III. Privileged polycylinders

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **29.04.2024**

Remark : This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi'"] \\ & S & \end{tikzcd} \qquad \mathcal{O}_{X \times_S Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_y .$$

We have proved that $\mathcal{O}_{W \times X}$ is $\mathcal{O}_W$-flat, so by scalar extension $\mathcal{O}_{S \times X}$ is $\mathcal{O}_S$ flat.

Corollary : If X and S are two manifolds and $\pi : X \to S$ is a submersion, then π is flat.

III. Privileged polycylinders

§ 1. *Banach vector bundles over an analytic space*

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X.

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E. If $U \subset \mathbf{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g : U \to E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X, i.e. X' is the support of the quotient sheaf $\mathcal{O}_U / J$, where $U \subset \mathbf{C}^n$ is open and J is a coherent sheaf of ideals of $\mathcal{O}_U$, then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \to \mathcal{H}(V, E) / J_V . \mathcal{H}(V, E)$ ($V \subset U$, V-open).

Remark : If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U.

The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models X' of X, i.e. $\mathcal{H}_X(E) | X' = \mathcal{H}_{X'}(E)$, for every local model X'.

Definition 1 : The set of *analytic morphisms* from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F.

Definition 2 : An analytic vector bundle morphism from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.

Let E be a topological space, X an analytic space, and $\pi : E \to X$ a continuous projection.

$$\begin{tikzcd} E & E|U_\iota \arrow[l, hook'] \arrow[r, "\phi_\iota"] \arrow[dr] & E_{\iota U_\iota} \arrow[d] \\ X \arrow[u, leftarrow, "\pi"'] & & U_\iota \arrow[ll, hook'] \end{tikzcd}$$

Suppose that X has an open covering $(U_\iota)_{\iota \varepsilon I}$, and that for every $\iota \in I$ there is given a trivial Banach space bundle $E_{\iota U_\iota}$ and a homeomosphism ϕ_ι, such that the following diagram is commutative:

We suppose further that for each pair $\iota, \kappa \in I$ there is given an analytic vector bundle morphism $\gamma_{\iota\kappa} : E_{\kappa U_\iota \cap U_\kappa} \to E_{\iota U_\iota \cap U_\kappa}$, with the underlying mapping $\phi_\iota \circ \phi_\kappa^{-1}$, such that:

$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa}\gamma_{\kappa\lambda}; \quad \gamma_{\iota\iota} = I, \qquad \text{for all} \quad \iota, \kappa, \gamma \in I .$$

This data gives a Banach vector bundle atlas on E and provides E with the structure of a Banach vector bundle over X (two atlases are equivalent if there exists an atlas containing both).

Remark: If X is reduced, the $\gamma_{\iota\kappa}$ are determined by their underlying map and the condition $\gamma_{\iota\lambda} = \gamma_{\iota\kappa}\gamma_{\kappa\lambda}$ is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

Proposition 1: Let $\phi : E \to F$ be a morphism of two Banach vector

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, dash] & & F \arrow[dl, dash] \\ & X & \end{tikzcd}$$

bundles E and F, and $x \in X$.

If $\phi_x \in \mathscr{L}(E(x), F(x))$ is an isomorphism, then there exists an open neighbourhood $U \subset X$ of x, such that $\phi|U : E|U \to F|U$ is a vector bundle isomorphism.

Proof: First we take a trivialisation $E|V = E_{0V}$, $F|V = F_{0V}$ at $x \in V \subset X$ (V-open).

The set Isom (E_0, F_0) of isomorphic mappings is an open subset of $\mathscr{L}(E_0, F_0)$ and the mapping $g \to q^{-1}$ is an analytic isomorphism:

$$\text{Isom}\,(E_0, F_0) \simeq \text{Isom}\,(F_0, E_0) .$$

So we have in an open neighbourhood $U \subset X$ of x an analytic morphism $y \to \phi_y^{-1} \in \mathscr{L}(F_0, E_0)$, which defines the inverse morphism $(\phi|U)^{-1} : F|U \to \to E|U$.

Definition 3 : Let E and F be two Banach spaces and f a continuous linear mapping from E into F. f is a *split mono-(epi) morphism,* if there exists a mapping $g \in \mathscr{L}(F, E)$ such that $g \circ f = I_E$. (Resp. $f \circ g = I_F$.)

Definirion 4 : Let E_1 and E_2 be two Banach vector bundles over an analytic space X, and f a vector bundle morphism from E_1 into E_2. f *is a split mono (epi) morphism,* if there exists a vector bundle morphism $g : E_2 \to E_1$ such that $g \circ f = I_{E_1}$. (Resp. $f \circ g = I_{E_2}$.)

