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FLATNESS AND PRIVILEGE

by A. DOUADY

I. FLAT MORPHISMS

§ 1. *Analytic subspaces of an analytic space*

Let Y_1 and Y_2 be closed analytic subspaces of an analytic space X , and let them be defined by the \mathcal{O}_X ideals J_1, J_2 .

Definition 1: We say that Y_1 is *analytically included* in Y_2 , and we write $Y_1 \subset Y_2$, when $J_1 \supset J_2$.

Remark: The analytic inclusion implies the set theoretic inclusion, but the converse is not true.

Example: $X = (\mathbb{C}, \mathcal{O}_{\mathbb{C}})$; $J_1 = (x)$, $J_2 = (x^2)$. The space Y_1 is a simple point, Y_2 is a double point, $Y_1 \not\subset Y_2$, while they have the same underlying set.

Definition 2: The subspace $Y_1 \cup Y_2$ is the smallest subspace of X containing Y_1 and Y_2 , and it is defined by $J_1 \cap J_2$. The subspace $Y_1 \cap Y_2$ is the biggest subspace of X contained in both Y_1 and Y_2 , and it is defined by $J_1 + J_2$.

Remark: The underlying set of $Y_1 \cup Y_2$ (Resp. $Y_1 \cap Y_2$) is the union (Resp. intersection) of the underlying sets of Y_1 and Y_2 . However \cup and \cap of analytic spaces do not satisfy the distributivity laws which hold in set-theory: $(Y_1 \cup Y_2) \cap Y_3$ contains $Y_1 \cap Y_3$ and $Y_2 \cap Y_3$, and therefore their union; similarly $(Y_1 \cap Y_2) \cup Y_3 \subset (Y_1 \cup Y_3) \cap (Y_2 \cup Y_3)$. In general the converse inclusions do not hold.

Example: Let $X = \mathbb{C}^2$ and Y_1, Y_2, Z be given by ideals $(x-y)$, $(x+y)$ and (x) respectively.

$(Y_1 \cup Y_2) \cap Z$ is $\{0\}$ provided with $\mathbb{C}\{y\}/(y^2)$, while $(Y_1 \cap Y_2) \cup (Y_2 \cap Z)$ is the reduced space $\{0\}$. On the other hand: $Y_1 \cap Y_2 \subset Z$, $(Y_1 \cap Y_2) \cup Z = Z$, while $(Y_1 \cup Z) \cap (Y_2 \cup Z)$ is the space defined by the ideal (x^2, xy) . Its local ring at the origin is $\mathbb{C}\{x, y\}/(x^2, xy)$ in which x is nilpotent.

Definition 3 : Let X', X be analytic spaces, Y a closed analytic subspace of X defined by J , and $f = (f_0, f^1) : X' \rightarrow X$ a morphism.

The inverse image of Y by $f, f^{-1}(Y)$, is the analytic subspace Y' of X' defined by the ideal $J' = f^1(J) \mathcal{O}_{X'}$.

The inverse image of a simple point x in X is called the f -fiber over x , and is denoted by $f^{-1}(x)$ or $X'(x)$.

Proposition 1 : If $f = (f_0, f^1) : X' \rightarrow X$ is a morphism of analytic spaces, and Y is a subspace of X , then $f^{-1}(Y) \simeq Y \times_X X'$.

Proof : Let T be any analytic space, and $g : T \rightarrow X'$ a morphism. Then g can be considered as a morphism from T to $f^{-1}(Y)$ if and only if $f \circ g$ can be considered as a morphism from T to Y . Thus $f^{-1}(Y)$ and $X' \times_X Y$ are solutions of the same universal problem.

§ 2. Analytic pull-back

In the following we want to generalize the notion of inverse image of a subspace.

We shall first recall the basic properties of the tensor product $E \otimes_A F$, where A is a commutative ring and E, F are two A -modules.

(1°) $E \otimes A^n = E^n$ ($n \in \mathbb{N}$)

(2°) If the sequence of A -modules $F' \rightarrow F \rightarrow F'' \rightarrow 0$ is exact, then also the sequence $E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$ is exact. (Right exactness of the tensor product)

(3°) If $(F_i)_{i \in I}; f_{ij} : F_j \rightarrow F_i$ is an inductive system, then

$$E \otimes \lim_{\rightarrow} F_i = \lim_{\rightarrow} (E \otimes F_i).$$

On the other hand these properties characterize completely the functor \otimes .

