

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 14 (1968)  
**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** FLATNESS AND PRIVILEGE  
**Autor:** Douady, A.  
**Kapitel:** §1. Banach vector bundles over an analytic space  
**DOI:** <https://doi.org/10.5169/seals-42343>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

*Remark :* This a particular case of the following proposition: if  $\pi$  and  $\pi'$  are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & \xrightarrow{\quad Y \quad} & \\ \pi \searrow & \swarrow \pi' & \\ S & & \end{array}, \quad \mathcal{O}_{X \times Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

We have proved that  $\mathcal{O}_{W \times X}$  is  $\mathcal{O}_W$ -flat, so by scalar extension  $\mathcal{O}_{S \times X}$  is  $\mathcal{O}_S$  flat.

*Corollary :* If  $X$  and  $S$  are two manifolds and  $\pi : X \rightarrow S$  is a submersion, then  $\pi$  is flat.

### III. PRIVILEGED POLYCYLINDERS

#### § 1. Banach vector bundles over an analytic space

Let  $E$  be a Banach space and  $X$  an analytic space. We denote then by  $E_X$  the trivial bundle  $X \times E$  over  $X$ .

To define bundle morphisms, we first define the sheaf  $\mathcal{H}_X(E)$  of germs of analytic morphisms from  $X$  to  $E$ . If  $U \subset \mathbb{C}^n$  is open, then the set  $\mathcal{H}(U, E)$  of analytic morphisms from  $U$  into  $E$  consists of all functions  $g : U \rightarrow E$  having at every point  $x \in U$  a converging power series expansion.

Let now  $X'$  be a local model for  $X$ , i.e.  $X'$  is the support of the quotient sheaf  $\mathcal{O}_U/J$ , where  $U \subset \mathbb{C}^n$  is open and  $J$  is a coherent sheaf of ideals of  $\mathcal{O}_U$ , then  $\mathcal{H}_{X'}(E)$  is the sheaf associated to the presheaf  $V \mapsto \mathcal{H}(V, E)/J_V \cdot \mathcal{H}(V, E)$  ( $V \subset U$ ,  $V$ -open).

*Remark :* If  $X'$  is reduced, the sections of  $\mathcal{H}_{X'}(E)$  are just the functions from  $X'$  to  $E$  which are locally induced by analytic functions on open sets in  $U$ .

The sheaf  $\mathcal{H}_X(E)$  is constructed with help of the local models  $X'$  of  $X$ , i.e.  $\mathcal{H}_X(E)|X' = \mathcal{H}_{X'}(E)$ , for every local model  $X'$ .

*Definition 1 :* The set of *analytic morphisms* from an analytic space  $X$  into a Banach space  $E$  is the set  $\mathcal{H}(X; E)$  of sections of the sheaf  $\mathcal{H}_X(E)$ .

Let  $\mathcal{L}(E, F)$  be the Banach space of all continuous linear mappings from the Banach space  $E$  into the Banach space  $F$ .

*Definition 2 :* An analytic vector bundle morphism from  $E_X$  into  $F_X$  is an analytic morphism from  $X$  into  $\mathcal{L}(E, F)$ .

Let  $E$  be a topological space,  $X$  an analytic space, and  $\pi : E \rightarrow X$  a continuous projection.

$$\begin{array}{ccccc} & & \Phi_\iota & & \\ E & \longleftrightarrow & E|_{U_\iota} & \xrightarrow{\Phi_\iota} & E_{\iota U_\iota} \\ \pi \downarrow & & \searrow & & \swarrow \\ X & \longleftrightarrow & U_\iota & & \end{array}$$

Suppose that  $X$  has an open covering  $(U_\iota)_{\iota \in I}$ , and that for every  $\iota \in I$  there is given a trivial Banach space bundle  $E_{\iota U_\iota}$  and a homeomorphism  $\phi_\iota$ , such that the following diagram is commutative:

We suppose further that for each pair  $\iota, \kappa \in I$  there is given an analytic vector bundle morphism  $\gamma_{\iota \kappa} : E_{\kappa U_\iota \cap U_\kappa} \rightarrow E_{\iota U_\iota \cap U_\kappa}$ , with the underlying mapping  $\phi_\iota \circ \phi_\kappa^{-1}$ , such that:

$$\gamma_{\iota \lambda} = \gamma_{\iota \kappa} \gamma_{\kappa \lambda}; \quad \gamma_{\iota \iota} = I, \quad \text{for all } \iota, \kappa, \lambda \in I.$$

This data gives a Banach vector bundle atlas on  $E$  and provides  $E$  with the structure of a Banach vector bundle over  $X$  (two atlases are equivalent if there exists an atlas containing both).

