

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FLATNESS AND PRIVILEGE
Autor: Douady, A.
Kapitel: III. Privileged polycylinders
DOI: <https://doi.org/10.5169/seals-42343>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Remark : This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ \pi \searrow & & \swarrow \pi' \\ & S & \end{array} \quad \mathcal{O}_{X \times_S Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

We have proved that $\mathcal{O}_{W \times X}$ is \mathcal{O}_W -flat, so by scalar extension $\mathcal{O}_{S \times X}$ is \mathcal{O}_S flat.

Corollary : If X and S are two manifolds and $\pi : X \rightarrow S$ is a submersion, then π is flat.

III. PRIVILEGED POLYCYLINDERS

§ 1. Banach vector bundles over an analytic space

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X .

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E . If $U \subset \mathbb{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g : U \rightarrow E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X , i.e. X' is the support of the quotient sheaf \mathcal{O}_U/J , where $U \subset \mathbb{C}^n$ is open and J is a coherent sheaf of ideals of \mathcal{O}_U , then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \rightarrow \mathcal{H}(V, E)/J_V \cdot \mathcal{H}(V, E)$ ($V \subset U$, V -open).

Remark : If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U .

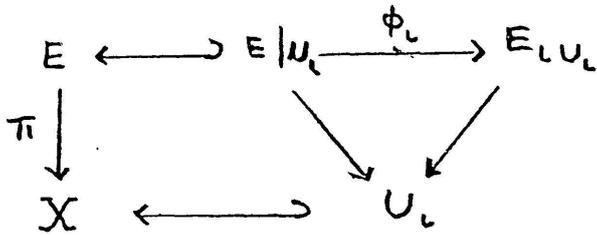
The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models X' of X , i.e. $\mathcal{H}_X(E)|_{X'} = \mathcal{H}_{X'}(E)$, for every local model X' .

Definition 1 : The set of *analytic morphisms* from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F .

Definition 2 : An *analytic vector bundle morphism* from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.

Let E be a topological space, X an analytic space, and $\pi : E \rightarrow X$ a continuous projection.



Suppose that X has an open covering $(U_\iota)_{\iota \in I}$, and that for every $\iota \in I$ there is given a trivial Banach space bundle $E|_{U_\iota}$ and a homeomorphism ϕ_ι , such that the following diagram is commutative:

We suppose further that for each pair $\iota, \kappa \in I$ there is given an analytic vector bundle morphism $\gamma_{\iota\kappa} : E|_{U_\iota \cap U_\kappa} \rightarrow E|_{U_\iota \cap U_\kappa}$, with the underlying mapping $\phi_\iota \circ \phi_\kappa^{-1}$, such that:

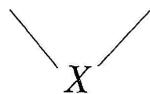
$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}; \quad \gamma_{\iota\iota} = I, \quad \text{for all } \iota, \kappa, \lambda \in I.$$

This data gives a Banach vector bundle atlas on E and provides E with the structure of a Banach vector bundle over X (two atlases are equivalent if there exists an atlas containing both).

Remark: If X is reduced, the $\gamma_{\iota\kappa}$ are determined by their underlying map and the condition $\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}$ is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

Proposition 1: Let $\phi : E \rightarrow F$ be a morphism of two Banach vector



bundles E and F , and $x \in X$.

If $\phi_x \in \mathcal{L}(E(x), F(x))$ is an isomorphism, then there exists an open neighbourhood $U \subset X$ of x , such that $\phi|_U : E|_U \rightarrow F|_U$ is a vector bundle isomorphism.

Proof: First we take a trivialisation $E|_V = E_0|_V, F|_V = F_0|_V$ at $x \in V \subset X$ (V -open).

The set $\text{Isom}(E_0, F_0)$ of isomorphic mappings is an open subset of $\mathcal{L}(E_0, F_0)$ and the mapping $g \rightarrow g^{-1}$ is an analytic isomorphism:

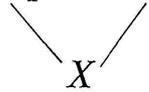
$$\text{Isom}(E_0, F_0) \simeq \text{Isom}(F_0, E_0).$$

So we have in an open neighbourhood $U \subset X$ of x an analytic morphism $y \rightarrow \phi_y^{-1} \in \mathcal{L}(F_0, E_0)$, which defines the inverse morphism $(\phi|_U)^{-1} : F|_U \rightarrow E|_U$.

Definition 3 : Let E and F be two Banach spaces and f a continuous linear mapping from E into F . f is a *split mono-(epi) morphism*, if there exists a mapping $g \in \mathcal{L}(F, E)$ such that $g \circ f = I_E$. (Resp. $f \circ g = I_F$.)

