Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FLATNESS AND PRIVILEGE
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Kapitel: §5. Geometrie applications of the flatness criterions
DOI: https://doi.org/10.5169/seals-42343

Proof: $\mathrm{Tor}_1^A(k, F) = \mathrm{Tor}_1^B(\bar{B}, F)$, by prop. 7.

§ 5. *Geometric applications of the flatness criterions*

A) *Flatness for finite morphisms*

Proposition 1: Let $\pi\colon X \to S$ be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then $\pi_*(\mathcal{O}_X)$ is a coherent analytic sheaf over S. The following conditions are equivalent:

(a) π is flat (i.e. for every $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, $s = \pi(x)$).

(b) For every s, $(\pi_* \mathcal{O}_X)_s$ is a flat $\mathcal{O}_{S,s}$-module.

(c) $\pi_* \mathcal{O}_X$ is a locally free sheaf.

Proof: Because π is finite $\pi_*(\mathcal{O}_X)_s = \bigoplus_{x \in \pi^{-1}(s)} \mathcal{O}_{X,x}$, thus the only point to prove is $(b) \Rightarrow (c)$.

Now if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module, then (by theorem 2) $\mathcal{O}_{X,x}$ is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

Proposition 2: Let S be a reduced analytic space and $\mathscr{E}$ a coherent $\mathcal{O}_s$-module. Let $E(s)$ be the finite dimensional vector space (over $\mathbf{C}$) $\mathscr{E}_s \otimes_{\mathcal{O}_{S,s}} \mathbf{C}_s$. $\mathscr{E}$ is a locally free $\mathcal{O}_{S,s}$-module if an only if $\dim_{\mathbf{C}} E(s)$ is locally constant.

Proof: If $\mathscr{E}$ is locally free, then $\dim_{\mathbf{C}} E(s)$ is locally constant. Suppose now that $\dim_{\mathbf{C}} E(s)$ is locally constant in an open set $U \subset S$, and that $\mathcal{O}_U^p \xrightarrow{d} \mathcal{O}_U^q \to \mathscr{E}_U \to 0$ is exact. d is determined by a $p \times q$ matrix of analytic functions on U, so it gives a morphism $\mathbf{C}_U^p \xrightarrow{d} \mathbf{C}_U^q$ of trivial vector bundles over U.

From the exact sequence $\mathcal{O}_s^p \xrightarrow{d_s} \mathcal{O}_s^q \to \mathscr{E}_s \to 0$, we get (by making tensor-products with $\mathbf{C}_s$) the exact sequence:

$$\mathbf{C}_s^p \xrightarrow{d(s)} \mathbf{C}_s^q \to E(s) \to 0\,,$$

which shows that d has constant rank in U. Thus $\mathrm{Ker}\, d$ and $\mathrm{Im}\, d$ are vector bundles, and we can write

$$\mathbf{C}_U^p = F_1 \oplus G_1\,, \qquad \mathbf{C}_U^q = F_0 \oplus G_0\,,$$

$$d : \begin{cases} F_1 \to 0 \\ G_1 \simeq F_0\,. \end{cases}$$

Now $\mathscr{E} \simeq$ the sheaf of analytic sections of G_0, therefore $\mathscr{E}$ is locally free.

Definition 1: Let $\pi : X \to S$ be a finite morphism of analytic spaces, and $s \in S$. For each $x \in X(s) = \pi^{-1}(s)$, $\mathcal{O}_{X(s),x} = \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$ is finite dimensional vectorspace over $\mathbf{C}$. Denote its dimension by $\nu(x)$. Then the degree $\nu(s)$ of s is defined by $\nu(s) = \sum_{x \varepsilon X(s)} \nu(x)$.

Theorem 1: Let $\pi : X \to S$ be a finite morphism of analytic space and let S be a reduced space. Then X is flat over S if and only if $\nu(s)$ is locally constant function of s.

Proof:
$$\nu(s) = \sum_{x \varepsilon X(s)} \dim_{\mathbf{C}} \mathcal{O}_{X(s),x} = \dim_{\mathbf{C}} \Big(\bigoplus_{x \varepsilon X(s)} \mathcal{O}_{X(s),x} \Big)$$
$$= \dim_{\mathbf{C}} \Big(\bigoplus_{x \varepsilon X(s)} (\mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}) \Big)$$
$$= \dim_{\mathbf{C}} \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \pi_* (\mathcal{O}_X)_s = \dim_{\mathbf{C}} E(s)\,.$$

The theorem follows from propositions 1 and 2.

