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Remark: This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$X \qquad Y \qquad \emptyset_{X \times Y} = \emptyset_X \otimes_{\emptyset_S} \emptyset_y.$$

We have proved that $\mathcal{O}_{W \times X}$ is \mathcal{O}_W -flat, so by scalar extension $\mathcal{O}_{S \times X}$ is \mathcal{O}_S flat.

Corollary: If X and S are two manifolds and $\pi: X \rightarrow S$ is a submersion, then π is flat.

III. PRIVILEGED POLYCYLINDERS

§ 1. Banach vector bundles over an analytic space

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X.

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E. If $U \subset \mathbb{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g: U \to E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X, i.e. X' is the support of the quotient sheaf \mathcal{O}_U/J , where $U \subset \mathbb{C}^n$ is open and J is a coherent sheaf of ideals of \mathcal{O}_U , then $\mathscr{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \to \mathscr{H}(V, E)/J_V \cdot \mathscr{H}(V, E)$ $(V \subset U, V\text{-open})$.

Remark: If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U.

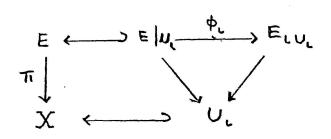
The sheaf $\mathscr{H}_X(E)$ is constructed with help of the local models X' of X, i.e. $\mathscr{H}_X(E)|X'=\mathscr{H}_{X'}(E)$, for every local model X'.

Definition 1: The set of analytic morphisms from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F.

Definition 2: An analytic vector bundle morphism from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.

Let E be a topological space, X an analytic space, and $\pi: E \rightarrow X$ a continuous projection.



Suppose that X has an open covering $(U_{\iota})_{\iota \in I}$, and that for every $\iota \in I$ there is given a trivial Banach space bundle $E_{\iota U_{\iota}}$ and a homeomosphism ϕ_{ι} , such that the following diagram is commutative:

We suppose further that for each pair ι , $\kappa \in I$ there is given an analytic vector bundle morphism $\gamma_{\iota\kappa}: E_{\kappa U_{\iota} \cap U_{\kappa}} \to E_{\iota U_{\iota} \cap U_{\kappa}}$, with the underlying mapping $\phi_{\iota} \circ \phi_{\kappa}^{-1}$, such that:

$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \, \gamma_{\kappa\lambda}; \quad \gamma \iota_{\iota} = I, \quad \text{ for all } \quad \iota, \kappa, \gamma \in I.$$

This data gives a Banach vector bundle atlas on E and provides E with the structure of a Banach vector bundle over X (two atlases are equivalent if there exists an atlas containing both).

Remark: If X is reduced, the $\gamma_{\iota\kappa}$ are determined by their underlying map and the condition $\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}$ is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

Proposition 1: Let $\phi: E \to F$ be a morphism of two Banach vector

bundles E and F, and $x \in X$.

If $\phi_x \in \mathcal{L}(E(x), F(x))$ is an isomorphism, then there exists an open neighbourhood $U \subset X$ of x, such that $\phi | U : E | U \to F | U$ is a vector bundle isomorphism.

Proof: First we take a trivialisation $E|V = E_{0V}$, $F|V = F_{0V}$ at $x \in V \subset X$ (V-open).

The set Isom (E_0, F_0) of isomorphic mappings is an open subset of $\mathcal{L}(E_0, F_0)$ and the mapping $g \rightarrow q^{-1}$ is an analytic isomorphism:

Isom
$$(E_0, F_0) \simeq \text{Isom } (F_0, E_0)$$
.

So we have in an open neighbourhood $U \subset X$ of x an analytic morphism $y \to \phi_y^{-1} \in \mathcal{L}(F_0, E_0)$, which defines the inverse morphism $(\phi | U)^{-1} : F | U \to E | U$.

Definition 3: Let E and F be two Banach spaces and f a continuous linear mapping from E into F. f is a split mono-(epi) morphism, if there exists a mapping $g \in \mathcal{L}(F, E)$ such that $g \circ f = I_E$. (Resp. $f \circ g = I_F$.)

Definition 4: Let E_1 and E_2 be two Banach vector bundles over an analytic space X, and f a vector bundle morphism from E_1 into E_2 . f is a split mono (epi) morphism, if there exists a vector bundle morphism $g: E_2 \rightarrow E_1$ such that $g \circ f = I_{E_1}$. (Resp. $f \circ g = I_{E_2}$.)

