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$$\mathcal{O}(\Delta \times V) \xrightarrow{\phi} \mathcal{H}(\Delta, B(K))$$

$$\pi \downarrow \qquad \qquad \downarrow \tilde{\pi}$$

$$\mathcal{O}(S \times V) \xrightarrow{\phi} \mathcal{H}(S, B(K))$$

is commutative;  $\phi$  is evidently an  $\mathcal{O}_S$ -algebra homomorphism.

## § 2. The flatness and privilege theorem

### Notation

Let  $S$  be an analytic space,  $U$  an open set in  $\mathbb{C}^n$ , and  $\pi : S \times U \rightarrow S$  the first projection.

If  $\mathcal{F}$  is an  $\mathcal{O}_{S \times U}$  module, then for every  $s \in S$  we denote by  $\mathcal{F}(s)$  the  $\mathcal{O}_U$ -module  $i_s^* \mathcal{F}$ , where  $i_s$  is the injective morphism  $x \rightarrow (s, x)$  from  $U$  into  $S \times U$ . If  $x \in U$

$$(\mathcal{F}(s))_x \simeq \mathcal{F}_{(s, x)} / m_s \cdot \mathcal{F}_{(s, x)} \simeq \mathcal{F}_{(s, x)} \otimes_{\mathcal{O}_{S, s}} \mathbb{C}_s.$$

*Theorem 1:* Let  $\mathcal{E}$  be a coherent and  $S$ -flat  $\mathcal{O}_{S \times U}$ -module, and  $K$  a poly-cylinder in  $U$ .

(a) When  $K$  is privileged for  $\mathcal{E}(s_0)$ ,  $s_0$  has a neighbourhood  $V$  such that  $K$  is  $\mathcal{E}(s)$ -privileged for each  $s \in V$ . In other words: the set  $S' = \{s \in S \mid K \text{ is } \mathcal{E}(s)\text{-privileged}\}$  is open in  $S$ .

(b) It is possible to define a Banach vector bundle over  $S'$  whose fibre at any  $s \in S'$  is  $B(K, \mathcal{E}(s))$ .

To prove the theorem we need:

*Lemma 1:* Under the conditions of the theorem, we can, for every  $s \in S$ , find a neighbourhood  $W$  of  $\{s\} \times K$  and a free resolution of finite length

$$0 \rightarrow \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathcal{L}_1 \xrightarrow{d_1} \mathcal{L}_0 \xrightarrow{\varepsilon} \mathcal{E} \rightarrow 0 \text{ in } W.$$

*Proof:* Let  $(s, x)$  be a point of  $S \times U$  and  $\mathcal{L}^*$  a finite resolution of  $\mathcal{F}(x)$  in a neighbourhood of  $x$  (there exists such one, by the theorem of syzygies). We shall show that there exists a resolution  $\mathcal{L}^*$  of  $\mathcal{F}$  in a neighbourhood of  $(s, x)$  such that  $\mathcal{L}^*(s) = \mathcal{L}^*_s$ ; if  $\mathcal{L}^*_i = \mathcal{O}_x^{r_i}$  define

$$\mathcal{L}_i = \mathcal{O}_{S \times U}^{r_i} \text{ and } \mathcal{K}_i^0 = \text{Ker } d_i^0 : \mathcal{L}_i^0 \rightarrow \mathcal{L}_{i-1}^0.$$

We shall construct by induction (with respect to  $i$ )  $d_i : \mathcal{L}_1 \rightarrow \mathcal{L}_{i-1}$  in a neighbourhood of  $(s, x)$  such that  $d_i(s) = d_i^0$ , and prove that  $\mathcal{K}_i = \text{Ker } d_i$  is  $S$ -flat and that  $\mathcal{K}_i(s) = \mathcal{K}_i^0$ .

Suppose that we have constructed  $d_i$  and proved the properties for  $\mathcal{K}_i$ . We can construct  $d_{i+1} : \mathcal{L}_{i+1} \rightarrow \mathcal{L}_i$  in a neighbourhood of  $(s, x)$  such that the diagram is commutative.

$$\begin{array}{ccc} \mathcal{L}_{i+1} & \xrightarrow{d_{i+1}} & \mathcal{K}_i \\ \downarrow & & \downarrow \\ \mathcal{L}_{i+1}^0 & \xrightarrow{d_{i+1}} & \mathcal{K}_i^0 \end{array}$$

Nakayama's lemma shows that  $\text{Im } d_{i+1} = \mathcal{K}_i$  at the point  $(s, x)$ , therefore in a neighbourhood of that point.

The exact sequence

$$0 \rightarrow \mathcal{K}_{i+1} \rightarrow \mathcal{L}_{i+1} \rightarrow \mathcal{K}_i \rightarrow 0,$$

where  $\mathcal{K}_i$  and  $\mathcal{L}_{i+1}$  are  $S$ -flat, shows that  $\mathcal{K}_{i+1}$  is  $S$ -flat, and that  $\mathcal{K}_{i+1}(s) = \mathcal{K}_{i+1}^0$ . The first step of the induction is analogous.