Examples of flat morphisms

Example 1: If $\pi : X \to S$ is a local isomorphism near x, then π is flat at x.

Example 2: Consider § 2, Ex. 1. Here $\nu(x) = 1$.

Examples of non-flat morphisms

Examples 1: If $X \subset S$ is a closed subspace, not open, $\nu(s)$ is not locally constant.

Example 2: Let X be a subspace of $\mathbf{C}^4$ defined by the ideal intersection of (x_3, x_4) and $(x_1 - x_1, x_4 - x_2)$ (which is equal to the product ideal) and let π be the projection onto the (x_1, x_2)-plane $\mathbf{C}^2$. Then X is a union of two 2-planes in $\mathbf{C}^4$, whose intersection is (0). When $s \neq 0$. $X(s)$ consists of two simple points, so $\nu(s) = 2$. $X(0)$ is given by the ideal $(x_1, x_2, x_3^2, x_3 x_4, x_4^2)$, thus $\nu(0) = 3$.

Example 3: Let $S = \{(u, v, w) \in \mathbf{C}^3 \mid v^2 = uw\}$ and $\pi : \mathbf{C}^2 \to S$ be the map $(x, y) \to (x^2, xy, y^2)$. This map identifies S with the quotient of $\mathbf{C}^2$ by the equivalence relation idenfying (x, y) with $(-x, -y)$. However, π is not flat, since for $s \in S$, $\nu(s) = 2$ if $s \neq 0$ and $\nu(s) = 3$ if $s = 0$.

B) *Projection of a product of analytic spaces*

Theorem 2: Let S and X be analytic spaces. If $\pi : S\times X\to S$ is the projection morphism, then π is flat, i.e. $\mathcal{O}_{S\times X,(s,x)}$ is a flat $\mathcal{O}_{S,s}$ module for every $(s, x) \in S\times X$.

To prove this theorem we need first some homological algebra. Then we shall show it in the particular case, when S is a manifold, and finally in the general case.

(a) *Koszul complex*

Let A be a ring, M an A-module and $h_1, \ldots, h_n$ homomorphisms $M\to M$, which commute with each other, i.e. $h_i h_j = h_j h_i$ for every i, j.

If $1 \leqq k \leqq n$, set $Q_k = M/h_1 (M)+\ldots+h_k (M)$, and $Q_0 = M$, thus, in particular, $Q_n = Q = M/ \sum\limits_{i=j}^{n} h_i (M)$,. Every h_k induces a map $\tilde{h}_k\, Q_{k-1} \to Q_{k-1}$.

Definition 2: The sequence $(h_1, \ldots, h_n)$ is called regular if each of the mappings $\tilde{h}_k$ $(1\leqq k\leqq n)$ is injective.

The Koszul complex of the module M and of the mappings h_k $(1 \leqq k \leqq n)$ $K. = K. [M; h_1, \ldots, h_n]$ is defined in the following way:

$$K_i = \wedge^{n+i} A^n \otimes M \simeq M^{\binom{n}{i}}, \qquad 0\leqslant i\leqslant n\,.$$

We define the homorphisms $d_i : K_i\to K_{i-1}$ $(i>0)$ by $\lambda\otimes x\to \sum\limits_i (e_i\wedge \lambda)\otimes \otimes h_i (x)$, where (e_i) is the natural base of A^n. We also define $\varepsilon : K_0\to Q$ as the natural map : $K_0 = M\to M/ \sum\limits_{i=1}^{n} h_i (M) = Q$. Using the fact that $h_1, \ldots, h_n$ commute with each other, it is easy to verify that

$$(d_{i-1} \circ d_i) (\lambda \otimes x) = \sum_{i,j} (e_j \wedge e_i \wedge \lambda) \otimes h_j (h_i (x)) = 0;$$

also $\varepsilon d_1 = 0$. Thus $K.$ is really a complex.

Theorem 3 (Poincaré-Koszul).

If $(h_1, \ldots, h_n)$ is a regular sequence, then

$$H_i(K.) = \begin{cases} Q & if \quad i = 0 \\ & . \\ 0 & if \quad i > 0 \end{cases}$$

If $h_i \in A$, it defines the map: $A \xrightarrow{h_i I} A$, which we denote also by h_i. We say that $(h_1, \dots, h_n)$ is a regular sequence of elements if $(h_1 I, \dots, h_n I)$ is a regular sequence.

