Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 14 (1968)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FLATNESS AND PRIVILEGE

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**Kapitel:** §4. Algebraic study of flatness

**DOI:** https://doi.org/10.5169/seals-42343

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The set of "nice" fibers is dense in X, so we cannot remove the z-axis and still get a closed subspace of  $\mathbb{C}_3$ .

# § 4. Algebraic study of flatness

In the following all rings are commutative, with 1, and all modules are unitary.

Definition 1: An A-module E is flat, if for every exact sequence of A-modules

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$$
,

the sequence  $0 \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$  is also exact. We can also say, because  $\otimes$  is right exact, that E is flat, if for every injective homomorphism  $F' \to F$ ,  $E \otimes F' \to E \otimes F$  is also injective.

Examples of modules which are not flat:

- (1) if  $A = \mathbb{Z}$ ,  $E = \mathbb{Z}_2 = \mathbb{Z}/2 \mathbb{Z}$ ,  $F = F' = \mathbb{Z}$ ; then the sequence  $0 \to \mathbb{Z} \to \mathbb{Z}$  ( $2I : x' \to 2x$ ) is exact. But now  $\mathbb{Z}_2 \otimes \mathbb{Z} = \mathbb{Z}_2$ , and the homomorphism  $\mathbb{Z}_2 \to \mathbb{Z}_2$  is the zero homomorphism, which is not injective. So  $\mathbb{Z}_2$  is not a flat  $\mathbb{Z}$  module.
- (2) If  $A = \mathbb{C}\{x\}$ ,  $E = \mathbb{C} = \mathbb{C}\{x\}/(x)$ ,  $F = F' = \mathbb{C}\{x\}$ , then the sequence  $0 \to F \xrightarrow{xI} F' (xI : p(x) \to xp(x))$  is exact. But the homomorphism  $E \xrightarrow{xI} E$  is not injective.

Proposition 1: If A is an integral domain and E a flat A-module, then E is torsion-free.

*Proof*: Let  $a \in A$ ,  $a \neq 0$ . Because A is an integral domain, the sequence  $0 \rightarrow AA \xrightarrow{aI} (aI : x \rightarrow ax)$  is exact. Since E is flat, the sequence  $0 \rightarrow E \xrightarrow{aI} E$  is also exact. In other words E has no torsion elements.

Proposition 2: If A is a principal-ideal domain, then E is flat if and only if E is torsionfree.

Proof: See corollary of prop. 6.

# Examples of flat modules:

(1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.

(2) Every free module is flat. In fact, if E is free and finite type, then  $E = A^n$  and  $E \otimes F = F^n$ . If  $F' \to F$  is injective, so is  $F'^n \to F^n$  too.

If E is an arbitrary free module, then it is an inductive limit of free modules of finite type, and the flatness of E follows from (1).

(3) Let S be a multiplicative system in A. Then the ring of fractions  $S^{-1}A$  is a flat A-module. In fact the ring  $S^{-1}A$  can be identified with an inductive limit of free modules, so it is flat ((1)(2)). We assume for simplicity that S has only regular elements. We can define in the set S a partial order in the following way:

$$s' \ge s \Leftrightarrow \exists t \in A$$
,  $ts = s'$  (such a t is then unique).

Let  $E_s = A$  for every  $s \in S$ , and if  $s' \ge s$  (i.e. s' = ts) then let  $f_s^{s'}$  be the homomorphism t.  $I_A : E_s \to E_{s'}$ . The family  $(E_s)_{s \in S}$  with the homomorphisms  $(f_s^{s'})$  is an inductive system.

Let  $E = \lim_{s \to \infty} E_s$  be the inductive limit of this system, and  $\varphi_s$  the canonical homomorphism  $E_s \to E$ . We shall define an isomorphism  $\psi : E \to S^{-1}A$ .