Equivalently, f:

$$\begin{tikzcd} E_1 \arrow[rr] \arrow[dr] & & E_2 \arrow[dl] \\ & X & \end{tikzcd}$$

is a split monomorphism if an only if E_2 can be decomposed in a direct sum $E_2 = F_2 \oplus G_2$ such that

$$f : \begin{cases} E_1 \simeq F_2 \\ 0 \to G_2 \end{cases}.$$

and f is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1 , \quad \text{such that} \quad f : \begin{cases} F_1 \to 0 \\ G_1 \simeq E_2 \end{cases}.$$

Proposition 2 : Let

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$

be a bundle morphism and $x \in X$.

If $\phi_x : E(x) \to F(x)$ is a split epi (mono) morphism, then the point x has an open neighbourhood $U \subset X$, such that $\phi|U : E|U \to F|U$ is a split vector bundle epi (mono) morphism.

Proof: Suppose that ϕ_x is a split epimorphism. We take first a trivilisation $E|V = E_{0V}$, $F|V = F_{0V}$ at x, so that there exists a mapping $\sigma \in \mathscr{L}(F_0, E_0)$, $\phi_x \circ \sigma = I_{F_0}$. If we define a morphism $\psi : F_{0V} \to E_{0V}$ by $x \to \sigma \in \mathscr{L}(F_0, E_0)$, the morphism $\gamma = \phi \circ \psi : F_{0V} \to F_{0V}$ has an isomorphic fibre mapping $\gamma_x = I_{F_0}$ in x. By proposition 1 we have an isomorphic restriction $\gamma|U$, $\phi|U \circ (\psi|U \circ (\gamma|U)^{-1}) = I_{F_{0U}}$.

When ϕ_x is a split monomorphism, the proof is similar.

Definition 5 : Let B_1, B_2, B_3 be Banach spaces, and $j, k : B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$ continuous linear mappings. This sequence forms *a complex,* if $k \circ j = 0$. This sequence is *split exact* if the space B_i can be decomposed in direct sums $B_i = C_i \oplus D_i$ such that

$$j : \begin{cases} C_1 \to 0 \\ D_1 \simeq C_2 \end{cases} \qquad k : \begin{cases} C_2 \to 0 \\ D_2 \simeq C_3 \end{cases}.$$

Definition 6: A Banach vector bundle morphism sequence

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[dr] & E_2 \arrow[r, "g"] \arrow[d] & E_3 \arrow[dl] \\ & X & \end{tikzcd}$$

is *a complex* if $g \circ f = 0$.

The sequence is *split exact*, if every E_i can be decomposed $E_i = F_i \oplus G_i$, such that:

$$f : \begin{cases} F_1 \to 0 \\ G_1 \simeq F_2 \end{cases} \qquad g : \begin{cases} F_2 \to 0 \\ G_2 \simeq F_3 \end{cases}.$$

Theorem 1: Let

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[dr] & E_2 \arrow[r, "g"] \arrow[d] & E_3 \arrow[dl] \\ & X & \end{tikzcd}$$

be a complex of Banach vector bundles and $x_0 \in X$.

If the sequence of Banach spaces $E_1(x_0) \xrightarrow{f x_0} E_2(x_0) \xrightarrow{f x_0} E_3(x_0)$ is split exact, then there exists an open neighbourhood $U \subset X$ of x_0, such that $E_1|U \xrightarrow{f|U} E_2|U \xrightarrow{g|U} E_3|U$ is a split exact sequence of Banach vector bundles.

Proof: We take a neighbourhood V of x, such that we have a complex $E_{1V} \xrightarrow{f|V} E_{2V} \xrightarrow{g|V} E_{3V}$ of trivial bundles. By assumption we have the decompositions $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$ with

$$f_{x_0} : \begin{cases} F_1(x_0) \to 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \qquad g_{x_0} : \begin{cases} F_2(x_0) \to 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases}.$$

By proposition 2, $f|V : G_{1V} \to E_{2V}$, $g|V : G_{2V} \to E_{3V}$ are both split monomorphisms in a neighbourhood $W \subset V$ of x_0 and the images $F_2 = f(G_{1W})$, $F_3 = g(G_{2W})$ are subbundles of E_{2W} esp. E_{3W}, such that

$$E_{2W} = F_2 \oplus G_{2W}, \qquad E_{3W} = F_3 \oplus G_{3W}.$$

By our construction

$$g \mid W : \begin{cases} F_2 & \to 0 \\ \\ G_2\, W \simeq F_3 \end{cases} .$$

If p: $E_{2W} \to F_2$ is the projection with kernel G_{2W}, the map, $p \circ f : E_{1W} \to F_2$ is a split epimorphism in x_0. Again by prop. 2 we have over an open eighbourhood $U \subset W$ of x_0 a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \operatorname{Ker} p \circ f$)

$$(p \circ f) \mid U : \begin{cases} F_1 & \to 0 \\ \\ G_{1U} \overset{\sim}{\to} F_{2U} \end{cases} .$$