Corollary. If $(h_1, \dots, h_n)$ is a regular sequence of elements, then the Koszul complex $K. = K. [A; h_1, \dots, h_n] = \{ \wedge^{n-1} A^n \simeq A^{\binom{n}{i}} \}$ is a free resolution of $Q = A/(h_i)$ ((h_i) is the ideal generated by $h_1, \dots, h_n$)).

Example: If $A = \mathbf{C}\{x_1, \dots, x_n\}$; $h_i = x_i$, then $Q_k = A/(x_1, \dots, x_k) = \mathbf{C}\{x_{k+1}, \dots, x_n\}$ and $Q = Q_n = \mathbf{C}$. The complex $K. = K. [A; x_1, \dots, x_n]$ is a free resolution of $\mathbf{C}$.

(b) *Proof of theorem 2, when S is a complex manifold*

In this case we can take $\mathcal{O}_{S,s} = \mathbf{C}\{t_1, \dots, t_m\} = A$ and if $\mathcal{O}_{X,x} = \mathbf{C}\{x_1, \dots, x_n\}/(f_1, \dots, f_p)$, then

$$\mathcal{O}_{S \times X, (s,x)} = \mathbf{C}\{t_1, \dots, t_m, x_1, \dots, x_n\}/(f_1, \dots, f_p) = B\,.$$

B is an A-module in a natural way.

By the corollary of the Poincaré-Koszul theorem $K. = K. [A; t_1, \dots, t_m]$ in a free resolution of $\mathbf{C}$. We want to compute the modules $\operatorname{Tor}_i^A(\mathbf{C}, B) = H_i(K. \otimes B)$ $(i > 0)$.

It's easily seen, that we can consider the complex $K. \otimes B$ as a Koszul complex $K'. = K. [B; t_1, \dots, t_m]$ (where $t_i : B \xrightarrow{t_i I} B$). But now the sequence $(t_1, \dots, t_m)$ is regular, thus by the Poincaré-Koszul theorem $H_i [K'.] = 0$ if $i > 0$.

In particular: $\operatorname{Tor}_1^A(\mathbf{C}, B) = H_1 [K. \otimes B] = H_1 [K'.] = 0$. By the second flatness criterion B is A-flat.

(c) *The general case*

The question being local, we can suppose that $S \subset W \subset \mathbf{C}^n$, where W is open, and S an analytic subspace of W. Let S be defined by $g_1, \dots, g_r$. Then $S \times X \subset W \times X$ and $\mathcal{O}_S = \mathcal{O}_W/(g_1, \dots, g_r)$. On the other hand $\mathcal{O}_{S \times X} = \mathcal{O}_{W \times X}/(g_1, \dots, g_r) = \mathcal{O}_S \otimes_{\mathcal{O}_W} \mathcal{O}_{W \times X}$. The last equality follows from the fact, that if $\pi : X \to S$ is a morphism, and $S' \subset S$ a subspace, $X' = \pi^{-1}(S')$, then $\mathcal{O}_{X'} = \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X$.

Remark : This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi'"] \\ & S & \end{tikzcd} \qquad \mathcal{O}_{X \underset{S}{\times} Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_y .$$

We have proved that $\mathcal{O}_{W \times X}$ is $\mathcal{O}_W$-flat, so by scalar extension $\mathcal{O}_{S \times X}$ is $\mathcal{O}_S$ flat.

Corollary : If X and S are two manifolds and $\pi : X \to S$ is a submersion, then π is flat.

III. Privileged polycylinders

§ 1. *Banach vector bundles over an analytic space*

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X.

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E. If $U \subset \mathbf{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g : U \to E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X, i.e. X' is the support of the quotient sheaf $\mathcal{O}_U / J$, where $U \subset \mathbf{C}^n$ is open and J is a coherent sheaf of ideals of $\mathcal{O}_U$, then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \to \mathcal{H}(V, E) / J_V \,.\, \mathcal{H}(V, E)$ ($V \subset U$, V-open).

Remark : If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U.

The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models X' of X, i.e. $\mathcal{H}_X(E) | X' = \mathcal{H}_{X'}(E)$, for every local model X'.

Definition 1 : The set of *analytic morphisms* from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F.

Definition 2 : An analytic vector bundle morphism from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.