*Remark:* If  $X$  is reduced, the  $\gamma_{\iota \kappa}$  are determined by their underlying map and the condition  $\gamma_{\iota \lambda} = \gamma_{\iota \kappa} \gamma_{\kappa \lambda}$  is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

*Proposition 1:* Let  $\phi : E \rightarrow F$  be a morphism of two Banach vector

$$\begin{array}{c} \swarrow \quad \searrow \\ X \end{array}$$

bundles  $E$  and  $F$ , and  $x \in X$ .

If  $\phi_x \in \mathcal{L}(E(x), F(x))$  is an isomorphism, then there exists an open neighbourhood  $U \subset X$  of  $x$ , such that  $\phi|_U : E|_U \rightarrow F|_U$  is a vector bundle isomorphism.

*Proof:* First we take a trivialisation  $E|_V = E_0|_V$ ,  $F|_V = F_0|_V$  at  $x \in V \subset X$  ( $V$ -open).

The set  $\text{Isom}(E_0, F_0)$  of isomorphic mappings is an open subset of  $\mathcal{L}(E_0, F_0)$  and the mapping  $g \mapsto g^{-1}$  is an analytic isomorphism:

$$\text{Isom}(E_0, F_0) \simeq \text{Isom}(F_0, E_0).$$

So we have in an open neighbourhood  $U \subset X$  of  $x$  an analytic morphism  $y \mapsto \phi_y^{-1} \in \mathcal{L}(F_0, E_0)$ , which defines the inverse morphism  $(\phi|_U)^{-1} : F|_U \rightarrow E|_U$ .

*Definition 3 :* Let  $E$  and  $F$  be two Banach spaces and  $f$  a continuous linear mapping from  $E$  into  $F$ .  $f$  is a *split mono-(epi) morphism*, if there exists a mapping  $g \in \mathcal{L}(F, E)$  such that  $g \circ f = I_E$ . (Resp.  $f \circ g = I_F$ .)

*Definirion 4 :* Let  $E_1$  and  $E_2$  be two Banach vector bundles over an analytic space  $X$ , and  $f$  a vector bundle morphism from  $E_1$  into  $E_2$ .  $f$  is a *split mono (epi) morphism*, if there exists a vector bundle morphism  $g : E_2 \rightarrow E_1$  such that  $g \circ f = I_{E_1}$ . (Resp.  $f \circ g = I_{E_2}$ .)

Equivalently,  $f : E_1 \rightarrow E_2$  is a split monomorphism if and only if  $E_2$  can

$$\begin{array}{ccc} & & \\ & \searrow & \swarrow \\ & X & \end{array}$$

be decomposed in a direct sum  $E_2 = F_2 \oplus G_2$  such that

$$f : \begin{cases} E_1 \simeq F_2 \\ 0 \rightarrow G_2 \end{cases} .$$

and  $f$  is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1, \quad \text{such that} \quad f : \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq E_2 \end{cases} .$$

*Proposition 2 :* Let  $E \xrightarrow{\varphi} F$  be a bundle morphism and  $x \in X$ .

$$\begin{array}{ccc} & & \\ & \searrow & \swarrow \\ & X & \end{array}$$

If  $\phi_x : E(x) \rightarrow F(x)$  is a split epi (mono) morphism, then the point  $x$  has an open neighbourhood  $U \subset X$ , such that  $\phi|U : E|U \rightarrow F|U$  is a split vector bundle epi (mono) morphism.

*Proof :* Suppose that  $\phi_x$  is a split epimorphism. We take first a trivialisation  $E|V = E_{0V}$ ,  $F|V = F_{0V}$  at  $x$ , so that there exists a mapping  $\sigma \in \mathcal{L}(F_0, E_0)$ ,  $\phi_x \circ \sigma = I_{F_0}$ . If we define a morphism  $\psi : F_{0V} \rightarrow E_{0V}$  by  $x \rightarrow \sigma \in \mathcal{L}(F_0, E_0)$ , the morphism  $\gamma = \phi \circ \psi : F_{0V} \rightarrow F_{0V}$  has an isomorphic fibre mapping  $\gamma_x = I_{F_{0V}}$  in  $x$ . By proposition 1 we have an isomorphic restriction  $\gamma|U$ ,  $\phi|U \circ (\psi|U)^{-1} = I_{F_{0U}}$ .

When  $\phi_x$  is a split monomorphism, the proof is similar.

