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whenever $x \in \ddot{K}$, we get $a = \inf_K |h(x)| > 0$. Hence $\|hf\| = \sup_K |hf(x)| \geqq$ $\geqq a \sup_K |f(x)| = a \|f\|$.

(i′) $\Rightarrow$ (ii). Suppose that $X \cap \ddot{K} \neq \varnothing$ and $x = (x_1, x_2) \in X \cap \ddot{K}$. We choose an analytic function $f_1 : U_1 \to \mathbf{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \to \mathbf{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \to f_1(z_1) f_2(z_2)$. Since $h(x) = 0$ it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in K.

Applying Dini's theorem we get $\|hf^n\| \to 0$. From the inequality $a \|f^n\| \leqq$ $\leqq \|hf^n\|$ we get $\|f^n\| \to 0$, which is a contradiction, because for every $n : f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h : B(K) \to B(K)$ is a split monomorphism?

IV. Flatness and privilege

§ 1. *Morphisms from an analytic space into $B(K)$*

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbf{C}^n$. We want to construct an $\mathcal{O}_S$-algebra homomorphism $\phi : \mathcal{O}_{S \times U}(S \times U) \to$ $\to \mathcal{H}(S; B(K))$.

(*a*) Consider first $S = U' \subset \mathbf{C}^m$, U'-open. If $h \in \mathcal{O}_{U' \times U}(U' \times U)$ and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s, x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand its obvious that ϕ is an $\mathcal{O}_{U'}$-algebra homomorphism.

(*b*) Let S have a special model in the polydisc Δ in $\mathbf{C}^m$, defined by a sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_\Delta$, and let $\mathcal{J}$ be generated by $f_1, \ldots, f_p$, V-a polycylinder neighbourhood of K in U. By Cartan's theorem B for a polycylinder,

the sequence $0 \to \mathcal{J}(\Delta \times V) \to \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \to 0$ is exact. If we denote by $\tilde{\pi}$ the projection $\mathcal{H}(\Delta, B(K)) \to \mathcal{H}(S, B(K))$, $(f_1, \ldots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset$ $\subset \operatorname{Ker} \tilde{\pi}$. Therefore, because π is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \to \mathcal{H}(S, B(K))$, such that the diagram

$$\begin{array}{ccc} \mathcal{O}(\Delta \times V) & \xrightarrow{\phi} & \mathscr{H}(\Delta, B(K)) \\ \pi\downarrow & & \downarrow\tilde{\pi} \\ \mathcal{O}(S \times V) & \xrightarrow{\phi} & \mathscr{H}(S, B(K)) \end{array}$$

is commutative; ϕ is evidently an $\mathcal{O}_S$-algebra homomorphism.

§ 2. *The flatness and privilege theorem*

Notation

Let S be an analytic space, U an open set in $\mathbf{C}^n$, and $\pi : S\times U \to S$ the first projection.

If $\mathscr{F}$ is an $\mathcal{O}_{S\times U}$ module, then for every $s \in S$ we denote by $\mathscr{F}(s)$ the $\mathcal{O}_U$-module $i_s^* \mathscr{F}$, where i_s is the injective morphism $x \to (s, x)$ from U into $S\times U$. If $x \in U$

$$(\mathscr{F}(s))_x \simeq \mathscr{F}_{(s,x)}/m_s \cdot \mathscr{F}_{(s,x)} \simeq \mathscr{F}_{(s,x)} \otimes_{\mathcal{O}_{S,s}} \mathbf{C}_s .$$

Theorem 1: Let $\mathscr{E}$ be a coherent and S-flat $\mathcal{O}_{S\times U}$-module, and K a polycylinder in U.

(*a*) When K is privileged for $\mathscr{E}(s_0)$, s_0 has a neighbourhood V such that K is $\mathscr{E}(s)$-privileged for each $s \in V$. In other words: the set $S' = \{s \in S \mid K$ is $\mathscr{E}(s)$-privileged$\}$ is open in S.

