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whenever $x \in K$, we get $a = \inf_{K} |h(x)| > 0$. Hence $||hf|| = \sup_{K} |hf(x)| \ge 2$ $\ge a \sup_{K} |f(x)| = a ||f||$.

(i') \Rightarrow (ii). Suppose that $X \cap K \neq \emptyset$ and $x = (x_1, x_2) \in X \cap K$. We choose an analytic function $f_1 : U_1 \to \mathbb{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \to \mathbb{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \to f_1(z_1) f_2(z_2)$. Since h(x) = 0 it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in K.

Applying Dini's theorem we get $||hf^n|| \to 0$. From the inequality $a||f^n|| \le$ $\le ||hf^n||$ we get $||f^n|| \to 0$, which is a contradiction, because for every $n: f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h: B(K) \rightarrow B(K)$ is a split monomorphism?

IV. FLATNESS AND PRIVILEGE

§ 1. Morphisms from an analytic space into B(K)

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbb{C}^n$. We want to construct an \mathcal{O}_S -algebra homomorphism $\phi : \mathcal{O}_{S \times U} (S \times U) \to \mathcal{H} (S; B(K))$.

- (a) Consider first $S = U' \subset \mathbb{C}^m$, U'-open. If $h \in \mathcal{O}_{U' \times U}$ ($U' \times U$) and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s,x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand its obvious that ϕ is an $\mathcal{O}_{U'}$ -algebra homomorphism.
- (b) Let S have a special model in the polydisc Δ in \mathbb{C}^m , defined by a sheaf \mathscr{J} of ideals of \mathscr{O}_{Δ} , and let \mathscr{J} be generated by $f_1, ..., f_p$, V-a polycylinder neighbourhood of K in U. By Cartan's theorem B for a polycylinder,

the sequence $0 \rightarrow \mathcal{J}(\Delta \times V) \rightarrow \mathcal{O}(\Delta \times V) \rightarrow \mathcal{O}(S \times V) \rightarrow 0$ is exact. If we denote by π the projection $\mathcal{H}(\Delta, B(K)) \rightarrow \mathcal{H}(S, B(K)), (f_1, ..., f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset$

 \subset Ker π . Therefore, because π is surjection, there exists a unique

 $\phi: \mathcal{O}(S \times V) \rightarrow \mathcal{H}(S, B(K))$, such that the diagram

$$\mathcal{O}\left(\Delta \times V\right) \xrightarrow{\phi} \mathcal{H}\left(\Delta, B\left(K\right)\right) \\
\pi \downarrow \qquad \qquad \downarrow \widetilde{\pi} \\
\mathcal{O}\left(S \times V\right) \xrightarrow{\phi} \mathcal{H}\left(S, B\left(K\right)\right)$$

is commutative; ϕ is evidently an \mathcal{O}_S -algebra homomorphism.

§ 2. The flatness and privilege theorem

Notation

Let S be an analytic space, U an open set in \mathbb{C}^n , and $\pi: S \times U \rightarrow S$ the first projection.

If \mathscr{F} is an $\mathscr{O}_{S\times U}$ module, then for every $s\in S$ we denote by $\mathscr{F}(s)$ the \mathscr{O}_U -module $i_s^*\mathscr{F}$, where i_s is the injective morphism $x\to(s,x)$ from U into $S\times U$. If $x\in U$

$$(\mathscr{F}(s))_x \simeq \mathscr{F}_{(s,x)}/m_s \cdot \mathscr{F}_{(s,x)} \simeq \mathscr{F}_{(s,x)} \otimes_{\mathscr{O}_{S,s}} \mathbf{C}_s.$$

Theorem 1: Let $\mathscr E$ be a coherent and S-flat $\mathscr O_{S\times U}$ -module, and K a polycylinder in U.

- (a) When K is privileged for $\mathscr{E}(s_0)$, s_0 has a neighbourhood V such that K is $\mathscr{E}(s)$ -privileged for each $s \in V$. In other words: the set $S' = \{s \in S \mid K \text{ is } \mathscr{E}(s)\text{-privileged}\}$ is open in S.
- (b) It is possible to define a Banach vector bundle over S' whose fibre at any $s \in S'$ is $B(K, \mathscr{E}(s))$.

To prove the theorem we need:

Lemma 1: Under the conditions of the theorem, we can, for every $s \in S$, find a neighbourhood W of $\{s\} \times K$ and a free resolution of finite length

$$0 \! \to \! \mathcal{L}_p \! \xrightarrow{d_p} \! \dots \xrightarrow{d_2} \! \mathcal{L}_1 \! \xrightarrow{d_1} \! \mathcal{L}_0 \! \xrightarrow{\varepsilon} \! \mathscr{E} \! \to \! 0 \; \text{in} \; W.$$

Proof: Let (s, x) be a point of $S \times U$ and \mathcal{L}^0_* a finite resolution of $\mathcal{F}(x)$ in a neighbourhood of x (there exists such one, by the theorem of syzygies). We shall show that that there exists a resolutin \mathcal{L}^* of \mathcal{F} in a neighbourhood of (s, x) such that $\mathcal{L}^*(s) = \mathcal{L}^0_*$; if $\mathcal{L}^0_i = \mathcal{O}^{r_i}_x$ define

$$\mathcal{L}_i = \mathcal{O}_{S \times U}^{r_i}$$
 and $\mathcal{K}_i^0 = \operatorname{Ker} d_i^0 : \mathcal{L}_i^0 \to \mathcal{L}_{i-1}^0$.