Definirion 4 : Let E_1 and E_2 be two Banach vector bundles over an analytic space X , and f a vector bundle morphism from E_1 into E_2 . f is a *split mono (epi) morphism*, if there exists a vector bundle morphism $g : E_2 \rightarrow E_1$ such that $g \circ f = I_{E_1}$. (Resp. $f \circ g = I_{E_2}$.)

Equivalently, $f : E_1 \rightarrow E_2$ is a split monomorphism if and only if E_2 can



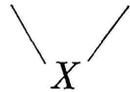
be decomposed in a direct sum $E_2 = F_2 \oplus G_2$ such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \rightarrow G_2 \end{cases}.$$

and f is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1, \quad \text{such that } f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq E_2 \end{cases}.$$

Proposition 2 : Let $E \xrightarrow{\phi} F$ be a bundle morphism and $x \in X$.



If $\phi_x : E(x) \rightarrow F(x)$ is a split epi (mono) morphism, then the point x has an open neighbourhood $U \subset X$, such that $\phi|_U : E|_U \rightarrow F|_U$ is a split vector bundle epi (mono) morphism.

Proof : Suppose that ϕ_x is a split epimorphism. We take first a trivilisation $E|_V = E_{0V}, F|_V = F_{0V}$ at x , so that there exists a mapping $\sigma \in \mathcal{L}(F_0, E_0)$, $\phi_x \circ \sigma = I_{F_0}$. If we define a morphism $\psi : F_{0V} \rightarrow E_{0V}$ by $x \rightarrow \sigma \in \mathcal{L}(F_0, E_0)$, the morphism $\gamma = \phi \circ \psi : F_{0V} \rightarrow F_{0V}$ has an isomorphic fibre mapping $\gamma_x = I_{F_0}$ in x . By proposition 1 we have an isomorphic restriction $\gamma|_U, \phi|_U \circ (\psi|_U \circ (\gamma|_U)^{-1}) = I_{F_{0U}}$.

When ϕ_x is a split monomorphism, the proof is similar.

Definition 5 : Let B_1, B_2, B_3 be Banach spaces, and $j, k : B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$ continuous linear mappings. This sequence forms a *complex*, if $k \circ j = 0$. This sequence is *split exact* if the space B_i can be decomposed in direct

sums $B_i = C_i \oplus D_i$ such that

$$j: \begin{cases} C_1 \rightarrow 0 \\ D_1 \simeq C_2 \end{cases} \quad k: \begin{cases} C_2 \rightarrow 0 \\ D_2 \simeq C_3 \end{cases} .$$

Definition 6: A Banach vector bundle morphism sequence

$$\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \\ & & X & & \end{array} \quad \text{is a complex if } g \circ f = 0.$$

The sequence is *split exact*, if every E_i can be decomposed $E_i = F_i \oplus G_i$, such that:

$$f: \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_2 \end{cases} \quad g: \begin{cases} F_2 \rightarrow 0 \\ G_2 \simeq F_3 \end{cases} .$$

Theorem 1: Let $\begin{array}{ccccc} E_1 & \xrightarrow{f} & E_2 & \xrightarrow{g} & E_3 \\ & \searrow & \downarrow X & \swarrow & \end{array}$ be a complex of Banach vector

bundles and $x_0 \in X$.

If the sequence of Banach spaces $E_1(x_0) \xrightarrow{f_{x_0}} E_2(x_0) \xrightarrow{g_{x_0}} E_3(x_0)$ is split exact, then there exists an open neighbourhood $U \subset X$ of x_0 , such that $E_1|_U \rightarrow E_2|_U \rightarrow E_3|_U$ is a split exact sequence of Banach vector bundles.

Proof: We take a neighbourhood V of x , such that we have a complex $E_{1V} \rightarrow E_{2V} \rightarrow E_{3V}$ of trivial bundles. By assumption we have the decompositions $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$ with

$$f_{x_0}: \begin{cases} F_1(x_0) \rightarrow 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \quad g_{x_0}: \begin{cases} F_2(x_0) \rightarrow 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases} .$$

By proposition 2, $f|_V : G_{1V} \rightarrow E_{2V}$, $g|_V : G_{2V} \rightarrow E_{3V}$ are both split monomorphisms in a neighbourhood $W \subset V$ of x_0 and the images $F_2 = f(G_{1W})$, $F_3 = g(G_{2W})$ are subbundles of E_{2W} esp. E_{3W} , such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W} .$$

By our construction

$$g|_W : \begin{cases} F_2 & \rightarrow 0 \\ G_2 W & \simeq F_3 \end{cases} .$$

If $p: E_{2W} \rightarrow F_2$ is the projection with kernel G_{2W} , the map, $p \circ f: E_{1W} \rightarrow F_2$ is a split epimorphism in x_0 . Again by prop. 2 we have over an open neighbourhood $U \subset W$ of x_0 a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \text{Ker } p \circ f$)

$$(p \circ f)|_U : \begin{cases} F_1 & \rightarrow 0 \\ G_{1U} & \xrightarrow{\sim} F_{2U} \end{cases} .$$