Equivalently, $f: E_1 \to E_2$ is a split monomorphism if an only if E_2 can

be decomposed in a direct sum $E_2 = F_2 \oplus G_2$ such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \to G_2 \end{cases}.$$

and f is a split epimorphism if correspondingly

$$E_1 \,=\, F_1 \oplus G_1 \;, \quad \text{such that} \quad f\!:\! \left\{ \begin{array}{l} F_1 \,\to\, 0 \\ \\ G_1 \,\simeq\! E_2 \end{array} \right. .$$

Proposition 2: Let $E \stackrel{\varphi}{\to} F$ be a bundle morphism and $x \in X$.



If $\phi_x : E(x) \to F(x)$ is a split epi (mono) morphism, then the point x has an open neighbourhood $U \subset X$, such that $\phi | U : E | U \to F | U$ is a split vector bundle epi (mono) morphism.

Proof: Suppose that ϕ_x is a split epimorphism. We take first a trivilisation $E|V=E_{0V},F|V=F_{0V}$ at x, so that there exists a mapping $\sigma\in\mathscr{L}(F_0,E_0)$, $\phi_x\circ\sigma=I_{F_0}$. If we define a morphism $\psi:F_{0V}\to E_{0V}$ by $x\to\sigma\in\mathscr{L}(F_0,E_0)$, the morphism $\gamma=\phi\circ\psi:F_{0V}\to F_{0V}$ has an isomorphic fibre mapping $\gamma_x=I_{F_0}$ in x. By proposition 1 we have an isomorphic restriction $\gamma|U,\phi|U\circ(\psi|U\circ(\gamma|U)^{-1})=I_{F_{0U}}$.

When ϕ_x is a split monomorphism, the proof is similar.

Definition 5: Let B_1 , B_2 , B_3 be Banach spaces, and $j, k : B_1 \rightarrow B_2 \rightarrow B_3$ continuous linear mappings. This sequence forms a complex, if $k \circ j = 0$. This sequence is *split exact* if the space B_i can be decomposed in direct

sums $B_i = C_i \oplus D_i$ such that

$$j: \begin{cases} C_1 \to 0 \\ D_1 \simeq C_2 \end{cases} \qquad k: \begin{cases} C_2 \to 0 \\ D_2 \simeq C_3 \end{cases}.$$

Definition 6: A Banach vector bundle morphism sequence

$$E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$$
 is a complex if $g \circ f = 0$.

The sequence is *split exact*, if every E_i can be decomposed $E_i = F_i \oplus G_i$, such that:

$$f \colon \begin{cases} F_1 \to 0 \\ G_1 \simeq F_2 \end{cases} \qquad g \colon \begin{cases} F_2 \to 0 \\ G_2 \simeq F_3 \end{cases}.$$

Theorem 1: Let $E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$ be a complex of Banach vector

bundles and $x_0 \in X$.

If the sequence of Banach spaces $E_1(x_0) \stackrel{fx_0}{\to} E_2(x_0) \stackrel{fx_0}{\to} E_3(x_0)$ is split exact, then there exists an open neighbourhood $U \subset X$ of x_0 , such that $E_1 U \rightarrow E_2 U \rightarrow E_3 U$ is a split exact sequence of Banach vector bundles.

Proof: We take a neighbourhood V of x, such that we have a complex $f \mid V g \mid V$ $E_{1V} \rightarrow E_{2V} \rightarrow E_{3V}$ of trivial bundles. By assumption we have the decompositions $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$ with

$$f_{x_0} : \begin{cases} F_1(x_0) \to 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \qquad g_{x_0} : \begin{cases} F_2(x_0) \to 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases}.$$

By proposition $2, f|V: G_{1V} \to E_{2V}, g|V: G_{2V} \to E_{3V}$ are both split monomorphisms in a neighbourhood $W \subset V$ of x_0 and the images $F_2 = f(G_{1W})$, $F_3 = g(G_{2W})$ are subbundles of E_{2W} esp. E_{3W} , such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W}.$$

By our construction

$$g \mid W : \begin{cases} F_2 & \to 0 \\ G_2 & W \simeq F_3 \end{cases}.$$

If $p: E_{2W} \to F_2$ is the projection with kernel G_{2W} , the map, $p \circ f: E_{1W} \to F_2$ is a split epimorphism in x_0 . Again by prop. 2 we have over an open eighbourhood $U \subset W$ of x_0 a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \text{Ker p} \circ f$)

$$(p \circ f) \mid U : \begin{cases} F_1 \to 0 \\ & \\ G_{1U} \to F_{2U} \end{cases}.$$

The image $f | U(F_1)$ is contained in G_{2U} . But $g | U \circ f | U = 0$ and $g | G_{2U}$ is a monomorphism hence $f | U : F_1 \rightarrow 0$. We get finally (restricting all our morphisms to U)

$$f \mid U : \begin{cases} F_{1U} \to 0 \\ G_{1U} \simeq F_{2U} \end{cases} \qquad g \mid U : \begin{cases} F_{2U} \to 0 \\ \tilde{G}_{2U} \to F_{3U} \end{cases}.$$

§ 2. Privileged polycylinders

Definition 1: A polycylinder in \mathbb{C}^n is a compact set K of the form $K = K_1 \times ... \times K_n$ where each K_i is a compact, convex subset of \mathbb{C} , with nonempty interior. If each K_i is a disc, then K is a polydisc. We first recall the following theorem of Cartan.