Definition 1 : Let $f = (f_0, f^1) : X' \rightarrow X$ be a morphism of analytic spaces, and \mathcal{E} an \mathcal{O}_X -module. Then $f_0^* \mathcal{E}$ is an $f_0^* \mathcal{O}_X$ -module and $\mathcal{O}_{X'}$ is also an $f_0^* \mathcal{O}_X$ -module (by $f^1 : f_0^* \mathcal{O}_X \rightarrow \mathcal{O}_{X'}$).

The analytic pull-back $f^* \mathcal{E}$ of \mathcal{E} by f is defined by scalar extension:

$$f^* \mathcal{E} = f_0^* \mathcal{E} \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'}$$

Remark : The inverse image is a particular case of the analytic pull-back.

In fact, if Y is a closed analytic subspace of X and $f : X' \rightarrow X$ is a morphism:

$$\begin{aligned} f^* \mathcal{O}_Y &= f_0^* (\mathcal{O}_X / J_Y) \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'} \simeq f_0^* \mathcal{O}_X / f_0^* J_Y \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'} \\ &\simeq \mathcal{O}_{X'} / f^1(J_Y) \cdot \mathcal{O}_{X'} \simeq \mathcal{O}_{f^{-1}(Y)} \end{aligned}$$

(The third isomorphism follows from the fact, that $A/I \otimes_A E \simeq E/IE$).

Elementary properties of the analytic pull-back :

- (a) $(f^* \mathcal{E})_{x'} = (f_0^* \mathcal{E})_{x'} \otimes_{(f_0^* \mathcal{O}_X)_{x'}} \mathcal{O}_{X', x'} \simeq \mathcal{E}_x \otimes_{\mathcal{O}_{X, x}} \mathcal{O}_{X', x'}$ where $x = f_0(x')$ (since \otimes commutes with inductive limits).
- (b) $f^* (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f^* \mathcal{E} \otimes_{\mathcal{O}_{X'}} f^* \mathcal{F}$, where \mathcal{E} and \mathcal{F} are \mathcal{O}_X -modules.
- (c) If \mathcal{E} is a coherent \mathcal{O}_X -module, then $f^* \mathcal{E}$ is a coherent $\mathcal{O}_{X'}$ -module.

In fact, \mathcal{E} has a locally finite presentation:

$\mathcal{O}_X^q \rightarrow \mathcal{O}_X^p \rightarrow \mathcal{E} \rightarrow 0$, and f^* is compatible with cokernels, $f^*(\mathcal{O}_X^r) = \mathcal{O}_{X'}^r$.

Special case : The pull-back of vector bundle. Let (E, π) be an analytic

$$\begin{array}{ccc} E \times X' & \xrightarrow{\bar{f}} & E \\ \downarrow \pi' & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array}$$

vector bundle over the analytic space X , and $f : X' \rightarrow X$ a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over X' , such that \bar{f} is a bundle morphism. We call this bundle E' .

Proposition 1 : Let \mathcal{E} (Resp. \mathcal{E}') be the sheaf of analytic sections of E (Resp. E'). Then $\mathcal{E}' = f^* \mathcal{E}$.

Proof (Sketch) : We have a $f_0^* \mathcal{O}_X$ linear morphism $f_0^* \mathcal{E} \rightarrow \mathcal{E}'$, which extends to a morphism $f^* \mathcal{E} \rightarrow \mathcal{E}'$. We can prove that this is an isomorphism. Since the question is local with respect to X' , we can suppose that E is a trivial bundle over X with fiber \mathbf{C}^r , then $\mathcal{E} = \mathcal{O}_X^r$. Also $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$. Therefore $f^* \mathcal{E} = \mathcal{E}'$.

§ 3. Introduction to flatness by examples

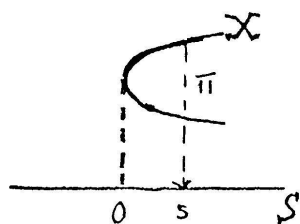
Let S be an analytic space. By analytic space over s we mean an analytic space X provided with a morphism $\pi : X \rightarrow S$. Let S be a simple point in S , and consider $X(s) = f^{-1}(s)$.

The main purpose of these lectures is to give a precise meaning to the expression:

“ $X(s)$ depends nicely on s ”, and to give a criterion for the “nice” behaviour.

We begin with some examples.

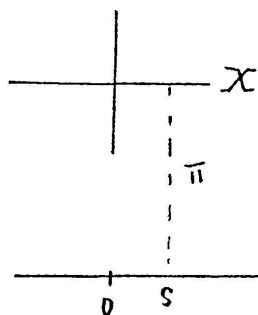
Example 1: X is the closed subspace on \mathbf{C}^2 defined by $(y^2 - x)$, $S = \mathbf{C}$ and $\pi = 1\text{st projection}$.



$$X(s) = \begin{cases} 2 \text{ simple points if } s \neq 0 \\ \text{double point if } s = 0. \end{cases}$$

Here the behaviour of $X(s)$ is nice.

Example 2: X is the closed subspace of \mathbf{C}^2 defined by (xy) , $S = \mathbf{C}$ and $\pi = 1\text{st projection}$.



$X(s)$ is given by $(x-s, xy)$, and

$$(x-s, xy) = \begin{cases} (x-s, y) & \text{if } s \neq 0 \\ (x) & \text{if } s = 0. \end{cases}$$

The first case is a simple point, the second one the y -axis.

A similar example is the map of a point into \mathbf{C} .

In both of these examples the dimension of the fiber makes a jump at one point. We notice, however, that the exceptional point corresponds to an irreducible component of X , and after removing this component π behaves nicely.

This kind of removing is not possible in general, as the following example shows:

Example 3: X is given in \mathbf{C}^3 by $(xz - y)$, and π is the projection on the (x, y) -plane.

If $s = (x_0, y_0)$, then the fiber $X(s)$ is defined by

$$(x-x_0, y-y_0, xz-y) = \begin{cases} \left(x-x_0, y-y_0, z-\frac{y_0}{x_0}\right) & \text{if } x_0 \neq 0 \\ (x, y) & \text{if } x_0 = y_0 = 0 \\ (1) & \text{if } x_0 = 0, y_0 \neq 0. \end{cases}$$

The set of “ nice ” fibers is dense in X , so we cannot remove the z -axis and still get a closed subspace of \mathbf{C}_3 .

§ 4. Algebraic study of flatness

In the following all rings are commutative, with 1, and all modules are unitary.

Definition 1: An A -module E is *flat*, if for every exact sequence of A -modules

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0,$$

the sequence $0 \rightarrow E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$ is also exact. We can also say, because \otimes is right exact, that E is flat, if for every injective homomorphism $F' \rightarrow F$, $E \otimes F' \rightarrow E \otimes F$ is also injective.

Examples of modules which are not flat:

- (1) if $A = \mathbf{Z}$, $E = \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$, $F = F' = \mathbf{Z}$; then the sequence $0 \rightarrow \mathbf{Z} \xrightarrow{2I} \mathbf{Z} (2I : x' \rightarrow 2x)$ is exact. But now $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$, and the homomorphism $\mathbf{Z}_2 \xrightarrow{2I} \mathbf{Z}_2$ is the zero homomorphism, which is not injective. So \mathbf{Z}_2 is not a flat \mathbf{Z} module.
- (2) If $A = \mathbf{C}\{x\}$, $E = \mathbf{C} = \mathbf{C}\{x\}/(x)$, $F = F' = \mathbf{C}\{x\}$, then the sequence $0 \rightarrow F \xrightarrow{xI} F' (xI : p(x) \rightarrow xp(x))$ is exact. But the homomorphism $E \xrightarrow{xI} E$ is not injective.

Proposition 1: If A is an integral domain and E a flat A -module, then E is torsion-free.

Proof: Let $a \in A$, $a \neq 0$. Because A is an integral domain, the sequence $0 \rightarrow AA \xrightarrow{aI} (aI : x \rightarrow ax)$ is exact. Since E is flat, the sequence $0 \rightarrow E \xrightarrow{aI} E$ is also exact. In other words E has no torsion elements.

Proposition 2: If A is a principal-ideal domain, then E is flat if and only if E is torsionfree.

Proof: See corollary of prop. 6.

Examples of flat modules:

- (1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.

- (2) Every free module is flat. In fact, if E is free and finite type, then $E = A^n$ and $E \otimes F = F^n$. If $F' \rightarrow F$ is injective, so is $F'^n \rightarrow F^n$ too.

If E is an arbitrary free module, then it is an inductive limit of free modules of finite type, and the flatness of E follows from (1).

- (3) Let S be a multiplicative system in A . Then the ring of fractions $S^{-1}A$ is a flat A -module. In fact the ring $S^{-1}A$ can be identified with an inductive limit of free modules, so it is flat ((1) (2)). We assume for simplicity that S has only regular elements. We can define in the set S a partial order in the following way:

$$s' \geq s \Leftrightarrow \exists t \in A, \quad ts = s' \quad (\text{such a } t \text{ is then unique}).$$

Let $E_s = A$ for every $s \in S$, and if $s' \geq s$ (i.e. $s' = ts$) then let $f_s^{s'}$ be the homomorphism $t \cdot I_A : E_s \rightarrow E_{s'}$. The family $(E_s)_{s \in S}$ with the homomorphisms $(f_s^{s'})$ is an inductive system.

Let $E = \lim_{\rightarrow} E_s$ be the inductive limit of this system, and φ_s the canonical homomorphism $E_s \rightarrow E$. We shall define an isomorphism $\psi : E \rightarrow S^{-1}A$.

We first define for every s a homomorphism $\psi_s : E_s = A \rightarrow S^{-1}A$; $x \mapsto x/s$. Now if $s' \geq s$, then

$$(\psi_{s'} \circ f_s^{s'})(x) = \psi_{s'}(tx) = \frac{tx}{s'} = \frac{tx}{ts} = \frac{x}{s} = \psi_s(x).$$

Therefore there exists a homomorphism $\psi : E \rightarrow S^{-1}A$, satisfying $\psi_s = \psi \circ \varphi_s$ for every $s \in S$.

Because every element of $S^{-1}A$ has the form a/s , ψ is surjective. On the other hand if $\psi(\varphi_s(x)) = 0$, then $\psi_s(x) = x/s = 0$. Thus $x = 0$, and ψ is also injective.

The above proof can be extended to the general case, not assuming that the elements of S are regular. The extended proof involves the notion of inductive limit of an inductive system indexed by a category instead of an ordered set.

From (1) and (2) above, any module which is the inductive limit of free modules, is flat. Conversely:

Theorem 1 : (Daniel, Lazard)

Any flat module is a inductive limit of free modules.

For the proof: See *C.R. Acad. Sci. Paris*, 258 (1964), pp. 6313-6316.

Some elementary properties of flat modules :

- (1) If E and F are flat A -modules, then $E \otimes_A F$ is also flat. In fact, if $G' \rightarrow G$ is injective, then $F \otimes_A G' \rightarrow F \otimes_A G$ is injective, and also $E \otimes_A (F \otimes_A G') \rightarrow E \otimes_A (F \otimes_A G)$ is injective. The result follows from the associativity of the tensor product.
- (2) Let $\phi : A \rightarrow B$ be a ring homomorphism, and E a flat A -module. The module $B \otimes_A E$ is a flat B -module.

If F is a B -module, then $F \otimes_B (B \otimes_A E) = (F \otimes_B B) \otimes_A E = F \otimes_A E$ further if F' and F are B -modules, and $F' \rightarrow F$ an injective homomorphism of B -modules, we can consider this homomorphism as an injective homomorphism of A -modules. Because E is A -flat,

$$F' \otimes_A E \rightarrow F \otimes_A E \text{ is injective.}$$

- (3) Let $\phi : A \rightarrow B$ be a ring homomorphism, such that B is a flat A -module. If F is a flat B -module, then F is a flat A -module. In fact: if $E' \rightarrow E$ is injective, then $E' \otimes_A B \rightarrow E \otimes_A B$ is injective, and also $(E' \otimes_A B) \otimes_B F' \rightarrow (E \otimes_A B) \otimes_B F$ is injective. But $(E' \otimes_A B) \otimes_B F' = E' \otimes_A F$; $(E \otimes_A B) \otimes_B F = E \otimes_A F$.
If an A -module E is not flat, we want to measure how far it is from being flat. For this purpose we introduce the functor Tor .

Definition 2 : A free resolution of E is an exact sequence: $\dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow E \rightarrow 0$, where all L_i are free A -modules.

The complex of the resolution is the sequence

$$(L.) \dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow 0.$$

Every module has a free resolution. Two resolutions are algebraically homotopy-equivalent. Forming the tensor products $L_i \otimes F$, we get

$$(L. \otimes F) \dots \rightarrow L_n \otimes F \rightarrow L_{n-1} \otimes F \rightarrow \dots \rightarrow L_1 \otimes F \rightarrow L_0 \otimes F \rightarrow 0.$$

Definition 3 :

$$\text{Tor}_n^A(E, F) = H_n(L. \otimes F) = \frac{\text{Ker}(L_n \otimes F \rightarrow L_{n-1} \otimes F)}{\text{Im}(L_{n+1} \otimes F \rightarrow L_n \otimes F)}$$

if $n \geq 1$, and $\text{Tor}_0^A(E, F) = \text{Coker}(L_1 \otimes F \rightarrow L_0 \otimes F) = E \otimes F$.