The image $f|U(F_1)$ is contained in G_{2U}. But $g|U \circ f|U = 0$ and $g|G_{2U}$ is a monomorphism hence $f|U : F_1 \to 0$. We get finally (restricting all our morphisms to U)

$$f \mid U : \begin{cases} F_{1U} \to 0 \\ \\ G_{1U} \simeq F_{2U} \end{cases} \qquad g \mid U : \begin{cases} F_{2U} \to 0 \\ \\ G_{2U} \overset{\sim}{\to} F_{3U} \end{cases} .$$

§ 2. *Privileged polycylinders*

Definition 1: A polycylinder in $\mathbf{C}^n$ is a compact set K of the form $K = K_1 \times \ldots \times K_n$ where each K_i is a compact, convex subset of $\mathbf{C}$, with nonempty interior. If each K_i is a disc, then K is a polydisc. We first recall the following theorem of Cartan.

Theorem 1: Let K be a polycylinder contained in an open subset U of $\mathbf{C}^n$. Let $\mathscr{F}$ be a coherent analytic sheaf on U.

(A) There exists an open neighbourhood of K over which $\mathscr{F}$ admits a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0 .$$

(B) $H^q(K, \mathscr{F}) = 0$ for $q > 0$.
(Reference: For instance Gunning and Rossi.)
We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$$

of a coherent sheaf $\mathscr{F}$, the sequence

$$0 \to \mathscr{L}_n(K) \to \ldots \to \mathscr{L}_0(K) \to \mathscr{F}(K) \to 0$$

is an $\mathcal{O}_U(K)$ - free resolution of $\mathscr{F}(K)$.

2) Given a short exact sequence of coherent sheaves

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0\,,$$

then the sequence

$$0 \to \mathscr{F}_{\prime}(K) \to \mathscr{F}(K) \to \mathscr{F}''(K) \to 0 \qquad \text{is exact.}$$

Let $\mathscr{F}$ be a coherent analytic sheaf on U, and let $K \subset U$ be a polycylinder If V is an open neighbourhood of K, then $\mathscr{F}(V)$ can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give $\mathscr{F}(K)$ the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from $\mathscr{F}(K)$ and by choosing K in a " privileged " way.

Let $B(K) = \{f : K \to \mathbf{C} \mid f$ continuous on K and analytic on $\mathring{K}\}$, then $B(K)$ is Banach algebra and $B(K) \subset C(K)$. The sections of $\mathcal{O}_U$ over K are elements of $B(K)$, and $B(K)$ is in fact the uniform closure of $\mathcal{O}_U(K)$ in $C(K)$.

If $\mathscr{L} = \mathcal{O}_U^r$, we define $B(K, \mathscr{L}) = B(K)^r$. Then $B(K; \mathscr{L})$ is a free $B(K)$-module, and since $\mathscr{L}(K) = \mathcal{O}_U(K)^r$, we have $B(K; \mathscr{L}) = B(K) \otimes_{\mathcal{O}_{U(K)}} \mathscr{L}(K)$.

We now assume that $\mathscr{F}$ is a coherent sheaf on U, where $U \subset \mathbf{C}^n$ is open. Consider a free resolution

$$(R) \qquad 0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0 \quad \text{of } \mathscr{F}.$$

From (R) we get an $\mathcal{O}_U(K)$-free resolution of $\mathscr{F}(K)$

$$(R') \qquad 0 \to \mathscr{L}_n(K) \to \ldots \to \to_1(K) \to \mathscr{L}_0(K) \to \mathscr{F}(K) \to 0\,.$$

Taking the tensorproduct $B(K) \otimes_{\mathcal{O}_U(K)}$ we get the complex

$$B(K; \mathscr{L}.) : 0 \to B(K; \mathscr{L}_n) \to \ldots \to B(K; \mathscr{L}_1) \to B(K; \mathscr{L}_0)\,.$$

Definition 2: The polycylinder K is called *$\mathscr{F}$-privileged* if the complex $B(K; \mathscr{L}.)$ is split-exact in every degree >0.

Remark: The property of being $\mathscr{F}$-privileged is independent of the resolution (R).

The exactnes of $B(K; \mathscr{L})$ can be expressed by $\mathrm{Tor}_i^{\mathcal{O}(K)}(B(K), \mathscr{F}(K)) = 0$, for every $i > 0$, and Tor is independent of the resolution (R). It is a little more complicated to show, that the splitting property is independent of (R), and this is omitted.

Since $B(K; \mathscr{L}_i)$ is a Banach space, the image and its complement are thus Banach spaces if K. is $\mathscr{F}$-privileged. In this case we define $B(K; \mathscr{F}) = \operatorname{Coker}(B(K, \mathscr{L}_1) \to B(K; \mathscr{L}_0)) = B(K) \otimes_{\mathcal{O}_U} \mathscr{F}(K)$ and we get a $B(K)$-module, which is a Banach-space.

