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Autor: Douady, A.
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Definition 3: Let X', X be analytic spaces, Y a closed analytic subspace of X defined by J , and $f = (f_0, f^1) : X' \rightarrow X$ a morphism.

The inverse image of Y by $f, f^{-1}(Y)$, is the analytic subspace Y' of X' defined by the ideal $J' = f^1(J) \mathcal{O}_{X'}$.

The inverse image of a simple point x in X is called the f -fiber over x , and is denoted by $f^{-1}(x)$ or $X'(x)$.

Proposition 1: If $f = (f_0, f^1) : X' \rightarrow X$ is a morphism of analytic spaces, and Y is a subspace of X , then $f^{-1}(Y) \simeq \underset{X}{Y \times X'}$.

Proof: Let T be any analytic space, and $g : T \rightarrow X'$ a morphism. Then g can be considered as a morphism from T to $f^{-1}(Y)$ if and only if $f \circ g$ can be considered as a morphism from T to Y . Thus $f^{-1}(Y)$ and $\underset{X}{X' \times X}$ are solutions of the same universal problem.

§ 2. Analytic pull-back

In the following we want to generalize the notion of inverse image of a subspace.

We shall first recall the basic properties of the tensor product $E \otimes_A F$, where A is a commutative ring and E, F are two A -modules.

(1°) $E \otimes A^n = E^n$ ($n \in \mathbb{N}$)

(2°) If the sequence of A -modules $F' \rightarrow F \rightarrow F'' \rightarrow 0$ is exact, then also the sequence $E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$ is exact. (Right exactness of the tensor product)

(3°) If $(F_i)_{i \in I}; f_{ij} : F_j \rightarrow F_i$ is an inductive system, then

$$E \otimes \lim_{\rightarrow} F_i = \lim_{\rightarrow} (E \otimes F_i).$$

On the other hand these properties characterize completely the functor \otimes .

Definition 1: Let $f = (f_0, f^1) : X' \rightarrow X$ be a morphism of analytic spaces, and \mathcal{E} an \mathcal{O}_X -module. Then $f_0^* \mathcal{E}$ is an $f_0^* \mathcal{O}_X$ -module and $\mathcal{O}_{X'}$ is also an $f_0^* \mathcal{O}_X$ -module (by $f^1 : f_0^* \mathcal{O}_X \rightarrow \mathcal{O}_{X'}$).

The analytic pull-back $f^* \mathcal{E}$ of \mathcal{E} by f is defined by scalar extension:

$$f^* \mathcal{E} = f_0^* \mathcal{E} \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'}$$

Remark : The inverse image is a particular case of the analytic pull-back.

In fact, if Y is a closed analytic subspace of X and $f : X' \rightarrow X$ is a morphism:

$$f^* \mathcal{O}_Y = f_0^* (\mathcal{O}_X / J_Y) \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'} \simeq f_0^* \mathcal{O}_X / f_0^* J_Y \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'}$$

$$\simeq \mathcal{O}_{X'} / f^1(J_Y) \cdot \mathcal{O}_{X'} \simeq \mathcal{O}_{f^{-1}(Y)}$$

(The third isomorphism follows from the fact, that $A/I \otimes_A E \simeq E/IE$).

Elementary properties of the analytic pull-back :

- (a) $(f^* \mathcal{E})_{x'} = (f_0^* \mathcal{E})_{x'} \otimes_{(f_0^* \mathcal{O}_X)_{x'}} \mathcal{O}_{X',x'} \simeq \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$ where $x = f_0(x')$ (since \otimes commutes with inductive limits).
- (b) $f^* (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f^* \mathcal{E} \otimes_{\mathcal{O}_{X'}} f^* \mathcal{F}$, where \mathcal{E} and \mathcal{F} are \mathcal{O}_X -modules.
- (c) If \mathcal{E} is a coherent \mathcal{O}_X -module, then $f^* \mathcal{E}$ is a coherent $\mathcal{O}_{X'}$ -module.

In fact, \mathcal{E} has a locally finite presentation:

$$\mathcal{O}_X^q \rightarrow \mathcal{O}_X^p \rightarrow \mathcal{E} \rightarrow 0, \text{ and } f^* \text{ is compatible with cokernels, } f^* (\mathcal{O}_X^r) = \mathcal{O}_{X'}^r.$$

Special case : The pull-back of vector bundle. Let (E, π) be an analytic

$$\begin{array}{ccc} E \times X' & \xrightarrow{\bar{f}} & E \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

vector bundle over the analytic space X , and $f : X' \rightarrow X$ a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over X' , such that \bar{f} is a bundle morphism. We call this bundle E' .

Proposition 1 : Let \mathcal{E} (Resp. \mathcal{E}') be the sheaf of analytic sections of E (Resp. E'). Then $\mathcal{E}' = f^* \mathcal{E}$.

Proof (Sketch) : We have a $f_0^* \mathcal{O}_X$ linear morphism $f_0^* \mathcal{E} \rightarrow \mathcal{E}'$, which extends to a morphism $f^* \mathcal{E} \rightarrow \mathcal{E}'$. We can prove that this is an isomorphism. Since the question is local with respect to X' , we can suppose that E is a trivial bundle over X with fiber \mathbf{C}^r , then $\mathcal{E} = \mathcal{O}_X^r$. Also $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$. Therefore $f^* \mathcal{E} = \mathcal{E}'$.

§ 3. Introduction to flatness by examples

Let S be an analytic space. By analytic space over s we mean an analytic space X provided with a morphism $\pi : X \rightarrow S$. Let S be a simple point in S , and consider $X(s) = f^{-1}(s)$.