact that the inverse image of a strongly pseudoconvex neighbourhood of x_0 is a strongly pseudoconvex neighbourhood of A.

We now introduce the concept of holomorphic line bundle.

Definition 1.5. Suppose X is a complex manifold. A holomorphic line bundle F on X is a complex manifold F together with a mapping π with the following properties:

(i) π: $F \to X$ is a holomorphic map (called projection) onto X.

(ii) For $x \in X$, $\pi^{-1}(x)$ has the structure of a one-dimensional vector space over the complex numbers.

(iii) For each $x \in X$ there exist a neighbourhood U of x and a holomorphic mapping h of $F \mid U = \pi^{-1}(U)$ onto $U \times \mathbf{C}$ such that h^{-1} is holomorphic and $h \mid \pi^{-1}(a)$ is a $\mathbf{C}$-isomorphism onto $\{ a \} \times \mathbf{C}$ for every $a \in U$.

Let $\{ U_i \}$ be an open covering of X such that for each i we have a mapping h_i of $F \mid U_i$ onto $U_i \times \mathbf{C}$ with the properties in (iii) above. If $U_i \cap U_j \neq \emptyset$, we get a mapping $h_i \circ h_j^{-1}$: $(U_i \cap U_j) \times \mathbf{C} \to (U_i \cap U_j) \times \mathbf{C}$. If $(x, c) \in (U_i \cap U_j) \times \mathbf{C}$, then the image of (x, c) under the mapping

$h_i \circ h_j^{-1}$ can be written $(x, \gamma'(x, c))$ where $\gamma'(x, c) \in \mathbf{C}$. According to the last property in (iii), for fixed $x \in U_i \cap U_j$ the mapping $c \to \gamma'(x, c)$ is a $\mathbf{C}$-isomorphism of $\mathbf{C}$ onto itself. Therefore

$$\gamma'(x, c) = g_{ij}(x) \cdot c \,, \text{ where } g_{ij}(x) \neq 0 \,, \tag{1.1}$$

and it is easily seen that g_{ij} is holomorphic in $U_i \cap U_j$.

The functions g_{ij} obviously satisfy the cocycle conditions

$$g_{ij}\, g_{jk}\, g_{ki} = 1 \quad \text{on} \quad U_i \cap U_j \cap U_k \,, \tag{1.2}$$

$$g_{ij}\, g_{ji} = 1 \quad \text{on} \quad U_i \cap U_j \,. \tag{1.3}$$

The g_{ij} are called transition functions corresponding to the line bundle F.

Conversely, it is easy to prove (cf. [4], p. 135) that given an open covering $\{ U_i \}$ and functions g_{ij} without zeros in $U_i \cap U_j$ which satisfy the cocycle conditions, we can construct a line bundle which has g_{ij} as transition functions.

Now, let F be a line bundle over a complex manifold X, and let π be the corresponding projection. We denote $\pi^{-1}(a)$ by F_a. Let F_a^* be the $\mathbf{C}$-dual of F_a. Then

$$F^* = \bigcup_{a \in X} F_a$$

is in a natural way a holomorphic line bundle over X, which is called the dual bundle of F. If F has transition functions $\{ g_{ij} \}$, then F^* has transition functions $\{ g_{ij}^{-1} \}$.

Definition 1.6. Let F be a holomorphic line bundle over a compact complex manifold. Then F is *negative* if the zero cross section $\mathfrak{o}$ of F can be blown down to a point. F is *positive* if the dual bundle is negative.

In the sequel we let $\underline{F}$ denote the sheaf of germs of analytic sections of a line bundle F.

2. The vanishing theorem of Kodaira

This is the following theorem, which is our first main result:

Theorem 2.1. Let X be a compact connected complex manifold and F a positive line bundle on X and S a coherent analytic sheaf on X. Then there exists an integer $k(S, F)$ such that for $k > k(S, F)$ we have $H^q(X, S \otimes \underline{F}^k) = 0\ (\forall q \geqslant 1)$.

The proof uses the following finiteness theorem: