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*Proof*:  $\operatorname{Tor}_{1}^{A}(k, F) = \operatorname{Tor}_{1}^{B}(\overline{B}, F)$ , by prop. 7.

## § 5. Geometric applications of the flatness criterions

# A) Flatness for finite morphisms

Proposition 1: Let  $\pi: X \to S$  be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then  $\pi_* (\mathcal{O}_X)$  is a coherent analytic sheaf over S. The following conditions are equivalent:

- (a)  $\pi$  is flat (i.e. for every  $x \in X$ ,  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module,  $s = \pi(x)$ ).
- (b) For every s,  $(\pi_* \mathcal{O}_X)_s$  is a flat  $\mathcal{O}_{S,s}$ -module.
- (c)  $\pi_* \mathcal{O}_X$  is a locally free sheaf.

*Proof*: Because  $\pi$  is finite  $\pi_* (\mathcal{O}_X)_s = \bigoplus_{x \in \pi^{-1}(s)} \mathcal{O}_{X,x}$ , thus the only point to prove is  $(b) \Rightarrow (c)$ .

Now if  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module, then (by theorem 2)  $\mathcal{O}_{X,x}$  is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

Proposition 2: Let S be a reduced analytic space and  $\mathscr E$  a coherent  $\mathscr O_s$ -module. Let E(s) be the finite dimensional vector space (over C)  $\mathscr E_s \otimes_{\mathscr O} C_s$ .  $\mathscr E$  is a locally free  $\mathscr O_{S,s}$ -module if an only if  $\dim_C E(s)$  is locally constant.

*Proof*: If  $\mathscr{E}$  is locally free, then  $\dim_{\mathbb{C}} E(s)$  is locally constant. Suppose now that  $\dim_{\mathbb{C}} E(s)$  is locally constant in an open set  $U \subset S$ , and that  $\mathcal{O}_U^p \to \mathcal{O}_U^q \to \mathcal{E}_U \to 0$  is exact. d is determined by a  $p \times q$  matrix of analytic functions on U, so it gives a morphism  $\mathbf{C}_U^p \to \mathbf{C}_U^q$  of trivial vector bundles over U.

From the exact sequence  $\mathcal{O}_s^p \to \mathcal{O}_s^q \to \mathcal{E}_s \to 0$ , we get (by making tensor-products with  $C_s$ ) the exact sequence:

$$\mathbf{C}_{s}^{p} \stackrel{d(s)}{\rightarrow} \mathbf{C}_{s}^{q} \rightarrow E(s) \rightarrow 0$$
,

which shows that d has constant rank in U. Thus Ker d and Im d are vector bundles, and we can write

$$\mathbf{C}_{\mathit{U}}^{\mathit{p}} = \mathit{F}_{1} \oplus \mathit{G}_{1} \;, \quad \mathbf{C}_{\mathit{U}}^{\mathit{q}} = \mathit{F}_{0} \oplus \mathit{G}_{0} \;,$$
 
$$d: \left\{ egin{aligned} F_{1} &\rightarrow 0 \\ G_{1} &\simeq \mathit{F}_{0} \;. \end{aligned} \right.$$

Now  $\mathscr{E} \simeq$  the sheaf of analytic sections of  $G_0$ , therefore  $\mathscr{E}$  is locally free.

Definition 1: Let  $\pi: X \to S$  be a finite morphism of analytic spaces, and  $s \in S$ . For each  $x \in X(s) = \pi^{-1}(s)$ ,  $\mathcal{O}_{X(s),x} = \mathbb{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$  is finite dimensional vectorspace over  $\mathbb{C}$ . Denote its dimension by v(x). Then the degree v(s) of s is defined by  $v(s) = \sum_{x \in X(s)} v(x)$ .

Theorem 1: Let  $\pi: X \to S$  be a finite morphism of analytic space and let S be a reduced space. Then X is flat over S if and only if v(s) is locally constant function of s.

$$Proof: v(s) = \sum_{x \in X(s)} \dim_{\mathbb{C}} \mathcal{O}_{X(s),x} = \dim_{\mathbb{C}} \left( \bigoplus_{x \in X(s)} \mathcal{O}_{X(s),x} \right)$$

$$= \dim_{\mathbb{C}} \left( \bigoplus_{x \in X(s)} \left( \mathbf{C} \bigotimes_{\mathscr{O}_{S,s}} \mathscr{O}_{X,x} \right) \right)$$

$$= \dim_{\mathbb{C}} \mathbf{C} \bigotimes_{\mathscr{O}_{S,s}} \pi_{*} (\mathscr{O}_{X})_{s} = \dim_{\mathbb{C}} E(s).$$

The theorem follows from propositions 1 and 2.

Examples of flat morphisms

Example 1: If  $\pi: X \to S$  is a local isomorphism near x, then  $\pi$  is flat at x.

Example 2: Consider § 2, Ex. 1. Here v(x) = 1.