The image $f|_U(F_1)$ is contained in G_{2U} . But $g|_U \circ f|_U = 0$ and $g|_{G_{2U}}$ is a monomorphism hence $f|_U: F_1 \rightarrow 0$. We get finally (restricting all our morphisms to U)

$$f|_U : \begin{cases} F_{1U} & \rightarrow 0 \\ G_{1U} & \simeq F_{2U} \end{cases} \qquad g|_U : \begin{cases} F_{2U} & \rightarrow 0 \\ G_{2U} & \xrightarrow{\sim} F_{3U} \end{cases} .$$

§ 2. *Privileged polycylinders*

Definition 1: A polycylinder in \mathbf{C}^n is a compact set K of the form $K = K_1 \times \dots \times K_n$ where each K_i is a compact, convex subset of \mathbf{C} , with nonempty interior. If each K_i is a disc, then K is a polydisc. We first recall the following theorem of Cartan.

Theorem 1: Let K be a polycylinder contained in an open subset U of \mathbf{C}^n . Let \mathcal{F} be a coherent analytic sheaf on U .

(A) There exists an open neighbourhood of K over which \mathcal{F} admits a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 .$$

(B) $H^q(K, \mathcal{F}) = 0$ for $q > 0$.

(Reference: For instance Gunning and Rossi.)

We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$$

of a coherent sheaf \mathcal{F} , the sequence

$$0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0$$

is an $\mathcal{O}_U(K)$ - free resolution of $\mathcal{F}(K)$.

2) Given a short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0,$$

then the sequence

$$0 \rightarrow \mathcal{F}'(K) \rightarrow \mathcal{F}(K) \rightarrow \mathcal{F}''(K) \rightarrow 0 \quad \text{is exact.}$$

Let \mathcal{F} be a coherent analytic sheaf on U , and let $K \subset U$ be a polycylinder. If V is an open neighbourhood of K , then $\mathcal{F}(V)$ can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give $\mathcal{F}(K)$ the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from $\mathcal{F}(K)$ and by choosing K in a "privileged" way.

Let $B(K) = \{f : K \rightarrow \mathbb{C} \mid f \text{ continuous on } K \text{ and analytic on } \overset{\circ}{K}\}$, then $B(K)$ is Banach algebra and $B(K) \subset C(K)$. The sections of \mathcal{O}_U over K are elements of $B(K)$, and $B(K)$ is in fact the uniform closure of $\mathcal{O}_U(K)$ in $C(K)$.

If $\mathcal{L} = \mathcal{O}_U^r$, we define $B(K, \mathcal{L}) = B(K)^r$. Then $B(K; \mathcal{L})$ is a free $B(K)$ -module, and since $\mathcal{L}(K) = \mathcal{O}_U(K)^r$, we have $B(K; \mathcal{L}) = B(K) \otimes_{\mathcal{O}_U(K)} \mathcal{L}(K)$.

We now assume that \mathcal{F} is a coherent sheaf on U , where $U \subset \mathbb{C}^n$ is open. Consider a free resolution

$$(R) \quad 0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 \quad \text{of } \mathcal{F}.$$

From (R) we get an $\mathcal{O}_U(K)$ -free resolution of $\mathcal{F}(K)$

$$(R') \quad 0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_1(K) \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0.$$

Taking the tensorproduct $B(K) \otimes_{\mathcal{O}_U(K)}$ we get the complex

$$B(K; \mathcal{L}.): 0 \rightarrow B(K; \mathcal{L}_n) \rightarrow \dots \rightarrow B(K; \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0).$$

Definition 2: The polycylinder K is called \mathcal{F} -privileged if the complex $B(K; \mathcal{L}.)$ is split-exact in every degree > 0 .

Remark: The property of being \mathcal{F} -privileged is independent of the resolution (R).

The exactness of $B(K; \mathcal{L}.)$ can be expressed by $\text{Tor}_i^{\mathcal{O}_U(K)}(B(K), \mathcal{F}(K)) = 0$, for every $i > 0$, and Tor is independent of the resolution (R). It is a little

more complicated to show, that the splitting property is independent of (R) , and this is omitted.

Since $B(K; \mathcal{L}_i)$ is a Banach space, the image and its complement are thus Banach spaces if K is \mathcal{F} -privileged. In this case we define $B(K; \mathcal{F}) = \text{Coker}(B(K, \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0)) = B(K) \otimes_{\mathcal{O}_U} \mathcal{F}(K)$ and we get a $B(K)$ -module, which is a Banach-space.