Theorem 1: Let K be a polycylinder contained in an open subset U of \mathbb{C}^n . Let \mathscr{F} be a coherent analytic sheaf on U.

(A) There exists an open neighbourhood of K over which \mathcal{F} admits a finite free resolution

$$0 \to \mathcal{L}_n \to \dots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0 \ .$$

- (B) $H^q(K, \mathcal{F}) = 0$ for q > 0. (Reference: For instance Gunning and Rossi.) We have the following consequences of this theorem:
- 1) Given a finite free resolution

$$0 \to \mathcal{L}_n \to \dots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

of a coherent sheaf \mathcal{F} , the sequence

$$0 \to \mathcal{L}_n(K) \to \dots \to \mathcal{L}_0(K) \to \mathcal{F}(K) \to 0$$

is an $\mathcal{O}_{U}(K)$ - free resolution of $\mathscr{F}(K)$.

2) Given a short exact sequence of coherent sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

then the sequence

$$0 \to \mathcal{F}_{L}(K) \to \mathcal{F}(K) \to \mathcal{F}''(K) \to 0$$
 is exact.

Let \mathscr{F} be a coherent analytic sheaf on U, and let $K \subset U$ be a polycylinder If V is an open neighbourhood of K, then $\mathscr{F}(V)$ can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give $\mathcal{F}(K)$ the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from $\mathcal{F}(K)$ and by choosing K in a "privileged" way.

Let $B(K) = \{f : K \rightarrow \mathbb{C} | f \text{ continuous on } K \text{ and analytic on } \mathring{K} \}$, then B(K) is Banach algebra and $B(K) \subset C(K)$. The sections of \mathcal{O}_U over K are elements of B(K), and B(K) is in fact the uniform closure of $\mathcal{O}_U(K)$ in C(K).

If $\mathcal{L} = \mathcal{O}_U^r$, we define $B(K, \mathcal{L}) = B(K)^r$. Then $B(K; \mathcal{L})$ is a free B(K)-module, and since $\mathcal{L}(K) = \mathcal{O}_U(K)^r$, we have $B(K; \mathcal{L}) = B(K) \otimes \mathcal{L}(K)$.

We now assume that \mathscr{F} is a coherent sheaf on U, where $U \subset \mathbb{C}^n$ is open. Consider a free resolution

$$(R) 0 \to \mathcal{L}_n \to \dots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0 \text{of } \mathcal{F}.$$

From (R) we get an $\mathcal{O}_U(K)$ -free resolution of $\mathscr{F}(K)$

$$(R') 0 \to \mathcal{L}_n(K) \to \dots \to \to_1(K) \to \mathcal{L}_0(K) \to \mathcal{F}(K) \to 0.$$

Taking the tensorproduct $B(K) \otimes_{\mathcal{O}_{I}(K)}$ we get the complex

$$B(K; \mathcal{L}_{\cdot}): 0 \rightarrow B(K; \mathcal{L}_{n}) \rightarrow \dots \rightarrow B(K; \mathcal{L}_{1}) \rightarrow B(K; \mathcal{L}_{0}).$$

Definition 2: The polycylinder K is called \mathscr{F} -privileged if the complex $B(K; \mathscr{L})$ is split-exact in every degree >0.

Remark: The property of being \mathcal{F} -privileged is independent of the resolution (R).

The exactnes of $B(K; \mathcal{L})$ can be expressed by $\operatorname{Tor}_{i}^{\mathfrak{O}(K)}(B(K), \mathcal{F}(K)) = 0$, for every i > 0, and Tor is independent of the resolution (R). It is a little

more complicated to show, that the splitting property is independent of (R), and this is omitted.

Since $B(K; \mathcal{L}_i)$ is a Banach space, the image and its complement are thus Banach spaces if K is \mathcal{F} -privileged. In this case we define $B(K; \mathcal{F}) = \operatorname{Coker}(B(K, \mathcal{L}_1) \to B(K; \mathcal{L}_0)) = B(K) \otimes_{\mathcal{O}} \mathcal{F}(K)$ and we get a B(K)-module, which is a Banach-space.

