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$$\begin{array}{ccc} \mathcal{O}(\Delta \times V) & \xrightarrow{\phi} & \mathcal{H}(\Delta, B(K)) \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ \mathcal{O}(S \times V) & \xrightarrow{\phi} & \mathcal{H}(S, B(K)) \end{array}$$

is commutative; $\phi$ is evidently an $\mathcal{O}_S$-algebra homomorphism.

## § 2. *The flatness and privilege theorem*

*Notation*

Let $S$ be an analytic space, $U$ an open set in $\mathbf{C}^n$, and $\pi : S \times U \to S$ the first projection.

If $\mathcal{F}$ is an $\mathcal{O}_{S \times U}$ module, then for every $s \in S$ we denote by $\mathcal{F}(s)$ the $\mathcal{O}_U$-module $i_s^* \mathcal{F}$, where $i_s$ is the injective morphism $x \to (s, x)$ from $U$ into $S \times U$. If $x \in U$

$$(\mathcal{F}(s))_x \simeq \mathcal{F}_{(s,x)}/m_s \cdot \mathcal{F}_{(s,x)} \simeq \mathcal{F}_{(s,x)} \otimes_{\mathcal{O}_{S,s}} \mathbf{C}_s .$$

*Theorem 1:* Let $\mathcal{E}$ be a coherent and $S$-flat $\mathcal{O}_{S \times U}$-module, and $K$ a polycylinder in $U$.

(*a*) When $K$ is privileged for $\mathcal{E}(s_0)$, $s_0$ has a neighbourhood $V$ such that $K$ is $\mathcal{E}(s)$-privileged for each $s \in V$. In other words: the set $S' = \{s \in S \mid K$ is $\mathcal{E}(s)$-privileged$\}$ is open in $S$.

(*b*) It is possible to define a Banach vector bundle over $S'$ whose fibre at any $s \in S'$ is $B(K, \mathcal{E}(s))$.

To prove the theorem we need:

*Lemma 1:* Under the conditions of the theorem, we can, for every $s \in S$, find a neighbourhood $W$ of $\{s\} \times K$ and a free resolution of finite length

$$0 \to \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathcal{L}_1 \xrightarrow{d_1} \mathcal{L}_0 \xrightarrow{\varepsilon} \mathcal{E} \to 0 \text{ in } W.$$

*Proof:* Let $(s, x)$ be a point of $S \times U$ and $\mathcal{L}^0_*$ a finite resolution of $\mathcal{F}(x)$ in a neighbourhood of $x$ (there exists such one, by the theorem of syzygies). We shall show that that there exists a resolutin $\mathcal{L}^*$ of $\mathcal{F}$ in a neighbourhood of $(s, x)$ such that $\mathcal{L}^*(s) = \mathcal{L}^0_*$; if $\mathcal{L}^0_i = \mathcal{O}_x^{r_i}$ define

$$\mathcal{L}_i = \mathcal{O}^{r_i}_{S \times U} \text{ and } \mathcal{K}^0_i = \operatorname{Ker} d^0_i : \mathcal{L}^0_i \to \mathcal{L}^0_{i-1}.$$

We shall construct by induction (with respect to $i$) $d_i : \mathscr{L}_1 \to \mathscr{L}_{i-1}$ in a neighbourhood of $(s, x)$ such that $d_i(s) = d_i^0$, and prove that $\mathscr{K}_i = \operatorname{Ker} d_i$ is $S$-flat and that $\mathscr{K}_i(s) = \mathscr{K}_i^0$.

$$\begin{array}{ccc} \mathscr{L}_{i+1} & \xrightarrow{d_{i+1}} & \mathscr{K}_i \\ \downarrow & & \downarrow \\ \mathscr{L}^0_{i+1} & \xrightarrow{d_{i+1}} & \mathscr{K}^0_i \end{array}$$

Suppose that we have constructed $d_i$ and proved the properties for $\mathscr{K}_i$. We can construct $d_{i+1} : \mathscr{L}_{i+1} \to \mathscr{L}_i$ in a neighbourhood of $(s, x)$ such that the diagram is commutative.

Nakayama's lemma shows that $\operatorname{Im} d_{i+1} = \mathscr{K}_i$ at the point $(s, x)$, therefore in a neighbourhood of that point.

The exact sequence

$$0 \to \mathscr{K}_{i+1} \to \mathscr{L}_{i+1} \to \mathscr{K}_i \to 0\,,$$

where $\mathscr{K}_i$ and $\mathscr{L}_{i+1}$ are $S$-flat, shows that $\mathscr{K}_{i+1}$ is $S$-flat, and that $\mathscr{K}_{i+1}(s) = \mathscr{K}^0_{i+1}$. The first step of the induction is analogous.