Examples of non-flat morphisms

Examples 1: If  $X \subset S$  is a closed subspace, not open, v(s) is not locally constant.

Example 2: Let X be a subspace of  $\mathbb{C}^4$  defined by the ideal intersection of  $(x_3, x_4)$  and  $(x_1 - x_1, x_4 - x_2)$  (which is equal to the product ideal) and let  $\pi$  be the projection onto the  $(x_1, x_2)$ -plane  $\mathbb{C}^2$ . Then X is a union of two 2-planes in  $\mathbb{C}^4$ , whose intersection is (0). When  $s \neq 0$ . X(s) consists of two simple points, so v(s) = 2. X(0) is given by the ideal  $(x_1, x_2, x_3^2, x_3x_4, x_4^2)$ , thus v(0) = 3.

Example 3: Let  $S = \{(u, v, w) \in \mathbb{C}^3 \mid v^2 = uw\}$  and  $\pi : \mathbb{C}^2 \to S$  be the map  $(x, y) \to (x^2, xy, y^2)$ . This map identifies S with the quotient of  $\mathbb{C}^2$  by the equivalence relation idenfying (x, y) with (-x, -y). However,  $\pi$  is not flat, since for  $s \in S$ , v(s) = 2 if  $s \neq 0$  and v(s) = 3 if s = 0.

# B) Projection of a product of analytic spaces

Theorem 2: Let S and X be analytic spaces. If  $\pi: S \times X \to S$  is the projection morphism, then  $\pi$  is flat, i.e.  $\mathcal{O}_{S \times X, 's, x}$  is a flat  $\mathcal{O}_{S, s}$  module for every  $(s, x) \in S \times X$ .

To prove this theorem we need first some homological algebra. Then we shall show it in the particular case, when S is a manifold, and finally in the general case.

### (a) Koszul complex

Let A be a ring, M an A-module and  $h_1, ..., h_n$  homomorphisms  $M \rightarrow M$ , which commute with each other, i.e.  $h_i h_j = h_j h_i$  for every i, j.

If  $1 \le k \le n$ , set  $Q_k = M/h_1(M) + ... + h_k(M)$ , and  $Q_0 = M$ , thus, in particular,  $Q_n = Q = M/\sum_{i=j}^n h_i(M)$ ,. Every  $h_k$  induces a map  $h_k Q_{k-1} \to Q_{k-1}$ .

Definition 2: The sequence  $(h_1, ..., h_n)$  is called regular if each of the mappings  $h_k$   $(1 \le k \le n)$  is injective.

The Koszul complex of the module M and of the mappings  $h_k$   $(1 \le k \le n)$  K = K.  $[M; h_1, ..., h_n]$  is defined in the following way:

$$K_i = \bigwedge^{n+i} A^n \otimes M \simeq M^{\binom{n}{i}}, \quad 0 \leqslant i \leqslant n.$$

We define the homorphisms  $d_i: K_i \to K_{i-1}$  (i > 0) by  $\lambda \otimes x \to \sum_i (e_i \wedge \lambda) \otimes \otimes h_i(x)$ , where  $(e_i)$  is the natural base of  $A^n$ . We also define  $\varepsilon: K_0 \to Q$  as the natural map  $: K_0 = M \to M / \sum_{i=1}^n h_i(M) = Q$ . Using the fact that  $h_1, ..., h_n$  commute with each other, it is easy to verify that

$$(d_{i-1} \circ d_i)(\lambda \otimes x) = \sum_{i,j} (e_j \wedge e_i \wedge \lambda) \otimes h_j(h_i(x)) = 0;$$

also  $\varepsilon d_1 = 0$ . Thus K. is really a complex.

Theorem 3 (Poincaré-Koszul).

If  $(h_1, ..., h_n)$  is a regular sequence, then

$$H_i(K.) = \begin{cases} Q & if & i = 0 \\ & & . \\ 0 & if & i > 0 \end{cases}$$

 $h_iI$ 

If  $h_i \in A$ , it defines the map:  $A \rightarrow A$ , which we denote also by  $h_i$ . We say that  $(h_1, ..., h_n)$  is a regular sequence of elements if  $(h_1, ..., h_n, I)$  is a regular sequence.

Corollary. If  $(h_1, ..., h_n)$  is a regular sequence of elements, then the Koszul complex K = K.  $[A; h_1, ..., h_n] = \{ \wedge^{n-1} A^n \simeq A^{\binom{n}{i}} \}$  is a free resolution of  $Q = A/(h_i)$  ( $(h_i)$  is the ideal generated by  $h_1, ..., h_n$ )).

Example: If  $A = \mathbb{C} \{x_1, ..., x_n\}$ ;  $h_i = x_i$ , then  $Q_k = A/(x_1, ..., x_k) = \mathbb{C} \{x_{k+1}, ..., x_n\}$  and  $Q = Q_n = \mathbb{C}$ . The complex K = K.  $[A; x_1, ..., x_n]$  is a free resolution of  $\mathbb{C}$ .