We first define for every s a homomorphism  $\psi_s: E_s = A \rightarrow S^{-1}A$ ;  $x \rightarrow x/s$ . Now if  $s' \geq s$ , then

$$(\psi_{s'} \circ f_{s'}^{s'})(x) = \psi_{s'}(tx) = \frac{tx}{s'} = \frac{tx}{ts} = \frac{x}{s} = \psi_{s}(x).$$

Therefore there exists a homomorphism  $\psi: E \to S^{-1}A$ , satisfying  $\psi_s = \psi \circ \varphi_s$  for every  $s \in S$ .

Because every element of  $S^{-1}A$  has the form a/s,  $\psi$  is surjective. On the other hand if  $\psi$  ( $\phi_s(x)$ ) = 0, then  $\psi_s(x) = x/s = 0$ . Thus x = 0, and  $\psi$  is also injective.

The above proof can be extended to the general case, not assuming that the elements of S are regular. The extended proof involves the notion of inductive limit of an inductive system indexed by a category instead of an ordered set.

From (1) and (2) above, any module which is the inductive limit of free modules, is flat. Conversely:

## Theorem 1: (Daniel, Lazard)

Any flat module is a inductive limit of free modules.

For the proof: See C.R. Acad. Sci. Paris, 258 (1964), pp. 6313-6316.

Some elementary properties of flat modules:

- (1) If E and F are flat A-modules, then  $E \otimes F$  is also flat. In fact, if  $G' \to G$  is injective, then  $F \otimes G' \to F \otimes G$  is injective, and also  $E \otimes (F \otimes G') \to E \otimes (F \otimes G)$  is injective. The result follows from the assosiativity of the tensor product.
- (2) Let  $\phi: A \rightarrow B$  be a ring homomorphism, and E a flat A-module. The module  $B \otimes E$  is a flat B-module.

If F is a B-module, then  $F \otimes (B \otimes E) = (F \otimes B) \otimes E = F \otimes E$  further if F' and F are B-modules, and  $F' \rightarrow F$  an injective homomorphism of B-modules, we can consider this homomorphism as an injective homomorphism of A-modules. Because E is A-flat,

$$F' \otimes_A E \rightarrow F \otimes_A E$$
 is injective.

(3) Let  $\phi: A \to B$  be a ring homomorphism, such that B is a flat A-module. If F is a flat B-module, then F is a flat A-module. In fact: if  $E' \to E$  is injective, then  $E' \otimes B \to E \otimes B$  is injective, and also  $(E \otimes B) \otimes F' \to (E \otimes B) \otimes F$  is injective. But  $(E' \otimes_A B)_B \otimes F' = E' \otimes_A F$ ;  $(E \otimes_A B) \otimes_B F = E \otimes_A F$ .

If an A-module E is not flat, we want to measure how far it is from being flat. For this purpose we introduce the functor Tor.

Definition 2: A free resolution of E is an exact sequence:  $... \rightarrow L_n \rightarrow L_{n-1} \rightarrow ... \rightarrow L_1 \rightarrow L_0 \rightarrow E \rightarrow 0$ , where all  $L_i$  are free A-modules.

The complex of the resolution is the sequence

(L.) ... 
$$\rightarrow L_n \rightarrow L_{n-1} \rightarrow ... \rightarrow L_1 \rightarrow L_0 \rightarrow 0$$
.

Every module has a free resolution. Two resolutions are algebraically homotopy-equivalent. Forming the tensorproducts  $L_i \otimes F$ , we get

$$(L. \otimes F) \dots \to L_n \otimes F \to L_{n-1} \otimes F \to \dots \to L_1 \otimes F \to L_0 \otimes F \to 0.$$

Definition 3:

$$\operatorname{Tor}_{n}^{A}(E, F) = H_{n}(L \otimes F) = \frac{\operatorname{Ker}(L_{n} \otimes F \to L_{n-1} \otimes F)}{\operatorname{Im}(L_{n+1} \otimes F \to L_{n} \otimes F)}$$

if 
$$n \ge 1$$
, and  $\operatorname{Tor}_0^A(E, F) = \operatorname{Coker}(L_1 \otimes F \to L_0 \otimes F) = E \otimes F$ .