*Proof of the theorem:* Let $s_0 \in S$ and

$$0 \to \mathscr{L}_p \xrightarrow{d_p} \ldots \xrightarrow{d_1} \mathscr{L}_0 \to \mathscr{E}|W \to 0$$

be a free $\mathscr{O}_{S\times U}$ resolution of $\mathscr{E}$ in a neighbourhood $W = V_1 \times V_2$ of $\{s_0\} \times K$. The sheaf $\mathscr{E}$ is $\mathscr{O}_S$-flat, so for each $s \in V_1$, the sequence

$$0 \to \mathscr{L}_p \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \ldots \to \mathscr{L}_1 \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \mathscr{L}_0 \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \mathscr{E}_{|W} \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to 0$$

is exact. So the sequence

$$(A) \qquad 0 \to \mathscr{L}_p(s) \xrightarrow{d_p(s)} \ldots \to \mathscr{L}_1(s) \xrightarrow{d_1(s)} \mathscr{L}_0(s) \xrightarrow{\varepsilon(s)} \mathscr{E}(s)_{|V_2} \to 0$$

is exact when $s \in V_1$. Now $\mathscr{L}_i(s) \simeq \mathscr{O}_{V_2}^{r_i}$ $(0 \leqq i \leqq p)$ and every $d_i(s)$ induces a continuous linear map:

$B(K, \mathscr{L}_i(s)) \to B(K, \mathscr{L}_{i-1}(s))$, which we also denote by $d_i(s)$. We can consider $d_i = (d_{ijk})$ as an $r_i \times r_{i-1}$-matrix with entries from $\mathscr{O}_{S\times U}(W)$.

By § 1 we have a $\mathscr{O}_S$-algebra homomorphism

$$\mathscr{O}_{S\times W}(S \times W) \to \mathscr{H}(S, B(K))\,.$$

From the matrix $(d_{ijk})$ we get by this homomorphism a morphism $\tilde{d}_i$:

$$V_0 \to \mathscr{L}(B(K)^{ri}, B(K)^{ri-1}) = \mathscr{L}(B(K, \mathscr{L}_i(s)), B(K, \mathscr{L}_{i-1}(s))\,.$$

(Here $V_0$ is some neighbourhood of $s_0$) such that $\tilde{d}_i(s) = d_i(s)$ for each $s \in V_0$. In other words we have a sequence of Banach vector bundle morphisms

$$(B) \qquad 0 \to B(K, \mathscr{L}_p) \xrightarrow{d_p} \dots \xrightarrow{\tilde{d}_1} B(K, \mathscr{L}_0).$$

Using the fact that $\mathcal{O}_{S \times U}(S \times U) \to \mathscr{H}(S, B(K))$ is an $\mathcal{O}_S$-algebra homomorphism, it easily follows that $(B)$ is complex of Banach vector bundles over $S$.

Now $K$ is $\mathscr{E}(s_0)$-privileged, thus

$$0 \to B(K, \mathscr{L}_p(s_0)) \xrightarrow{d_p(s_0)} \dots \xrightarrow{d_1(s_0)} B(K, \mathscr{L}_0(s_0))$$

is split exact, so by theorem III.1

$$0 \to B(K, \mathscr{L}_p)_{|V} \xrightarrow{\tilde{d}_p|V} \dots \xrightarrow{\tilde{d}_i|V} B(K, \mathscr{L}_0)_{|V}$$

is split exact for some neighbourhood $V$ of $s_0$.

Because $\tilde{d}_i(s) = d_i(s)$ and the sequence $(A)$ is exact part $(a)$ of the theorem follows.

$(b)$ $B(K. \mathscr{L}_0)|V$ splits as the direct sum of im $\tilde{d}_1$ and a bundle $E_V$, such that $E_{V,s} \simeq B(K, \mathscr{E}(s))$, for each $s \in V$. We must show that these bundle structures fit together globally.

Suppose therefore that $V$ is open in $S'$ and that

$$0 \to \mathscr{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathscr{L}_1 \xrightarrow{d_1} \mathscr{L}_0 \to \mathscr{E}_{|V \times V_2} \to 0$$

$$0 \to \mathscr{L}'_p \xrightarrow{d'_p} \dots \xrightarrow{d'_2} \mathscr{L}'_1 \xrightarrow{d'_1} \mathscr{L}'_0 \to \mathscr{E}_{|V \times V_2} \to 0$$

are free resolutions of $\xi$ over $V \times V_2$.