Warning: In the definition of split-exactnes, the subspaces are splitting vector spaces, but they are not splitting B(K)-modules in general.

We have the following important theorem about the existence of privileged polycylinders:

Theorem 2: Let U be an open subset of \mathbb{C}^n , and let \mathscr{F} be a coherent analytic sheaf on U. For any $x \in U$ there exists a fundamental system of neighbourhoods of x in U, which are \mathscr{F} -privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

Example: (Curves in \mathbb{C}^2) Let $U \subset \mathbb{C}^2$ be an open connected neighbour hood of the origin, and let $h: U \to \mathbb{C}$ be analytic and $h \neq 0$.

Let X be the curve given by h, that is $X = h^{-1}(0)$, $\mathcal{O}_X = \mathcal{O}_U/(h)$. We have an exact sequence $0 \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_X \rightarrow 0$. Consider a polycylinder $K = K_1 \times K_2 \subset U$. By definition K is \mathcal{O}_X -priviledged if and only if $h: B(K) \rightarrow B(K)$ is a split monomorphism.

Let K_j denote the boundary of K_j , and define $K = K_1 \times K_2$ (K is called the Silov Boundary of K).

Proposition 1: (a) The following conditions are equivalent:

- (i) $h: B(K) \rightarrow B(K)$ is a monomorphism.
- (i') $\exists a > 0$ such that $||hf|| \ge a||f||$, $\forall f \in B(K)$.
- (ii) $X \cap K = \emptyset$.
- (b) If $(K_1 \times K_2) \cap X = \emptyset$, then h is a split monomorphism (i.e. K is \mathcal{O}_X privileged).

Proof: (a) (i) \Leftrightarrow (i') is a well known fact from the theory of normed vector spaces.

(ii) \Rightarrow (i'). Assume $X \cap K = \emptyset$. If $f \in B(K)$, then it follows from the maximum principle that $||f|| = \sup_{K} |f(x)| = \sup_{K} |f(x)|$. Since $h(x) \neq 0$

whenever $x \in K$, we get $a = \inf_{K} |h(x)| > 0$. Hence $||hf|| = \sup_{K} |hf(x)| \ge 2$ $\ge a \sup_{K} |f(x)| = a ||f||$.

(i') \Rightarrow (ii). Suppose that $X \cap K \neq \emptyset$ and $x = (x_1, x_2) \in X \cap K$. We choose an analytic function $f_1: U_1 \to \mathbb{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2: U_2 \to \mathbb{C}$, with the same properties. Consider the function $f \in B(K): (z_1, z_2) \to f_1(z_1) f_2(z_2)$. Since h(x) = 0 it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in K.

Applying Dini's theorem we get $||hf^n|| \to 0$. From the inequality $a||f^n|| \le$ $\le ||hf^n||$ we get $||f^n|| \to 0$, which is a contradiction, because for every $n: f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h: B(K) \rightarrow B(K)$ is a split monomorphism?

IV. FLATNESS AND PRIVILEGE

§ 1. Morphisms from an analytic space into B(K)

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbb{C}^n$. We want to construct an \mathcal{O}_S -algebra homomorphism $\phi : \mathcal{O}_{S \times U} (S \times U) \to \mathcal{H} (S; B(K))$.

- (a) Consider first $S = U' \subset \mathbb{C}^m$, U'-open. If $h \in \mathcal{O}_{U' \times U}$ ($U' \times U$) and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s,x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand its obvious that ϕ is an $\mathcal{O}_{U'}$ -algebra homomorphism.
- (b) Let S have a special model in the polydisc Δ in \mathbb{C}^m , defined by a sheaf \mathscr{J} of ideals of \mathscr{O}_{Δ} , and let \mathscr{J} be generated by $f_1, ..., f_p$, V-a polycylinder neighbourhood of K in U. By Cartan's theorem B for a polycylinder,

the sequence $0 \rightarrow \mathcal{J}(\Delta \times V) \rightarrow \mathcal{O}(\Delta \times V) \rightarrow \mathcal{O}(S \times V) \rightarrow 0$ is exact. If we denote by π the projection $\mathcal{H}(\Delta, B(K)) \rightarrow \mathcal{H}(S, B(K)), (f_1, ..., f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset$

 \subset Ker π . Therefore, because π is surjection, there exists a unique

 $\phi: \mathcal{O}(S \times V) \rightarrow \mathcal{H}(S, B(K))$, such that the diagram