*Proof of the theorem:* Let  $s_0 \in S$  and

$$0 \rightarrow \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_1} \mathcal{L}_0 \rightarrow \mathcal{E}|_W \rightarrow 0$$

be a free  $\mathcal{O}_{S \times U}$  resolution of  $\mathcal{E}$  in a neighbourhood  $W = V_1 \times V_2$  of  $\{s_0\} \times K$ . The sheaf  $\mathcal{E}$  is  $\mathcal{O}_S$ -flat, so for each  $s \in V_1$ , the sequence

$$0 \rightarrow \mathcal{L}_p \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow \dots \rightarrow \mathcal{L}_1 \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow \mathcal{L}_0 \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow \mathcal{E}|_W \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow 0$$

is exact. So the sequence

$$(A) \quad 0 \rightarrow \mathcal{L}_p(s) \xrightarrow{d_p(s)} \dots \xrightarrow{d_1(s)} \mathcal{L}_0(s) \xrightarrow{\varepsilon(s)} \mathcal{E}(s)|_{V_2} \rightarrow 0$$

is exact when  $s \in V_1$ . Now  $\mathcal{L}_i(s) \simeq \mathcal{O}_{V_2}^{r_i}$  ( $0 \leq i \leq p$ ) and every  $d_i(s)$  induces a continuous linear map:

$B(K, \mathcal{L}_i(s)) \rightarrow B(K, \mathcal{L}_{i-1}(s))$ , which we also denote by  $d_i(s)$ . We can consider  $d_i = (d_{ijk})$  as an  $r_i \times r_{i-1}$ -matrix with entries from  $\mathcal{O}_{S \times U}(W)$ .

By § 1 we have a  $\mathcal{O}_S$ -algebra homomorphism

$$\mathcal{O}_{S \times W}(S \times W) \rightarrow \mathcal{H}(S, B(K)).$$

From the matrix  $(d_{ijk})$  we get by this homomorphism a morphism  $\tilde{d}_i$ :

$$V_0 \rightarrow \mathcal{L}(B(K)^{r_i}, B(K)^{r_{i-1}}) = \mathcal{L}(B(K, \mathcal{L}_i(s)), B(K, \mathcal{L}_{i-1}(s))).$$

(Here  $V_0$  is some neighbourhood of  $s_0$ ) such that  $\tilde{d}_i(s) = d_i(s)$  for each  $s \in V_0$ . In other words we have a sequence of Banach vector bundle morphisms

$$(B) \quad \begin{array}{c} d_p \quad \tilde{d}_1 \\ 0 \rightarrow B(K, \mathcal{L}_p) \rightarrow \dots \rightarrow B(K, \mathcal{L}_0). \end{array}$$

Using the fact that  $\mathcal{O}_{S \times U}(S \times U) \rightarrow \mathcal{H}(S, B(K))$  is an  $\mathcal{O}_S$ -algebra homomorphism, it easily follows that (B) is complex of Banach vector bundles over  $S$ .

Now  $K$  is  $\mathcal{E}(s_0)$ -privileged, thus

$$0 \rightarrow B(K, \mathcal{L}_p(s_0)) \xrightarrow{d_p(s_0)} \dots \xrightarrow{d_1(s_0)} B(K, \mathcal{L}_0(s_0))$$

is split exact, so by theorem III.1

$$0 \rightarrow B(K, \mathcal{L}_p)|_V \xrightarrow{\tilde{d}_p|_V} \dots \xrightarrow{\tilde{d}_1|_V} B(K, \mathcal{L}_0)|_V$$

is split exact for some neighbourhood  $V$  of  $s_0$ .

Because  $\tilde{d}_i(s) = d_i(s)$  and the sequence (A) is exact part (a) of the theorem follows.

(b)  $B(K, \mathcal{L}_0)|_V$  splits as the direct sum of  $\text{im } \tilde{d}_1$  and a bundle  $E_V$ , such that  $E_{V,s} \simeq B(K, \mathcal{E}(s))$ , for each  $s \in V$ . We must show that these bundle structures fit together globally.

Suppose therefore that  $V$  is open in  $S'$  and that

$$\begin{array}{c} d_p \quad d_2 \quad d_1 \\ 0 \rightarrow \mathcal{L}_p \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{E}|_{V \times V_2} \rightarrow 0 \\ d'_p \quad d'_2 \quad d'_1 \\ 0 \rightarrow \mathcal{L}'_p \rightarrow \dots \rightarrow \mathcal{L}'_1 \rightarrow \mathcal{L}'_0 \rightarrow \mathcal{E}|_{V \times V_2} \rightarrow 0 \end{array}$$

are free resolutions of  $\xi$  over  $V \times V_2$ .