(b) Proof of theorem 2, when S is a complex manifold

In this case we can take  $\mathcal{O}_{S,s} = \mathbb{C}\{t_1,...,t_m\} = A$  and if  $\mathcal{O}_{X,x} = \mathbb{C}\{x_1,...,x_n\}/(f_1,...,f_p)$ , then

$$\mathcal{O}_{S \times X,(s,x)} = \mathbb{C} \{t_1, ..., t_m, x_1, ..., x_n\} / (f_1, ..., f_p) = B.$$

B is an A-module in a natural way.

By the corollary of the Poincaré-Koszul theorem K = K.  $[A; t_1, ..., t_m]$  in a free resolution of  $\mathbb{C}$ . We want to compute the modules  $\operatorname{Tor}_i^A(\mathbb{C}, B) = H_i(K \otimes B)$  (i > 0).

It's easily seen, that we can consider the complex  $K \cdot \otimes B$  as a Koszul

complex K' = K.  $[B; t_1, ..., t_m]$  (where  $t_i : B \rightarrow B$ ). But now the sequence  $(t_1, ..., t_m)$  is regular, thus by the Poincaré-Koszul theorem  $H_i[K'] = 0$  if i > 0.

In particular:  $\operatorname{Tor}_{1}^{A}(\mathbb{C}, B) = H_{1}[K \otimes B] = H_{1}[K'] = 0$ . By the second flatness criterion B is A-flat.

# (c) The general case

The question being local, we can suppose that  $S \subset W \subset \mathbb{C}^n$ , where W is open, and S an analytic subspace of W. Let S be defined by  $g_1, ..., g_r$ . Then

 $S \times X \subset W \times X$  and  $\mathscr{O}_S = \mathscr{O}_W/(g_1, ..., g_r)$ . On the other hand  $\mathscr{O}_{S \times X} = \mathscr{O}_{W \times X}/(g_1, ..., g_r) = \mathscr{O}_S \otimes \mathscr{O}_{W \times X}$ . The last equality follows from

the fact, that if  $\pi: X \to S$  is a morphism, and  $S' \subset S$  a subspace,  $X' = \pi^{-1}(S')$ ,

then 
$$\mathscr{O}_{X'} = \mathscr{O}_{S'} \otimes_{\mathscr{O}_{S}} \mathscr{O}_{X}$$
.

Remark: This a particular case of the following proposition: if  $\pi$  and  $\pi'$  are two morphisms of which at least one is finite, then

$$X \qquad Y \qquad \emptyset_{X \times Y} = \emptyset_X \otimes_{\emptyset_S} \emptyset_y.$$

We have proved that  $\mathcal{O}_{W \times X}$  is  $\mathcal{O}_W$ -flat, so by scalar extension  $\mathcal{O}_{S \times X}$  is  $\mathcal{O}_S$  flat.

Corollary: If X and S are two manifolds and  $\pi: X \rightarrow S$  is a submersion, then  $\pi$  is flat.

### III. PRIVILEGED POLYCYLINDERS

### § 1. Banach vector bundles over an analytic space

Let E be a Banach space and X an analytic space. We denote then by  $E_X$  the trivial bundle  $X \times E$  over X.

To define bundle morphisms, we first define the sheaf  $\mathcal{H}_X(E)$  of germs of analytic morphisms from X to E. If  $U \subset \mathbb{C}^n$  is open, then the set  $\mathcal{H}(U, E)$  of analytic morphisms from U into E consists of all functions  $g: U \to E$  having at every point  $x \in U$  a converging power series expansion.

Let now X' be a local model for X, i.e. X' is the support of the quotient sheaf  $\mathcal{O}_U/J$ , where  $U \subset \mathbb{C}^n$  is open and J is a coherent sheaf of ideals of  $\mathcal{O}_U$ , then  $\mathscr{H}_{X'}(E)$  is the sheaf associated to the presheaf  $V \to \mathscr{H}(V, E)/J_V \cdot \mathscr{H}(V, E)$   $(V \subset U, V\text{-open})$ .

Remark: If X' is reduced, the sections of  $\mathcal{H}_{X'}(E)$  are just the functions from X' to E which are locally induced by analytic functions on open sets in U.

The sheaf  $\mathscr{H}_X(E)$  is constructed with help of the local models X' of X, i.e.  $\mathscr{H}_X(E)|X'=\mathscr{H}_{X'}(E)$ , for every local model X'.

Definition 1: The set of analytic morphisms from an analytic space X into a Banach space E is the set  $\mathcal{H}(X; E)$  of sections of the sheaf  $\mathcal{H}_X(E)$ .

Let  $\mathcal{L}(E, F)$  be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F.

Definition 2: An analytic vector bundle morphism from  $E_X$  into  $F_X$  is an analytic morphism from X into  $\mathcal{L}(E, F)$ .