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whenever  $x \in K$ , we get  $a = \inf_K |h(x)| > 0$ . Hence  $\|hf\| = \sup_K |hf(x)| \geq \geq a \sup_K |f(x)| = a \|f\|$ .

(i')  $\Rightarrow$  (ii). Suppose that  $X \cap K \neq \emptyset$  and  $x = (x_1, x_2) \in X \cap K$ . We choose an analytic function  $f_1 : U_1 \rightarrow \mathbf{C}$ , where  $U_1 \supset K_1$ , and  $U_1$  is open, such that  $f_1(x_1) = 1$ ,  $|f_1(z)| < 1$  if  $z \in K_1$ ,  $z \neq x_1$ . Similarly we choose an analytic function  $f_2 : U_2 \rightarrow \mathbf{C}$ , with the same properties. Consider the function  $f \in B(K) : (z_1, z_2) \mapsto f_1(z_1)f_2(z_2)$ . Since  $h(x) = 0$  it follows that the sequence  $\{hf^n\}$  converges pointwise to 0 in  $K$ .

Applying Dini's theorem we get  $\|hf^n\| \rightarrow 0$ . From the inequality  $a \|f^n\| \leq \|hf^n\|$  we get  $\|f^n\| \rightarrow 0$ , which is a contradiction, because for every  $n : f^n(x) = 1$ .

(b) Use the Weierstrass preparation theorem (extended form).

*Question.* Does the condition (ii) imply that  $h : B(K) \rightarrow B(K)$  is a split monomorphism?

#### IV. FLATNESS AND PRIVILEGE

##### § 1. *Morphisms from an analytic space into $B(K)$*

Let  $S$  be an analytic space and  $K$  a polycylinder in an open set  $U \subset \mathbf{C}^n$ . We want to construct an  $\mathcal{O}_S$ -algebra homomorphism  $\phi : \mathcal{O}_{S \times U}(S \times U) \rightarrow \mathcal{H}(S; B(K))$ .

- (a) Consider first  $S = U' \subset \mathbf{C}^m$ ,  $U'$ -open. If  $h \in \mathcal{O}_{U' \times U}(U' \times U)$  and  $s \in U'$ ,  $x \in K$ , define  $(\phi(h)(s))(x) = h(s, x)$ . Using the Cauchy integral, one can show that  $\phi(h)$  is analytic. On the other hand its obvious that  $\phi$  is an  $\mathcal{O}_{U'}$ -algebra homomorphism.
- (b) Let  $S$  have a special model in the polydisc  $\Delta$  in  $\mathbf{C}^m$ , defined by a sheaf  $\mathcal{J}$  of ideals of  $\mathcal{O}_\Delta$ , and let  $\mathcal{J}$  be generated by  $f_1, \dots, f_p$ ,  $V$ -a polycylinder neighbourhood of  $K$  in  $U$ . By Cartan's theorem  $B$  for a polycylinder,

the sequence  $0 \rightarrow \mathcal{J}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(\Delta \times V) \rightarrow \mathcal{O}(S \times V) \rightarrow 0$  is exact. If we denote by  $\tilde{\pi}$  the projection  $\mathcal{H}(\Delta, B(K)) \rightarrow \mathcal{H}(S, B(K))$ ,  $(f_1, \dots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset \subset \text{Ker } \tilde{\pi}$ . Therefore, because  $\pi$  is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \rightarrow \mathcal{H}(S, B(K))$ , such that the diagram

$$\mathcal{O}(\Delta \times V) \xrightarrow{\phi} \mathcal{H}(\Delta, B(K))$$

$$\pi \downarrow \quad \quad \quad \downarrow \tilde{\pi}$$

$$\mathcal{O}(S \times V) \xrightarrow{\phi} \mathcal{H}(S, B(K))$$

is commutative;  $\phi$  is evidently an  $\mathcal{O}_S$ -algebra homomorphism.

## § 2. The flatness and privilege theorem

### Notation

Let  $S$  be an analytic space,  $U$  an open set in  $\mathbf{C}^n$ , and  $\pi : S \times U \rightarrow S$  the first projection.

If  $\mathcal{F}$  is an  $\mathcal{O}_{S \times U}$  module, then for every  $s \in S$  we denote by  $\mathcal{F}(s)$  the  $\mathcal{O}_U$ -module  $i_s^* \mathcal{F}$ , where  $i_s$  is the injective morphism  $x \mapsto (s, x)$  from  $U$  into  $S \times U$ . If  $x \in U$

$$(\mathcal{F}(s))_x \simeq \mathcal{F}_{(s, x)} / m_s \cdot \mathcal{F}_{(s, x)} \simeq \mathcal{F}_{(s, x)} \otimes_{\mathcal{O}_{S, s}} \mathbf{C}_s.$$

*Theorem 1:* Let  $\mathcal{E}$  be a coherent and  $S$ -flat  $\mathcal{O}_{S \times U}$ -module, and  $K$  a poly-cylinder in  $U$ .

- (a) When  $K$  is privileged for  $\mathcal{E}(s_0)$ ,  $s_0$  has a neighbourhood  $V$  such that  $K$  is  $\mathcal{E}(s)$ -privileged for each  $s \in V$ . In other words: the set  $S' = \{s \in S \mid K$  is  $\mathcal{E}(s)$ -privileged $\}$  is open in  $S$ .
- (b) It is possible to define a Banach vector bundle over  $S'$  whose fibre at any  $s \in S'$  is  $B(K, \mathcal{E}(s))$ .

To prove the theorem we need:

*Lemma 1:* Under the conditions of the theorem, we can, for every  $s \in S$ , find a neighbourhood  $W$  of  $\{s\} \times K$  and a free resolution of finite length

$$0 \rightarrow \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathcal{L}_1 \xrightarrow{d_1} \mathcal{L}_0 \xrightarrow{\varepsilon} \mathcal{E} \rightarrow 0 \text{ in } W.$$

*Proof:* Let  $(s, x)$  be a point of  $S \times U$  and  $\mathcal{L}_*^0$  a finite resolution of  $\mathcal{F}(x)$  in a neighbourhood of  $x$  (there exists such one, by the theorem of syzygies). We shall show that there exists a resolution  $\mathcal{L}^*$  of  $\mathcal{F}$  in a neighbourhood of  $(s, x)$  such that  $\mathcal{L}^*(s) = \mathcal{L}_*^0$ ; if  $\mathcal{L}_i^0 = \mathcal{O}_x^{r_i}$  define

$$\mathcal{L}_i = \mathcal{O}_{S \times U}^{r_i} \text{ and } \mathcal{K}_i^0 = \text{Ker } d_i^0 : \mathcal{L}_i^0 \rightarrow \mathcal{L}_{i-1}^0.$$