We shall construct by induction (with respect to i) $d_i: \mathcal{L}_1 \to \mathcal{L}_{i-1}$ in a neighbourhood of (s, x) such that $d_i(s) = d_i^0$, and prove that $\mathcal{K}_i = \operatorname{Ker} d_i$ is S-flat and that $\mathcal{K}_i(s) = \mathcal{K}_i^0$.

Nakayama's lemma shows that Im $d_{i+1} = \mathcal{K}_i$ at the point (s, x), therefore in a neighbourhood of that point.

The exact sequence

$$0 \to \mathcal{K}_{i+1} \to \mathcal{L}_{i+1} \to \mathcal{K}_i \to 0$$
,

where \mathcal{K}_i and \mathcal{L}_{i+1} are S-flat, shows that \mathcal{K}_{i+1} is S-flat, and that $\mathcal{K}_{i+1}(s) = \mathcal{K}_{i+1}^0$. The first step of the induction is analogous.

Proof of the theorem: Let $s_0 \in S$ and

$$\begin{array}{cc} d_p & d_1 \\ 0 \rightarrow \mathcal{L}_p \rightarrow \dots \rightarrow \mathcal{L}_0 \rightarrow \mathcal{E} | W \rightarrow 0 \end{array}$$

be a free $\mathscr{O}_{S\times U}$ resolution of \mathscr{E} in a neighbourhood $W=V_1\times V_2$ of $\{s_0\}\times K$. The sheaf \mathscr{E} is \mathscr{O}_S -flat, so for each $s\in V_1$, the sequence

$$0 \to \mathcal{L}_p \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \dots \to \mathcal{L}_1 \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \mathcal{L}_0 \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \mathscr{E}_{|W} \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to 0$$

is exact. So the sequence

$$\begin{array}{ccc} d_p(s) & d_1(s) & \varepsilon(s) \\ (A) & 0 \rightarrow \mathcal{L}_p(s) & \rightarrow \dots \rightarrow \mathcal{L}_1(s) & \rightarrow \mathcal{L}_0(s) & \rightarrow \mathcal{E}(s)_{|V_2} \rightarrow 0 \end{array}$$

is exact when $s \in V_1$. Now $\mathcal{L}_i(s) \simeq \mathcal{O}_{V_2}^{r_i}$ $(0 \le i \le p)$ and every $d_i(s)$ induces a continuous linear map:

 $B(K, \mathcal{L}_i(s)) \rightarrow B(K, \mathcal{L}_{i-1}(s))$, which we also denote by $d_i(s)$. We can consider $d_i = (d_{ijk})$ as an $r_i \times r_{i-1}$ -matrix with entries from $\mathcal{O}_{S \times U}(W)$.

By § 1 we have a \mathcal{O}_S -algebra homomorphism

$$\mathcal{O}_{S\times W}(S\times W) \rightarrow \mathcal{H}(S, B(K))$$
.

From the matrix (d_{ijk}) we get by this homomorphism a morphism d_i :

$$V_0 \to \mathcal{L}\left(B\left(K\right)^{r_i}, B\left(K\right)^{r_{i-1}}\right) = \mathcal{L}\left(B\left(K, \mathcal{L}_i(s)\right), B\left(K, \mathcal{L}_{i-1}(s)\right)\right).$$

(Here V_0 is some neighbourhood of s_0) such that $d_i(s) = d_i(s)$ for each $s \in V_0$. In other words we have a sequence of Banach vector bundle morphisms

$$\begin{array}{ccc} d_p & \widetilde{d}_1 \\ 0 \rightarrow B(K, \mathcal{L}_p) \rightarrow \dots \rightarrow B(K, \mathcal{L}_0). \end{array}$$

Using the fact that $\mathcal{O}_{S\times U}(S\times U)\to \mathcal{H}(S,B(K))$ is an \mathcal{O}_S -algebra homomorphism, it easily follows that (B) is complex of Banach vector bundles over S.

Now K is $\mathscr{E}(s_0)$ -privileged, thus

$$0 \to B\left(K, \, \mathcal{L}_p\left(s_0\right)\right) \stackrel{d_1(s_0)}{\to} \dots \stackrel{}{\to} B\left(K, \, \mathcal{L}_0\left(s_0\right)\right)$$

is split exact, so by theorem III.1

$$\tilde{d}_{p}|V \quad \tilde{d}_{i}|V
0 \to B(K, \mathcal{L}_{p})|V \to \dots \to B(K, \mathcal{L}_{0})|V$$

is split exact for some neighbourhood V of s_0 .

