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Definition 3 : Let X', X be analytic spaces, Y a closed analytic subspace of X defined by J, and $f = (f_0, f^1) : X' \to X$ a morphism.

The inverse image of Y by f, $f^{-1}(Y)$, is the analytic subspace Y' of X' defined by the ideal $J' = f^1(J)\,\mathcal{O}_{X'}$.

The inverse image of a simple point x in X is called the f-fiber over x, and is denoted by $f^{-1}(x)$ or $X'(x)$.

Proposition 1 : If $f = (f_0, f^1) : X' \to X$ is a morphism of analytic spaces, and Y is a subspace of X, then $f^{-1}(Y) \simeq Y \times_X X'$.

Proof : Let T be any analytic space, and $g : T \to X'$ a morphism. Then g can be considered as a morphism from T to $f^{-1}(Y)$ if and only if $f \circ g$ can be considered as a morphism from T to Y. Thus $f^{-1}(Y)$ and $X' \times_X X$ are solutions of the same universal problem.

§ 2. *Analytic pull-back*

In the following we want to generalize the notion of inverse image of a subspace.

We shall first recall the basic properties of the tensor product $E \otimes_A F$, where A is a commutative ring and E, F are two A-modules.

(1°) $E \otimes A^n = E^n$ $(n \in N)$

(2°) If the sequence of A-modules $F' \to F \to F'' \to 0$ is exact, then also the sequence $E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$ is exact. (Right exactness of the tensor product)

(3°) If $(F_i)_{i \in I}$; $f_{ij} : F_j \to F_i$ is an inductive system, then

$$E \otimes \varinjlim F_i = \varinjlim (E \otimes F_i).$$

On the other hand these properties characterize completely the functor $\otimes$.

Definition 1 : Let $f = (f_0\ f^1) : X' \to X$ be a morphism of analytic spaces, and $\mathcal{E}$ an $\mathcal{O}_X$-module. Then $f_0^*\mathcal{E}$ is an $f_0^*\,\mathcal{O}_X$-module and $\mathcal{O}_{X'}$ is also an $f_0^*\,\mathcal{O}_X$-module (by $f^1 : f_0^*\,\mathcal{O}_X \to \mathcal{O}_{X'}$).

The analytic pull-back $f^*\mathcal{E}$ of $\mathcal{E}$ by f is defined by scalar extension:

$$f^*\mathcal{E} = f_0^*\mathcal{E} \otimes_{f_0^*\mathcal{O}_X} \mathcal{O}_{X'}$$

Remark: The inverse image is a particular case of the analytic pull-back.

In fact, if Y is a closed analytic subspace of X and $f: X' \to X$ is a morphism:

$$f^* \mathcal{O}_Y = f_0^* (\mathcal{O}_X / J_Y) \underset{f_0^* \mathcal{O}_X}{\otimes} \mathcal{O}_{X'} \simeq f_0^* \mathcal{O}_X / f_0^* J_Y \underset{f_0^* \mathcal{O}_X}{\otimes} \mathcal{O}_{X'}$$

$$\simeq \mathcal{O}_{X'} / f^1 (J_Y) . \mathcal{O}_{X'} \simeq \mathcal{O}_{f^{-1}(Y)}$$

(The third isomorphism follows from the fact, that $A/I \underset{A}{\otimes} E \simeq E/IE$).

Elementary properties of the analytic pull-back:

(*a*) $(f^* \mathscr{E})_{x'} = (f_0^* \mathscr{E})_{x'} \otimes_{(f_0^* \mathcal{O}_X)_{x'}} \mathcal{O}_{X',x'} \simeq \mathscr{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ where $x = f_0(x')$ (since $\otimes$ commutes with inductive limits).

(*b*) $f^* (\mathscr{E} \underset{\mathcal{O}X}{\otimes} \mathscr{F}) = f^* \mathscr{E} \underset{\mathcal{O}X'}{\otimes} f^* \mathscr{F}$, where $\mathscr{E}$ and $\mathscr{F}$ are $\mathcal{O}_X$-modules.

(*c*) If $\mathscr{E}$ is a coherent $\mathcal{O}_X$-module, then $f^* \mathscr{E}$ is a coherent $\mathcal{O}_{X'}$-module.

In fact, $\mathscr{E}$ has a locally finite presentation: $\mathcal{O}_X^q \to \mathcal{O}_X^p \to \mathscr{E} \to 0$, and f^* is compatible with cokernels, $f^* (\mathcal{O}_X^r) = \mathcal{O}_{X'}^r$.

Special case: The pull-back of vector bundle. Let (E, π) be an analytic vector bundle over the analytic space X, and $f: X' \to X$ a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over X', such that $\bar{f}$ is a bundle morphism. We call this bundle E'.

$$\begin{array}{ccc} E \underset{X}{\times} X' & \xrightarrow{\bar{f}} & E \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

Proposition 1: Let $\mathscr{E}$ (Resp. $\mathscr{E}'$) be the sheaf of analytic sections of E (Resp. E'). Then $\mathscr{E}' = f^* \mathscr{E}$.

Proof (Sketch): We have a $f_0^* \mathcal{O}_X$ linear morphism $f_0^* \mathscr{E} \to \mathscr{E}'$, which extends to a morphism $f^* \mathscr{E} \to \mathscr{E}'$. We can prove that this is an isomorphism. Since the question is local with respect to X', we can suppose that E is a trivial bundle over X with fiber $\mathbf{C}^r$, then $\mathscr{E} = \mathcal{O}_X^r$. Also $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$. Therefore $f^* \mathscr{E} = \mathscr{E}'$.

§ 3. *Introduction to flatness by examples*

Let S be an analytic space. By analytic space over s we mean an analytic space X provided with a morphism $\pi: X \to S$. Let S be a simple point in S, and consider $X(s) = f^{-1}(s)$.