Basic properties of Tor:

(1)  $\operatorname{Tor}_n(E, F)$  is independent of the choice of the resolution (up to a canonical isomorphism).

- (2) If we take a free resolution of F, we get  $\operatorname{Tor}_n(F, E) = \operatorname{Tor}_n(E, F)$  (Symmetry of the Tor). We can also define  $\operatorname{Tor}_n(E, F)$  by taking two free resolutions, one of E and one of F.
- (3) If  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$  is a short exact sequence, then we get a long exact sequence:

$$\operatorname{Tor}_{n}(E',F) \to \operatorname{Tor}_{n}(E,F) \to \operatorname{Tor}_{n}(E'',F) \to \\ \to \operatorname{Tor}_{n-1}(E',F) \to \operatorname{Tor}_{n-1}(E,F) \to \operatorname{Tor}_{n-1}(E'',F) \to \\ \to - - - - - - - - - - - - - - - - \to \\ \to \operatorname{Tor}_{1}(E',F) \to \operatorname{Tor}_{1}(E,F) \to \operatorname{Tor}_{1}(E'',F) \to \\ \to E' \otimes F \to E \otimes F \to E'' \otimes F \to 0.$$

- (4) Tor is compatible with inductive limit, i.e. if  $E = \lim_{\longrightarrow} (E_i)$ , then  $Tor_n(\lim_{\longrightarrow} E_i, F) = \lim_{\longrightarrow} (Tor_n(E_i, F))$ .
- (5) We can define Tor<sub>n</sub> (E, F) by taking a flat resolution of E.
  Proposition 3: Let E be an A-module. Then the following conditions are equivalent:
- (a) E is flat.
- (b) For all A-modules F, and for all  $n \ge 1$ ,  $\operatorname{Tor}_n(E, F) = 0$ .
- (c) For all A-modules F,  $Tor_1(E, F) = 0$ .

*Proof*: (a)  $\Rightarrow$  (b). If ...  $\rightarrow L_n \rightarrow L_{n-1} \rightarrow ... \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0$  is a free resolution of F, then the sequence

$$\dots \to E \otimes L_n \to E \otimes L_{n-1} \to \dots \to E \otimes L_1 \to E \otimes L_0 \to E \otimes F \to 0$$

is exact, thus  $\operatorname{Tor}_n(E, F) = 0$  for all  $n \ge 1$ .

 $(b)\Rightarrow (c)$  clear.  $(c)\Rightarrow (a)$ : If the sequence  $0\to F'\to F\to F''\to 0$  is exact, so is also (by (3) above)  $\operatorname{Tor}_1(E,F'')\to E\otimes F'\to E\otimes F\to E\otimes F''\to 0$ . Now  $\operatorname{Tor}_1(E,F'')=0$ , thus E is flat.

Proposition 4: If I and J are two ideals in A, then  $\operatorname{Tor}_1^A(A/I, A/J) = I \cap J/I$ . J.

*Proof*: From the exact sequence  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ , we get the exact sequence:

 $\operatorname{Tor}_1(A,A/J) \to \operatorname{Tor}_1(A/I,A/J) \to I \otimes A/J \to A \otimes A/J \to A/I \otimes A/J \to 0.$  But now  $\operatorname{Tor}_1(A,A/J) = 0 \quad (A \text{ beeing } A\text{-free}), \text{ and } I \otimes A/J = I/I \cdot J;$   $A \otimes A/J = A/J. \text{ Therefore the sequence } 0 \to \operatorname{Tor}_1(A/I,A/J) \to I/I \cdot J \to A/J \text{ is exact, and } \operatorname{Tor}_1(A/I,A/J) = \operatorname{Ker}(I/I \cdot J \to A/J) = I \cap J/I \cdot J.$ 