(*b*) It is possible to define a Banach vector bundle over S' whose fibre at any $s \in S'$ is $B(K, \mathscr{E}(s))$.

To prove the theorem we need:

Lemma 1: Under the conditions of the theorem, we can, for every $s \in S$, find a neighbourhood W of $\{s\}\times K$ and a free resolution of finite length

$$0 \to \mathscr{L}_p \xrightarrow{d_p} \ldots \xrightarrow{d_2} \mathscr{L}_1 \xrightarrow{d_1} \mathscr{L}_0 \xrightarrow{\varepsilon} \mathscr{E} \to 0 \text{ in } W.$$

Proof: Let (s, x) be a point of $S\times U$ and $\mathscr{L}^0_*$ a finite resolution of $\mathscr{F}(x)$ in a neighbourhood of x (there exists such one, by the theorem of syzygies). We shall show that that there exists a resolutin $\mathscr{L}^*$ of $\mathscr{F}$ in a neighbourhood of (s, x) such that $\mathscr{L}^*(s) = \mathscr{L}^0_*$; if $\mathscr{L}^0_i = \mathcal{O}_x^{r_i}$ define

$$\mathscr{L}_i = \mathcal{O}^{r_i}_{S\times U} \text{ and } \mathscr{K}^0_i = \operatorname{Ker} d^0_i \colon \mathscr{L}^0_i \to \mathscr{L}^0_{i-1}.$$

We shall construct by induction (with respect to i) $d_i : \mathscr{L}_1 \to \mathscr{L}_{i-1}$ in a neighbourhood of (s, x) such that $d_i(s) = d_i^0$, and prove that $\mathscr{K}_i = \operatorname{Ker} d_i$ is S-flat and that $\mathscr{K}_i(s) = \mathscr{K}_i^0$.

$$\begin{array}{ccc} \mathscr{L}_{i+1} & \xrightarrow{d_{i+1}} & \mathscr{K}_i \\ \downarrow & & \downarrow \\ \mathscr{L}^0_{i+1} & \xrightarrow{d_{i+1}} & \mathscr{K}^0_i \end{array}$$

Suppose that we have constructed d_i and proved the properties for $\mathscr{K}_i$. We can construct $d_{i+1} : \mathscr{L}_{i+1} \to \mathscr{L}_i$ in a neighbourhood of (s, x) such that the diagram is commutative.

Nakayama's lemma shows that $\operatorname{Im} d_{i+1} = \mathscr{K}_i$ at the point (s, x), therefore in a neighbourhood of that point.

The exact sequence

$$0 \to \mathscr{K}_{i+1} \to \mathscr{L}_{i+1} \to \mathscr{K}_i \to 0 ,$$

where $\mathscr{K}_i$ and $\mathscr{L}_{i+1}$ are S-flat, shows that $\mathscr{K}_{i+1}$ is S-flat, and that $\mathscr{K}_{i+1}(s) = \mathscr{K}^0_{i+1}$. The first step of the induction is analogous.

Proof of the theorem: Let $s_0 \in S$ and

$$0 \to \mathscr{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_1} \mathscr{L}_0 \to \mathscr{E} | W \to 0$$

be a free $\mathscr{O}_{S \times U}$ resolution of $\mathscr{E}$ in a neighbourhood $W = V_1 \times V_2$ of $\{s_0\} \times K$. The sheaf $\mathscr{E}$ is $\mathscr{O}_S$-flat, so for each $s \in V_1$, the sequence

$$0 \to \mathscr{L}_p \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \dots \to \mathscr{L}_1 \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \mathscr{L}_0 \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \mathscr{E}_{|W} \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to 0$$

is exact. So the sequence

$$(A) \qquad 0 \to \mathscr{L}_p(s) \xrightarrow{d_p(s)} \dots \to \mathscr{L}_1(s) \xrightarrow{d_1(s)} \mathscr{L}_0(s) \xrightarrow{\varepsilon(s)} \mathscr{E}(s)_{|V_2} \to 0$$

is exact when $s \in V_1$. Now $\mathscr{L}_i(s) \simeq \mathscr{O}_{V_2}^{r_i}$ $(0 \leqq i \leqq p)$ and every $d_i(s)$ induces a continuous linear map:

$B(K, \mathscr{L}_i(s)) \to B(K, \mathscr{L}_{i-1}(s))$, which we also denote by $d_i(s)$. We can consider $d_i = (d_{ijk})$ as an $r_i \times r_{i-1}$-matrix with entries from $\mathscr{O}_{S \times U}(W)$.