If  $V_1, V_2$  are open polycylinders, we can find an  $\mathcal{O}_{S \times U}$ -homomorphism  $\phi_0 : \mathcal{L}_0 \rightarrow \mathcal{L}'_0$  such that

$$\begin{array}{ccc} & \varepsilon' & \\ \mathcal{L}'_0 & \xrightarrow{\quad} & \mathcal{E}|_{V \times V_2} \rightarrow 0 \\ \phi_0 \uparrow & \parallel & \\ & \varepsilon & \\ \mathcal{L}_0 & \xrightarrow{\quad} & \mathcal{E}|_{V \times V_2} \rightarrow 0 \end{array}$$

commutes.  $\phi_0$  determines a bundle morphism  $\tilde{\phi}_0: B(K, \mathcal{L}_0) \rightarrow B(K, \mathcal{L}'_0)$ .  $B(K, \mathcal{L}_0)$  (resp.  $B(K, \mathcal{L}'_0)$ ) splits as  $(\text{im } \tilde{d}_1) \otimes E_V$  [Resp.  $(\text{im } \tilde{d}'_1) \otimes E'_V$ ].

Let  $p'$  be the projection morphism:  $B(K, \mathcal{L}_0) \rightarrow E'_V$  with kernel  $\text{im } \tilde{d}'_1$ , and put  $\tilde{\phi} = p' \circ \phi_0|_{E_V}$ .

The commutative diagram

$$\begin{array}{ccc}
 B(K, \mathcal{L}_0(s)) & \xrightarrow{\tilde{\phi}_0} & B(K, \mathcal{L}'_0(s)) \\
 \downarrow \varepsilon & \swarrow \tilde{\phi} & \downarrow \varepsilon' \\
 & E_{V,s} & \xrightarrow{\tilde{\phi}} E'_{V,s} \\
 & \nwarrow \varepsilon \simeq \alpha \circ \varepsilon & \nearrow \varepsilon' \simeq \alpha' \circ \varepsilon' \\
 B(K, \mathcal{E}(s)) & \xleftarrow{|\alpha|} & B(K, \mathcal{E}'(s))
 \end{array}$$

and the open mapping theorem shows that  $\tilde{\phi}(s)$  is an isomorphism of Banach spaces for each  $s \in V$ , so  $\tilde{\phi}: E_V \rightarrow E'_V$  is a bundle isomorphism. We also notice that  $\tilde{\phi}$  depends only on the choice of splittings in  $B(K, \mathcal{L}_0)$  and  $B(K, \mathcal{L}'_0)$ , and not on the choice of  $\tilde{\phi}_0$ . This ends the proof of the theorem.

*Remark:* Consider the general situation where  $X$  and  $S$  are analytic spaces, and  $\pi: X \rightarrow S$  is a morphism,  $\mathcal{E}$  an  $\mathcal{O}_X$ -module. To study the local dependence of  $\mathcal{E}$  on  $S$ , one can imbed an open set  $X'$  in  $X$  in the open set  $U \subset \mathbb{C}^n$ . The morphism  $\phi: X' \rightarrow U$ ,  $\pi: X' \rightarrow S$  determine the imbedding  $\pi \times \phi: X' \rightarrow S \times U$  such that the diagram commutes.  $\mathcal{E}$  can be extended by zero into a sheaf  $\mathcal{E}'$  over  $U \times S$ . Obviously this sheaf  $\mathcal{E}'$  is  $S$ -flat iff  $\mathcal{E}$  is  $S$ -flat.

Therefore theorem 1 makes clear also this general situation.

*Corollary:* If  $\pi: X \rightarrow S$  is a morphism and  $\mathcal{E}$  a coherent  $\mathcal{O}_X$ -module. Then  $\pi|_{\text{Supp}(\mathcal{E})}$  is an open map.

*Proof:* Suppose as above that  $X$  is imbedded in  $S \times U$ , and  $\mathcal{E}$  is extended by zero to  $S \times U$ . Let  $x_0 \in \text{Supp } \mathcal{E}$ , and  $V$  be a neighbourhood of  $x_0$  in  $S \times U$ . Let  $s_0 = \pi(x_0)$  and choose an  $\mathcal{E}(s_0)$ -privileged polycylinder  $K$  in  $U$ , such that  $\{s_0\} \times K \subset V$ , over some neighbourhood  $W$  of  $s_0$ . We have the Banach bundle  $B(K, \mathcal{E}|_{\pi^{-1}(W)})$ , whose fiber over  $s$  is  $B(K, \mathcal{E}(s))$ . Since  $x_0 \in \text{Supp } \mathcal{E}(s_0)$  and  $K$  is a neighbourhood of  $x_0$ ,  $B(K; \mathcal{E}(s_0)) \neq 0$ . As all the fibers are isomorphic, then for all  $s \in U$ ,  $B(K; \mathcal{E}(s)) \neq 0$  and therefore  $\{s\} \times K \cap \text{Supp } \mathcal{E} \neq \emptyset$ , and  $s \in \pi(\text{Supp } \mathcal{E})$ . This proves that  $\pi$  is open.