Because $d_i(s) = d_i(s)$ and the sequence (A) is exact part (a) of the theorem follows.

(b) $B(K, \mathcal{L}_0)|V$ splits as the direct sum of im d_1 and a bundle E_V , such that $E_{V,s} \simeq B(K, \mathcal{E}(s))$, for each $s \in V$. We must show that these bundle structures fit together globally.

Suppose therefore that V is open in S' and that

$$\begin{aligned} d_{p} & d_{2} & d_{1} \\ 0 \rightarrow \mathcal{L}_{p} \rightarrow \dots \rightarrow \mathcal{L}_{1} \rightarrow \mathcal{L}_{0} \rightarrow \mathcal{E}_{|V \times V_{2}} \rightarrow 0 \\ d_{p}^{'} & d_{2}^{'} & d_{1}^{'} \\ 0 \rightarrow \mathcal{L}_{p}^{'} \rightarrow \dots \rightarrow \mathcal{L}_{1}^{'} \rightarrow \mathcal{L}_{0}^{'} \rightarrow \mathcal{E}_{|V \times V_{2}} \rightarrow 0 \end{aligned}$$

are free resolutions of ξ over $V \times V_2$.

If V_1 , V_2 are open polycylinders, we can find an $\mathcal{O}_{S\times U}$ -homomorphism $\phi_0: \mathcal{L}_0 \to \mathcal{L}_0'$ such that

$$\mathcal{L}_{0}^{'} \xrightarrow{\varepsilon'} \mathcal{E}_{|V \times V_{2}} \to 0$$

$$\phi_{0} \uparrow \qquad ||$$

$$\mathcal{L}_{0} \xrightarrow{\varepsilon} \mathcal{E}_{|V \times V_{2}} \to 0$$

commutes. ϕ_0 determines a bundle morphism $\overset{\sim}{\phi}_0: B(K, \mathcal{L}_0) \to B(K, \mathcal{L}_0')$. $B(K, \mathcal{L}_0')$ (resp. $B(K, \mathcal{L}_0')$) splits as $(\text{im } \tilde{d}_1) \otimes E_V$ [Resp. $(\text{im } \tilde{d}_1') \otimes E_V'$].

Let p' be the projection morphism: $B(K, \mathcal{L}_0) \to E_V'$ with kernel im d_1' , and put $\phi = p' \circ \phi_0 | E_V$.

The commutative diagram

and the open mapping theorem shows that $\phi(s)$ is an isomorphism of Banach spaces for each $s \in V$, so $\widetilde{\phi}: E_V \to E_V'$ is a bundle isomorphism. We also notice that $\widetilde{\phi}$ depends only on the choice of splittings in $B(K, \mathcal{L}_0)$ and $B(K, \mathcal{L}_0')$, and not on the choice of $\widetilde{\phi}_0$. This ends the proof of the theorem.

Remark: Consider the general situation where X and S are analytic spaces, and $\pi: X \to S$ is a morphism, $\mathscr E$ an $\mathscr O_X$ -module. To study the local dependence of $\mathscr E$ on S, one can imbed an open set X' in X in the open set $U \subset \mathbb C^n$. The morphism $\phi: X' \to U$, $\pi: X' \to S$ determine the imbedding $\pi \times \phi: X' \to S \times U$ such that the diagram commutes. $\mathscr E$ can be extended by zero into a sheaf $\mathscr E'$ over $U \times S$. Obviously this sheaf $\mathscr E'$ is S-flat iff $\mathscr E$ is S-flat.

Therefore theorem 1 makes clear also this general situation.

Corollary: If $\pi: X \to S$ is a morphism and $\mathscr E$ a coherent $\mathscr O_X$ -module. Then $\pi \mid \operatorname{Supp}\,(\mathscr E)$ is an open map.

Proof: Suppose as above that X is imbedded in $S \times U$, and \mathscr{E} in extended by zero to $S \times U$. Let $x_0 \in \text{Supp } \mathscr{E}$, and V be a neighbourhood of x_0 in $S \times U$. Let $s_0 = \pi(x_0)$ and choose an $\mathscr{E}(s_0)$ -privileged polycylinder K in U, such that $\{s_0\} \times K \subset V$, over some neighbourhood W of s_0 . We have the Banach bundle $B(K, \mathscr{E}|\pi^{-1}(W))$, whose fiber over s is $B(K, \mathscr{E}(s))$. Since $x_0 \in \text{Supp } \mathscr{E}(s_0)$ and K is a neighbourhood of x_0 , $B(K; \mathscr{E}(s)) \neq 0$. As all the fibers are isomorphic, then for all $s \in U$, $B(K; \mathscr{E}(s)) \neq 0$ and therefore $\{s\} \times K \cap \text{Supp } \mathscr{E} \neq 0$, and $s \in \pi$ (Supp \mathscr{E}). This proves that π is open.