Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 14 (1968)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FLATNESS AND PRIVILEGE

Autor: Douady, A.

Kapitel: §1. Morphisms from an analytic space into B (K)

DOI: https://doi.org/10.5169/seals-42343

whenever $x \in \ddot{K}$, we get $a = \inf_K |h(x)| > 0$. Hence $\|hf\| = \sup_K |hf(x)| \geqq$
$\geqq a \sup_K |f(x)| = a \|f\|$.

(i') $\Rightarrow$ (ii). Suppose that $X \cap \ddot{K} \neq \varnothing$ and $x = (x_1, x_2) \in X \cap \ddot{K}$. We choose an analytic function $f_1 : U_1 \to \mathbf{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \to \mathbf{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \to f_1(z_1) f_2(z_2)$. Since $h(x) = 0$ it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in K.

Applying Dini's theorem we get $\|hf^n\| \to 0$. From the inequality $a \|f^n\| \leqq$ $\leqq \|hf^n\|$ we get $\|f^n\| \to 0$, which is a contradiction, because for every $n : f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h : B(K) \to B(K)$ is a split monomorphism?

IV. Flatness and privilege

§ 1. *Morphisms from an analytic space into $B(K)$*

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbf{C}^n$. We want to construct an $\mathcal{O}_S$-algebra homomorphism $\phi : \mathcal{O}_{S \times U}(S \times U) \to$ $\to \mathcal{H}(S; B(K))$.

(*a*) Consider first $S = U' \subset \mathbf{C}^m$, U'-open. If $h \in \mathcal{O}_{U' \times U}(U' \times U)$ and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s, x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand its obvious that ϕ is an $\mathcal{O}_{U'}$-algebra homomorphism.

(*b*) Let S have a special model in the polydisc Δ in $\mathbf{C}^m$, defined by a sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_\Delta$, and let $\mathcal{J}$ be generated by $f_1, \ldots, f_p$, V-a polycylinder neighbourhood of K in U. By Cartan's theorem B for a polycylinder,

the sequence $0 \to \mathcal{J}(\Delta \times V) \to \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \to 0$ is exact. If we denote by $\tilde{\pi}$ the projection $\mathcal{H}(\Delta, B(K)) \to \mathcal{H}(S, B(K))$, $(f_1, \ldots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset$ $\subset \operatorname{Ker} \tilde{\pi}$. Therefore, because π is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \to \mathcal{H}(S, B(K))$, such that the diagram

$$\begin{array}{ccc} \mathcal{O}(\Delta \times V) & \xrightarrow{\phi} & \mathcal{H}(\Delta, B(K)) \\ \pi\downarrow & & \downarrow\tilde{\pi} \\ \mathcal{O}(S \times V) & \xrightarrow{\phi} & \mathcal{H}(S, B(K)) \end{array}$$

is commutative; ϕ is evidently an $\mathcal{O}_S$-algebra homomorphism.

§ 2. *The flatness and privilege theorem*

Notation

Let S be an analytic space, U an open set in $\mathbf{C}^n$, and $\pi : S\times U\to S$ the first projection.

If $\mathcal{F}$ is an $\mathcal{O}_{S\times U}$ module, then for every $s\in S$ we denote by $\mathcal{F}(s)$ the $\mathcal{O}_U$-module $i_s^* \mathcal{F}$, where i_s is the injective morphism $x\to(s, x)$ from U into $S\times U$. If $x\in U$

$$(\mathcal{F}(s))_x \simeq \mathcal{F}_{(s,x)}/m_s \cdot \mathcal{F}_{(s,x)} \simeq \mathcal{F}_{(s,x)} \otimes_{\mathcal{O}_{S,s}} \mathbf{C}_s\,.$$

Theorem 1: Let $\mathscr{E}$ be a coherent and S-flat $\mathcal{O}_{S\times U}$-module, and K a polycylinder in U.

(*a*) When K is privileged for $\mathscr{E}(s_0)$, s_0 has a neighbourhood V such that K is $\mathscr{E}(s)$-privileged for each $s\in V$. In other words: the set $S' = \{s\in S \mid K$ is $\mathscr{E}(s)$-privileged$\}$ is open in S.

(*b*) It is possible to define a Banach vector bundle over S' whose fibre at any $s\in S'$ is $B(K, \mathscr{E}(s))$.

To prove the theorem we need:

Lemma 1: Under the conditions of the theorem, we can, for every $s\in S$, find a neighbourhood W of $\{s\}\times K$ and a free resolution of finite length

$$0\to\mathcal{L}_p\xrightarrow{d_p}\ldots\xrightarrow{d_2}\mathcal{L}_1\xrightarrow{d_1}\mathcal{L}_0\xrightarrow{\varepsilon}\mathscr{E}\to 0 \text{ in } W.$$

Proof: Let (s, x) be a point of $S\times U$ and $\mathcal{L}^0_*$ a finite resolution of $\mathcal{F}(x)$ in a neighbourhood of x (there exists such one, by the theorem of syzygies). We shall show that that there exists a resolutin $\mathcal{L}^*$ of $\mathcal{F}$ in a neighbourhood of (s, x) such that $\mathcal{L}^*(s) = \mathcal{L}^0_*$; if $\mathcal{L}^0_i = \mathcal{O}_x^{r_i}$ define

$$\mathcal{L}_i = \mathcal{O}^{r_i}_{S\times U} \text{ and } \mathcal{K}^0_i = \operatorname{Ker} d^0_i\colon \mathcal{L}^0_i \to \mathcal{L}^0_{i-1}.$$