By § 1 we have a $\mathscr{O}_S$-algebra homomorphism

$$\mathscr{O}_{S \times W}(S \times W) \to \mathscr{H}(S, B(K)) .$$

From the matrix (d_{ijk}) we get by this homomorphism a morphism $\tilde{d}_i$:

$$V_0 \to \mathscr{L}(B(K)^{ri}, B(K)^{ri-1}) = \mathscr{L}(B(K, \mathscr{L}_i(s)), B(K, \mathscr{L}_{i-1}(s)) .$$

(Here V_0 is some neighbourhood of s_0) such that $\tilde{d}_i(s) = d_i(s)$ for each $s \in V_0$. In other words we have a sequence of Banach vector bundle morphisms

$$(B) \qquad 0 \to B(K, \mathscr{L}_p) \xrightarrow{d_p} \dots \xrightarrow{\tilde{d}_1} B(K, \mathscr{L}_0).$$

Using the fact that $\mathcal{O}_{S\times U}(S\times U) \to \mathscr{H}(S, B(K))$ is an $\mathcal{O}_S$-algebra homomorphism, it easily follows that (B) is complex of Banach vector bundles over S.

Now K is $\mathscr{E}(s_0)$-privileged, thus

$$0 \to B(K, \mathscr{L}_p(s_0)) \xrightarrow{d_p(s_0)} \dots \xrightarrow{d_1(s_0)} B(K, \mathscr{L}_0(s_0))$$

is split exact, so by theorem III.1

$$0 \to B(K, \mathscr{L}_p)_{|V} \xrightarrow{\tilde{d}_p|V} \dots \xrightarrow{\tilde{d}_i|V} B(K, \mathscr{L}_0)_{|V}$$

is split exact for some neighbourhood V of s_0.

Because $\tilde{d}_i(s) = d_i(s)$ and the sequence (A) is exact part (a) of the theorem follows.

(b) $B(K. \mathscr{L}_0)|V$ splits as the direct sum of im $\tilde{d}_1$ and a bundle E_V, such that $E_{V,s} \simeq B(K, \mathscr{E}(s))$, for each $s \in V$. We must show that these bundle structures fit together globally.

Suppose therefore that V is open in S' and that

$$0 \to \mathscr{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathscr{L}_1 \xrightarrow{d_1} \mathscr{L}_0 \to \mathscr{E}_{|V\times V_2} \to 0$$

$$0 \to \mathscr{L}'_p \xrightarrow{d'_p} \dots \xrightarrow{d'_2} \mathscr{L}'_1 \xrightarrow{d'_1} \mathscr{L}'_0 \to \mathscr{E}_{|V\times V_2} \to 0$$

are free resolutions of ξ over $V \times V_2$.

If V_1, V_2 are open polycylinders, we can find an $\mathcal{O}_{S\times U}$-homomorphism $\phi_0 : \mathscr{L}_0 \to \mathscr{L}'_0$ such that

$$\begin{array}{ccccc} \mathscr{L}'_0 & \xrightarrow{\varepsilon'} & \mathscr{E}_{|V\times V_2} & \to & 0 \\ \phi_0 \uparrow & & \| & & \\ \mathscr{L}_0 & \xrightarrow{\varepsilon} & \mathscr{E}_{|V\times V_2} & \to & 0 \end{array}$$

commutes. ϕ_0 determines a bundle morphism $\tilde{\phi}_0\colon B(K, \mathscr{L}_0)\to B(K, \mathscr{L}'_0)$. $B(K, \mathscr{L}_0)$ (resp. $B(K, \mathscr{L}'_0)$) splits as (im $\tilde{d}_1)\otimes E_V$ [Resp. (im $\tilde{d}'_1)\otimes E'_V$].