*Definition 5 :* Let  $B_1, B_2, B_3$  be Banach spaces, and  $j, k : B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$  continuous linear mappings. This sequence forms a *complex*, if  $k \circ j = 0$ . This sequence is *split exact* if the space  $B_i$  can be decomposed in direct

sums  $B_i = C_i \oplus D_i$  such that

$$j : \begin{cases} C_1 \rightarrow 0 \\ D_1 \simeq C_2 \end{cases} \quad k : \begin{cases} C_2 \rightarrow 0 \\ D_2 \simeq C_3 \end{cases} .$$

*Definition 6:* A Banach vector bundle morphism sequence

$$\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array} \quad \text{is a complex if } g \circ f = 0.$$

The sequence is *split exact*, if every  $E_i$  can be decomposed  $E_i = F_i \oplus G_i$ , such that:

$$f : \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_2 \end{cases} \quad g : \begin{cases} F_2 \rightarrow 0 \\ G_2 \simeq F_3 \end{cases} .$$

*Theorem 1:* Let  $\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$  be a complex of Banach vector

bundles and  $x_0 \in X$ .

If the sequence of Banach spaces  $E_1(x_0) \xrightarrow{f_{x_0}} E_2(x_0) \xrightarrow{g_{x_0}} E_3(x_0)$  is split exact, then there exists an open neighbourhood  $U \subset X$  of  $x_0$ , such that  $E_1|U \xrightarrow{f|U} E_2|U \xrightarrow{g|U} E_3|U$  is a split exact sequence of Banach vector bundles.

*Proof:* We take a neighbourhood  $V$  of  $x_0$ , such that we have a complex  $E_{1V} \rightarrow E_{2V} \rightarrow E_{3V}$  of trivial bundles. By assumption we have the decompositions  $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$  with

$$f_{x_0} : \begin{cases} F_1(x_0) \rightarrow 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \quad g_{x_0} : \begin{cases} F_2(x_0) \rightarrow 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases} .$$

By proposition 2,  $f|V : G_{1V} \rightarrow E_{2V}$ ,  $g|V : G_{2V} \rightarrow E_{3V}$  are both split monomorphisms in a neighbourhood  $W \subset V$  of  $x_0$  and the images  $F_2 = f(G_{1W})$ ,  $F_3 = g(G_{2W})$  are subbundles of  $E_{2W}$  esp.  $E_{3W}$ , such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W}.$$

By our construction

$$g|_W : \begin{cases} F_2 \rightarrow 0 \\ G_2 W \simeq F_3 \end{cases} .$$

If  $p: E_{2W} \rightarrow F_2$  is the projection with kernel  $G_{2W}$ , the map,  $p \circ f: E_{1W} \rightarrow F_2$  is a split epimorphism in  $x_0$ . Again by prop. 2 we have over an open neighbourhood  $U \subset W$  of  $x_0$  a decomposition  $E_{1U} = F_1 \oplus G_{1U}$  (with  $F_1 = \text{Ker } p \circ f$ )

$$(p \circ f)|_U : \begin{cases} F_1 \rightarrow 0 \\ G_{1U} \xrightarrow{\sim} F_{2U} \end{cases} .$$

The image  $f|_U(F_1)$  is contained in  $G_{2U}$ . But  $g|_U \circ f|_U = 0$  and  $g|_{G_{2U}}$  is a monomorphism hence  $f|_U: F_1 \rightarrow 0$ . We get finally (restricting all our morphisms to  $U$ )

$$f|_U : \begin{cases} F_{1U} \rightarrow 0 \\ G_{1U} \simeq F_{2U} \end{cases} \quad g|_U : \begin{cases} F_{2U} \rightarrow 0 \\ G_{2U} \xrightarrow{\sim} F_{3U} \end{cases} .$$

## § 2. Privileged polycylinders

*Definition 1:* A polycylinder in  $\mathbf{C}^n$  is a compact set  $K$  of the form  $K = K_1 \times \dots \times K_n$  where each  $K_i$  is a compact, convex subset of  $\mathbf{C}$ , with nonempty interior. If each  $K_i$  is a disc, then  $K$  is a polydisc. We first recall the following theorem of Cartan.

*Theorem 1:* Let  $K$  be a polycylinder contained in an open subset  $U$  of  $\mathbf{C}^n$ . Let  $\mathcal{F}$  be a coherent analytic sheaf on  $U$ .

- (A) There exists an open neighbourhood of  $K$  over which  $\mathcal{F}$  admits a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 .$$

- (B)  $H^q(K, \mathcal{F}) = 0$  for  $q > 0$ .

(Reference: For instance Gunning and Rossi.)

We have the following consequences of this theorem:

- 1) Given a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$$

of a coherent sheaf  $\mathcal{F}$ , the sequence