Warning: In the definition of split-exactnes, the subspaces are splitting vector spaces, but they are not splitting $B(K)$-modules in general.

We have the following important theorem about the existence of privileged polycylinders:

Theorem 2: Let U be an open subset of $\mathbf{C}^n$, and let $\mathscr{F}$ be a coherent analytic sheaf on U. For any $x \in U$ there exists a fundamental system of neighbourhoods of x in U, which are $\mathscr{F}$-privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

Example: (Curves in $\mathbf{C}^2$) Let $U \subset \mathbf{C}^2$ be an open connected neighbourhood of the origin, and let $h: U \to \mathbf{C}$ be analytic and $h \neq 0$.

Let X be the curve given by h, that is $X = h^{-1}(0)$, $\mathcal{O}_X = \mathcal{O}_U/(h)$. We have an exact sequence $0 \to \mathcal{O}_U \xrightarrow{h.I} \mathcal{O}_U \to \mathcal{O}_X \to 0$. Consider a polycylinder $K = K_1 \times K_2 \subset U$. By definition K is $\mathcal{O}_X$-priviledged if and only if $h: B(K) \to B(K)$ is a split monomorphism.

Let $\dot{K}_j$ denote the boundary of K_j, and define $\ddot{K} = \dot{K}_1 \times \dot{K}_2$ ($\ddot{K}$ is called the Šilov Boundary of K).

Proposition 1: (a) The following conditions are equivalent:

(i) $h: B(K) \to B(K)$ is a monomorphism.

(i′) $\exists\, a > 0$ such that $\|hf\| \geqq a\|f\|$, $\forall f \in B(K)$.

(ii) $X \cap \ddot{K} = \varnothing$.

(b) If $(K_1 \times \dot{K}_2) \cap X = \varnothing$, then h is a split monomorphism (i.e. K is $\mathcal{O}_X$ privileged).

Proof: (a) (i) $\Leftrightarrow$ (i′) is a well known fact from the theory of normed vector spaces.

(ii) $\Rightarrow$ (i′). Assume $X \cap \ddot{K} = \varnothing$. If $f \in B(K)$, then it follows from the maximum principle that $\|f\| = \sup_K |f(x)| = \sup_{\ddot{K}} |f(x)|$. Since $h(x) \neq 0$

whenever $x \in \ddot{K}$, we get $a = \inf_K |h(x)| > 0$. Hence $||hf|| = \sup_K |hf(x)| \geqq$ $\geqq a \sup_K |f(x)| = a\,||f||$.

(i') $\Rightarrow$ (ii). Suppose that $X \cap \ddot{K} \neq \varnothing$ and $x = (x_1, x_2) \in X \cap \ddot{K}$. We choose an analytic function $f_1 : U_1 \to \mathbf{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \to \mathbf{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \to f_1(z_1) f_2(z_2)$. Since $h(x) = 0$ it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in K.

Applying Dini's theorem we get $||hf^n|| \to 0$. From the inequality $a\,||f^n|| \leqq$ $\leqq ||hf^n||$ we get $||f^n|| \to 0$, which is a contradiction, because for every $n : f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h : B(K) \to B(K)$ is a split monomorphism?

IV. Flatness and privilege

§ 1. *Morphisms from an analytic space into $B(K)$*

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbf{C}^n$. We want to construct an $\mathcal{O}_S$-algebra homomorphism $\phi : \mathcal{O}_{S\times U}(S \times U) \to$ $\to \mathcal{H}(S; B(K))$.

(*a*) Consider first $S = U' \subset \mathbf{C}^m$, U'-open. If $h \in \mathcal{O}_{U'\times U}(U' \times U)$ and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s, x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand its obvious that ϕ is an $\mathcal{O}_{U'}$-algebra homomorphism.

(*b*) Let S have a special model in the polydisc Δ in $\mathbf{C}^m$, defined by a sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_\Delta$, and let $\mathcal{J}$ be generated by $f_1, \ldots, f_p$, V-a polycylinder neighbourhood of K in U. By Cartan's theorem B for a polycylinder,

the sequence $0 \to \mathcal{J}(\Delta \times V) \to \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \to 0$ is exact. If we denote by $\tilde{\pi}$ the projection $\mathcal{H}(\Delta, B(K)) \to \mathcal{H}(S, B(K))$, $(f_1, \ldots, f_p) \,.\, \mathcal{H}(\Delta, B(K)) \subset$ $\subset \operatorname{Ker} \tilde{\pi}$. Therefore, because π is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \to \mathcal{H}(S, B(K))$, such that the diagram