Example: Let U be an open set in  $\mathbb{C}^n$ , and  $x \in U$ . Further let  $X, Y \subset U$  be two hypersurfaces, defined by I = (f) and J = (g). Supposing that f and g do not have common factors:  $I_x \cap J_x = I_x J_x$ , and

$$\operatorname{Tor}_{1}\left(\mathcal{O}_{X,x},\mathcal{O}_{Y,x}\right) = \operatorname{Tor}_{1}\left(\mathcal{O}_{U,x}/I_{x}, \quad \mathcal{O}_{U,x}/J_{x}\right) = \frac{I_{x} \cap J_{x}}{I_{x} \cdot J_{x}} = 0.$$

Heuristic remark: The formula  $\operatorname{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_Y, x) = 0$  expresses the fact that X and Y are "in general position". If for example X and Y are two linears subspaces in  $\mathbb{C}^n$  of dimensions p and q, we have  $\operatorname{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$  if  $\dim(X \cap Y) = p + q - n$ , and  $\operatorname{Tor}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) \neq 0$  otherwise.

Next we shall prove an elementary flatness criterion.

Proposition 5: Let E be an A-module. The following conditions are equivalent:

- (a) E is flat.
- (b) For all finitely generated ideals I of A,  $Tor_1(E, A/I) = 0$ .
- (c) For all monogenous A-modules F,  $Tor_1(E, F) = 0$ .

*Proof*:  $(a) \Rightarrow (b)$ , by prop. 3.

- $(b) \Rightarrow (c)$ : Because Tor is compatible with inductive limit, we can suppose, that  $\text{Tor}_1(E, A/I) = 0$  for an arbitrary ideal I of A. But every monogenous A-module F can be represented by A/I.
- $(c) \Rightarrow (a)$ . By prop. 3 it is sufficient to prove that  $Tor_1(E, F) = 0$  for any A-module F.

First consider the case, where F is finitely generated. We use induction, supposing that  $\operatorname{Tor}_1(E,F)=0$ , when F has n generators. Let F have (n+1) generators  $x_1, ..., x_n, x_{n+1}$ . If F' is the submodule generated by  $\{x_1, ..., x_n\}$ , then  $F' \subset F$  and F'' = F/F' is monogenous. The exact sequence  $0 \to F' \to F \to F'' \to 0$  gives the exact sequence  $\operatorname{Tor}_1(E, F') \to \operatorname{Tor}_1(E, F) \to \operatorname{Tor}_1(E, F'')$ . Now  $\operatorname{Tor}_1(E, F') = \operatorname{Tor}_1(E, F'') = 0$ , thus  $\operatorname{Tor}_1(E, F) = 0$ . In the general case, F can be considered as an inductive limit of finitely generated modules, and because  $\operatorname{Tor}_1(E, F) = 0$ . (E, F) = 0.

Proposition 6: Let A be an integral domain, and E an A-module. Then E is torsionfree if and only if  $Tor_1(E, A/(a)) = 0$ , for any element  $a \in A$ .

*Proof*: If E is A-module,  $a \in A$ , then the exact sequence  $0 \to A \to A \to A \to A \to A \to A/(a) \to 0$  gives the exact sequence  $0 \to Tor_1(E, A/(a)) \to E \to E$ . In other words  $Tor_1(E, A/(a)) = \{x \in E \mid ax = 0\}$ , from which the result follows.

Corollary: Let A be a principal ideal domain. E is flat if and only if E is torsionfree.

*Proof*: We have already proved that, if E is flat, then it is torsion free. The converse follows from prop. 6 and prop. 5.

The first flatness criterion for noetherian local rings is the following:

Theorem 2: Let A be a noetherian local ring with maximal ideal m; k = A/m, and E a finitely generated A-module. The following conditions are equivalent:

- (a) E is free.
- (b) E is flat.
- (c)  $\operatorname{Tor}_{1}^{A}(E, k) = 0$ .

*Proof*: We have already proved  $(a) \Rightarrow (b) \Rightarrow (c)$ .