Warning: In the definition of split-exactness, the subspaces are splitting vector spaces, but they are not splitting $B(K)$ -modules in general.

We have the following important theorem about the existence of privileged polycylinders:

Theorem 2: Let U be an open subset of \mathbf{C}^n , and let \mathcal{F} be a coherent analytic sheaf on U . For any $x \in U$ there exists a fundamental system of neighbourhoods of x in U , which are \mathcal{F} -privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

Example: (Curves in \mathbf{C}^2) Let $U \subset \mathbf{C}^2$ be an open connected neighbourhood of the origin, and let $h: U \rightarrow \mathbf{C}$ be analytic and $h \neq 0$.

Let X be the curve given by h , that is $X = h^{-1}(0)$, $\mathcal{O}_X = \mathcal{O}_U / (h)$. We have an exact sequence $0 \rightarrow \mathcal{O}_U \xrightarrow{h} \mathcal{O}_U \rightarrow \mathcal{O}_X \rightarrow 0$. Consider a polycylinder $K = K_1 \times K_2 \subset U$. By definition K is \mathcal{O}_X -privileged if and only if $h: B(K) \rightarrow B(K)$ is a split monomorphism.

Let \dot{K}_j denote the boundary of K_j , and define $\ddot{K} = \dot{K}_1 \times \dot{K}_2$ (\ddot{K} is called the Šilov Boundary of K).

Proposition 1: (a) The following conditions are equivalent:

- (i) $h: B(K) \rightarrow B(K)$ is a monomorphism.
- (i') $\exists a > 0$ such that $\|hf\| \geq a\|f\|$, $\forall f \in B(K)$.
- (ii) $X \cap \ddot{K} = \emptyset$.

(b) If $(K_1 \times K_2) \cap X = \emptyset$, then h is a split monomorphism (i.e. K is \mathcal{O}_X -privileged).

Proof: (a) (i) \Leftrightarrow (i') is a well known fact from the theory of normed vector spaces.

(ii) \Rightarrow (i'). Assume $X \cap \ddot{K} = \emptyset$. If $f \in B(K)$, then it follows from the maximum principle that $\|f\| = \sup_K |f(x)| = \sup_{\ddot{K}} |f(x)|$. Since $h(x) \neq 0$

whenever $x \in \overset{\circ}{K}$, we get $a = \inf_K |h(x)| > 0$. Hence $\|hf\| = \sup_K |hf(x)| \geq \geq a \sup_K |f(x)| = a \|f\|$.

(i') \Rightarrow (ii). Suppose that $X \cap \overset{\circ}{K} \neq \emptyset$ and $x = (x_1, x_2) \in X \cap \overset{\circ}{K}$. We choose an analytic function $f_1 : U_1 \rightarrow \mathbf{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \rightarrow \mathbf{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \rightarrow f_1(z_1)f_2(z_2)$. Since $h(x) = 0$ it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in K .

Applying Dini's theorem we get $\|hf^n\| \rightarrow 0$. From the inequality $a \|f^n\| \leq \|hf^n\|$ we get $\|f^n\| \rightarrow 0$, which is a contradiction, because for every $n : f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h : B(K) \rightarrow B(K)$ is a split monomorphism?

IV. FLATNESS AND PRIVILEGE

§ 1. Morphisms from an analytic space into $B(K)$

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbf{C}^n$. We want to construct an \mathcal{O}_S -algebra homomorphism $\phi : \mathcal{O}_{S \times U}(S \times U) \rightarrow \mathcal{H}(S; B(K))$.

(a) Consider first $S = U' \subset \mathbf{C}^m$, U' -open. If $h \in \mathcal{O}_{U' \times U}(U' \times U)$ and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s, x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand it's obvious that ϕ is an $\mathcal{O}_{U'}$ -algebra homomorphism.

(b) Let S have a special model in the polydisc Δ in \mathbf{C}^m , defined by a sheaf \mathcal{I} of ideals of \mathcal{O}_Δ , and let \mathcal{J} be generated by f_1, \dots, f_p , V -a polycylinder neighbourhood of K in U . By Cartan's theorem B for a polycylinder,

the sequence $0 \rightarrow \mathcal{I}(\Delta \times V) \rightarrow \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \rightarrow 0$ is exact. If we denote by $\tilde{\pi}$ the projection $\mathcal{H}(\Delta, B(K)) \rightarrow \mathcal{H}(S, B(K))$, $(f_1, \dots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset \subset \text{Ker } \tilde{\pi}$. Therefore, because π is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \rightarrow \mathcal{H}(S, B(K))$, such that the diagram