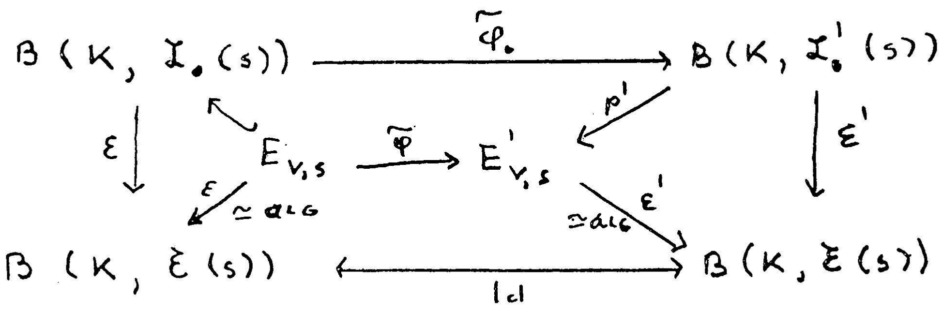
If $V_1, V_2$ are open polycylinders, we can find an $\mathcal{O}_{S \times U}$-homomorphism $\phi_0 : \mathscr{L}_0 \to \mathscr{L}'_0$ such that

$$\begin{array}{ccccc} \mathscr{L}'_0 & \xrightarrow{\varepsilon'} & \mathscr{E}_{|V \times V_2} & \to & 0 \\ \phi_0 \uparrow & & \| & & \\ \mathscr{L}_0 & \xrightarrow{\varepsilon} & \mathscr{E}_{|V \times V_2} & \to & 0 \end{array}$$

commutes. $\phi_0$ determines a bundle morphism $\tilde{\phi}_0 : B(K, \mathscr{L}_0) \to B(K, \mathscr{L}'_0)$. $B(K, \mathscr{L}_0)$ (resp. $B(K, \mathscr{L}'_0)$) splits as (im $\tilde{d}_1) \otimes E_V$ [Resp. (im $\tilde{d}'_1) \otimes E'_V$].

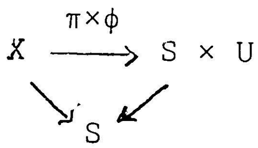
Let $p'$ be the projection morphism: $B(K, \mathscr{L}_0) \to E'_V$ with kernel im $\tilde{d}'_1$, and put $\tilde{\phi} = p' \circ \tilde{\phi}_0 | E_V$.

The commutative diagram




and the open mapping theorem shows that $\tilde{\phi}(s)$ is an isomorphism of Banach spaces for each $s \in V$, so $\tilde{\phi} : E_V \to E'_V$ is a bundle isomorphism. We also notice that $\tilde{\phi}$ depends only on the choice of splittings in $B(K, \mathscr{L}_0)$ and $B(K, \mathscr{L}'_0)$, and not on the choice of $\tilde{\phi}_0$. This ends the proof of the theorem.

*Remark:* Consider the general situation where $X$ and $S$ are analytic spaces, and $\pi : X \to S$ is a morphism, $\mathscr{E}$ an $\mathscr{O}_X$-module. To study the local dependence of $\mathscr{E}$ on $S$, one can imbed an open set $X'$ in $X$ in the open set $U \subset \mathbf{C}^n$. The morphism $\phi : X' \to U, \pi : X' \to S$ determine the imbedding $\pi \times \phi : X' \to S \times U$ such that the diagram




commutes. $\mathscr{E}$ can be extended by zero into a sheaf $\mathscr{E}'$ over $U \times S$. Obviously this sheaf $\mathscr{E}'$ is $S$-flat iff $\mathscr{E}$ is $S$-flat.

Therefore theorem 1 makes clear also this general situation.

*Corollary:* If $\pi : X \to S$ is a morphism and $\mathscr{E}$ a coherent $\mathscr{O}_X$-module. Then $\pi \,|\, \mathrm{Supp}\,(\mathscr{E})$ is an open map.

*Proof:* Suppose as above that $X$ is imbedded in $S \times U$, and $\mathscr{E}$ in extended by zero to $S \times U$. Let $x_0 \in \mathrm{Supp}\,\mathscr{E}$, and $V$ be a neighbourhood of $x_0$ in $S \times U$. Let $s_0 = \pi(x_0)$ and choose an $\mathscr{E}(s_0)$-privileged polycylinder $K$ in $U$, such that $\{s_0\} \times K \subset V$, over some neighbourhood $W$ of $s_0$. We have the Banach bundle $B(K, \mathscr{E} | \pi^{-1}(W))$, whose fiber over $s$ is $B(K, \mathscr{E}(s))$. Since $x_0 \in \mathrm{Supp}\,\mathscr{E}(s_0)$ and $K$ is a neighbourhood of $x_0$, $B(K; \mathscr{E}(s_0)) \neq 0$. As all the fibers are isomorphic, then for all $s \in U$, $B(K; \mathscr{E}(s)) \neq 0$ and therefore $\{s\} \times K \cap \mathrm{Supp}\,\mathscr{E} \neq 0$, and $s \in \pi$ (Supp $\mathscr{E}$). This proves that $\pi$ is open.