 $(c) \Rightarrow (a)$ : We recall first Nakayma's lemma. If A is a local ring with maximal ideal m; k=A/m, and E is a finitely generated A-module, such that  $k\otimes E=E/mE=0$ , then E=0.

The module  $\overline{E} = k \otimes E = E/mE$  is a finitely generated vector space over k. Let  $\{\overline{x}_1, ..., \overline{x}_r\}$  be a base of  $\overline{E}$  (over k), and  $\{x_1, ..., x_r\}$  E representatives of  $\overline{x}_i$ : s. Consider the homomorphism  $\phi: A^r \to E$ ,  $\phi(a_1, ..., a_r) = \sum a_i x_i$ . Denoting by R and Q the kernel and the cokernel of  $\phi$ , we get an exact sequence:

$$(*) 0 \to R \to A^r \to E \to Q \to 0$$

and R, Q are finitely generated A-modules. From (\*) we get the exact sequence

$$A^r \underset{A}{\otimes} k \to E \underset{A}{\otimes} k \to Q \underset{A}{\otimes} k \to 0$$
.

But  $\overline{E} = E \otimes k \simeq k^r = A^r \otimes k$ , so  $Q \otimes k = 0$ , and by Nakayama's lemma Q = 0.

Therefore ge have an exact sequence

$$0 \rightarrow R \rightarrow A^r \rightarrow E \rightarrow 0$$
.

From this we get:  $\operatorname{Tor}_1(E, k) \to k \otimes R \to k^r \to \overline{E} \to 0$  (exact). Now:  $\overline{E} \simeq k^r$ ,  $\operatorname{Tor}_1(E, k) = 0$  (by assumption). Therefore  $k \otimes R = 0$ , and once more by Nakayama's lemma R = 0, thus  $E \simeq A^r$ , i.e. E is free.

Proposition 7: Let  $\phi: A \to B$  be a ring homomorphism, and let B be A-flat. If I is an ideal of A, we write  $\overline{A} = A/I$ ,  $\overline{B} = B/IB = \overline{A} \otimes B$ . Let F be a B-module, then:  $\operatorname{Tor}_{i}^{A}(\overline{A}, F) = \operatorname{Tor}_{i}^{B}(\overline{B}, F)$   $(i \ge 0)$ .

*Proof*: We choose first a B-free resolution of F

$$\rightarrow L_{n+1} \rightarrow L_n \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0$$
.

If L. is the respective complex of resolution, then

$$\overline{B} \underset{B}{\otimes} L. = B/IB \underset{B}{\otimes} L. = \overline{A} \underset{A}{\otimes} (B \underset{B}{\otimes} L.) = \overline{A} \underset{A}{\otimes} L.$$

Because every  $L_i$  is B-free, and B is A-flat, every  $L_i$  is A-flat (Property 3 after Th. 1). Thus L. is a flat A-resolution, and

$$\operatorname{Tor}_{i}^{A}(\overline{A}, F) = H_{i}(\overline{A} \otimes L.) = H_{i}(\overline{B} \otimes L.) = \operatorname{Tor}_{i}^{B}(\overline{B}, F).$$

We shall next state the second flatness criterion for noetherian local rings.

Theorem 3: Let A and B be two noetherian local rings, with maximal ideals  $\underline{m}$ ,  $\underline{n}$ ;  $k = A/\underline{m}$ . If  $\phi : A \rightarrow B$  is a local homomorphism (i.e.  $\phi (\underline{m}) \subset \underline{n}$ ), and F finitely generated B module then

$$F$$
 is A-flat  $\Leftrightarrow$  Tor  $\frac{A}{1}(k, F) = 0$ .

The proof of this theorem is much more difficult than that of th. 20 see for example:

Bourbaki: Algèbre commutative, Chapter III § 5, th1,  $(i) \Leftrightarrow (iii)$ , p. 98.