Let p' be the projection morphism: $B(K, \mathscr{L}_0)\to E'_V$ with kernel im $\tilde{d}'_1$, and put $\tilde{\phi} = p' \circ \tilde{\phi}_0|E_V$.

The commutative diagram

$$\begin{tikzcd}
B(K, \mathscr{L}_0(s)) \arrow[rrr, "\tilde{\varphi}_0"] \arrow[ddd, "\varepsilon"'] & & & B(K, \mathscr{L}'_0(s')) \arrow[ddl, "p'"'] \arrow[ddd, "\varepsilon'"] \\
& & & \\
& E_{V,s} \arrow[uul] \arrow[r, "\tilde{\varphi}"] \arrow[dl, "\varepsilon", "\simeq \text{alg}"'] & E'_{V,s} \arrow[dr, "\varepsilon'", "\simeq \text{alg}"'] & \\
B(K, \mathscr{E}(s)) \arrow[rrr, leftrightarrow, "\text{Id}"'] & & & B(K, \mathscr{E}(s'))
\end{tikzcd}$$

and the open mapping theorem shows that $\tilde{\phi}(s)$ is an isomorphism of Banach spaces for each $s\in V$, so $\tilde{\phi}\colon E_V\to E'_V$ is a bundle isomorphism. We also notice that $\tilde{\phi}$ depends only on the choice of splittings in $B(K, \mathscr{L}_0)$ and $B(K, \mathscr{L}'_0)$, and not on the choice of $\tilde{\phi}_0$. This ends the proof of the theorem.

Remark: Consider the general situation where X and S are analytic spaces, and $\pi : X\to S$ is a morphism, $\mathscr{E}$ an $\mathscr{O}_X$-module. To study the local dependence of $\mathscr{E}$ on S, one can imbed an open set X' in X in the open set $U\subset \mathbf{C}^n$. The morphism $\phi\colon X'\to U$, $\pi\colon X'\to S$ determine the imbedding $\pi\times\phi\colon X'\to S\times U$ such that the diagram

$$\begin{tikzcd}
X \arrow[rr, "\pi\times\phi"] \arrow[dr] & & S\times U \arrow[dl] \\
& S &
\end{tikzcd}$$

commutes. $\mathscr{E}$ can be extended by zero into a sheaf $\mathscr{E}'$ over $U\times S$. Obviously this sheaf $\mathscr{E}'$ is S-flat iff $\mathscr{E}$ is S-flat.

Therefore theorem 1 makes clear also this general situation.

Corollary: If $\pi : X\to S$ is a morphism and $\mathscr{E}$ a coherent $\mathscr{O}_X$-module. Then $\pi \mid \text{Supp}(\mathscr{E})$ is an open map.

Proof: Suppose as above that X is imbedded in $S\times U$, and $\mathscr{E}$ in extended by zero to $S\times U$. Let $x_0\in \text{Supp}\,\mathscr{E}$, and V be a neighbourhood of x_0 in $S\times U$. Let $s_0 = \pi(x_0)$ and choose an $\mathscr{E}(s_0)$-privileged polycylinder K in U, such that $\{s_0\}\times K\subset V$, over some neighbourhood W of s_0. We have the Banach bundle $B(K, \mathscr{E}|\pi^{-1}(W))$, whose fiber over s is $B(K, \mathscr{E}(s))$. Since $x_0\in \text{Supp}\,\mathscr{E}(s_0)$ and K is a neighbourhood of x_0, $B(K; \mathscr{E}(s_0)) \neq 0$. As all the fibers are isomorphic, then for all $s\in U$, $B(K; \mathscr{E}(s)) \neq 0$ and therefore $\{s\}\times K\cap \text{Supp}\,\mathscr{E} \neq 0$, and $s\in\pi(\text{Supp}\,\mathscr{E})$. This proves that π is open.