The conditions in Bourbaki's theorem are here fullfilled:

- 1° A finitely generated module F over a noetherian local ring B is idealwise separated for n. (Ibid., § 5. 1. Ex. 1, p. 97.)
- 2° If  $\phi: A \to B$  is a local homomorphism, F is also idealwise separated for  $\underline{m}$ . (*Ibid.*, § 5, prop. 2, p. 101.)
- $3^{\circ}$  Also the flatness condition is fulfilled, because k is a field.

Remark: The main interest of the theorem lies in the fact, that it is true without any assumption of finitness on B.

Corollary: If the assumptions are the same as in the theorem 3, and if moreover B is A-flat, then

$$F ext{ is } A ext{-flat} \Leftrightarrow \operatorname{Tor}_1^B(\overline{B}, F) = 0$$

where  $\overline{B} = B/mB$ .

*Proof*:  $\operatorname{Tor}_{1}^{A}(k, F) = \operatorname{Tor}_{1}^{B}(\overline{B}, F)$ , by prop. 7.

### § 5. Geometric applications of the flatness criterions

### A) Flatness for finite morphisms

Proposition 1: Let  $\pi: X \to S$  be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then  $\pi_*(\mathcal{O}_X)$  is a coherent analytic sheaf over S. The following conditions are equivalent:

- (a)  $\pi$  is flat (i.e. for every  $x \in X$ ,  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module,  $s = \pi(x)$ ).
- (b) For every s,  $(\pi_* \mathcal{O}_X)_s$  is a flat  $\mathcal{O}_{S,s}$ -module.
- (c)  $\pi_* \mathcal{O}_X$  is a locally free sheaf.

*Proof*: Because  $\pi$  is finite  $\pi_* (\mathcal{O}_X)_s = \bigoplus_{x \in \pi^{-1}(s)} \mathcal{O}_{X,x}$ , thus the only point to prove is  $(b) \Rightarrow (c)$ .

Now if  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module, then (by theorem 2)  $\mathcal{O}_{X,x}$  is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

Proposition 2: Let S be a reduced analytic space and  $\mathscr E$  a coherent  $\mathscr O_s$ -module. Let E(s) be the finite dimensional vector space (over C)  $\mathscr E_s \otimes_{\mathscr O} C_s$ .  $\mathscr E$  is a locally free  $\mathscr O_{S,s}$ -module if an only if  $\dim_C E(s)$  is locally constant.

*Proof*: If  $\mathscr{E}$  is locally free, then  $\dim_{\mathbb{C}} E(s)$  is locally constant. Suppose now that  $\dim_{\mathbb{C}} E(s)$  is locally constant in an open set  $U \subset S$ , and that  $\mathcal{O}_U^p \to \mathcal{O}_U^q \to \mathcal{E}_U \to 0$  is exact. d is determined by a  $p \times q$  matrix of analytic functions on U, so it gives a morphism  $\mathbf{C}_U^p \to \mathbf{C}_U^q$  of trivial vector bundles over U.

From the exact sequence  $\mathcal{O}_s^p \to \mathcal{O}_s^q \to \mathcal{E}_s \to 0$ , we get (by making tensor-products with  $C_s$ ) the exact sequence:

$$\mathbf{C}_{s}^{p} \stackrel{d(s)}{\rightarrow} \mathbf{C}_{s}^{q} \rightarrow E(s) \rightarrow 0$$
,

which shows that d has constant rank in U. Thus Ker d and Im d are vector bundles, and we can write

$$\mathbf{C}_{\mathit{U}}^{\mathit{p}} = \mathit{F}_{1} \oplus \mathit{G}_{1} \;, \quad \mathbf{C}_{\mathit{U}}^{\mathit{q}} = \mathit{F}_{0} \oplus \mathit{G}_{0} \;,$$
 
$$d: \left\{ egin{aligned} F_{1} &\rightarrow 0 \\ G_{1} &\simeq \mathit{F}_{0} \;. \end{aligned} \right.$$