Basic properties of Tor :

- (1) $\text{Tor}_n(E, F)$ is independent of the choice of the resolution (up to a canonical isomorphism).

- (2) If we take a free resolution of F , we get $\text{Tor}_n(F, E) = \text{Tor}_n(E, F)$ (Symmetry of the Tor). We can also define $\text{Tor}_n(E, F)$ by taking two free resolutions, one of E and one of F .
- (3) If $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ is a short exact sequence, then we get a long exact sequence:

$$\begin{aligned} & \text{Tor}_n(E', F) \rightarrow \text{Tor}_n(E, F) \rightarrow \text{Tor}_n(E'', F) \rightarrow \\ & \rightarrow \text{Tor}_{n-1}(E', F) \rightarrow \text{Tor}_{n-1}(E, F) \rightarrow \text{Tor}_{n-1}(E'', F) \rightarrow \\ & \rightarrow \text{Tor}_1(E', F) \rightarrow \text{Tor}_1(E, F) \rightarrow \text{Tor}_1(E'', F) \rightarrow \\ & \rightarrow E' \otimes F \rightarrow E \otimes F \rightarrow E'' \otimes F \rightarrow 0. \end{aligned}$$

- (4) Tor is compatible with inductive limit, i.e. if $E = \lim (E_i)$, then

$$\text{Tor}_n(\lim E_i, F) = \lim (\text{Tor}_n(E_i, F)).$$

- (5) We can define $\text{Tor}_n(E, F)$ by taking a flat resolution of E .

Proposition 3: Let E be an A -module. Then the following conditions are equivalent:

- (a) E is flat.
- (b) For all A -modules F , and for all $n \geq 1$, $\text{Tor}_n(E, F) = 0$.
- (c) For all A -modules F , $\text{Tor}_1(E, F) = 0$.

Proof: (a) \Rightarrow (b). If $\dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0$ is a free resolution of F , then the sequence

$$\dots \rightarrow E \otimes L_n \rightarrow E \otimes L_{n-1} \rightarrow \dots \rightarrow E \otimes L_1 \rightarrow E \otimes L_0 \rightarrow E \otimes F \rightarrow 0$$

is exact, thus $\text{Tor}_n(E, F) = 0$ for all $n \geq 1$.

(b) \Rightarrow (c) clear. (c) \Rightarrow (a): If the sequence $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ is exact, so is also (by (3) above) $\text{Tor}_1(E, F'') \rightarrow E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$. Now $\text{Tor}_1(E, F'') = 0$, thus E is flat.

Proposition 4: If I and J are two ideals in A , then $\text{Tor}_1^A(A/I, A/J) = I \cap J / I \cdot J$.

Proof: From the exact sequence $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$, we get the exact sequence:

$$\text{Tor}_1(A, A/J) \rightarrow \text{Tor}_1(A/I, A/J) \rightarrow I \otimes A/J \rightarrow A \otimes A/J \rightarrow A/I \otimes A/J \rightarrow 0.$$

But now $\text{Tor}_1(A, A/J) = 0$ (A being A -free), and $I \otimes A/J = I/I \cdot J$; $A \otimes A/J = A/J$. Therefore the sequence $0 \rightarrow \text{Tor}_1(A/I, A/J) \rightarrow I/I \cdot J \rightarrow A/J$ is exact, and $\text{Tor}_1(A/I, A/J) = \text{Ker}(I/I \cdot J \rightarrow A/J) = I \cap J / I \cdot J$.

Example : Let U be an open set in \mathbb{C}^n , and $x \in U$. Further let $X, Y \subset U$ be two hypersurfaces, defined by $I = (f)$ and $J = (g)$. Supposing that f and g do not have common factors: $I_x \cap J_x = I_x J_x$, and

$$\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = \text{Tor}_1(\mathcal{O}_{U,x}/I_x, \mathcal{O}_{U,x}/J_x) = \frac{I_x \cap J_x}{I_x \cdot J_x} = 0.$$

Heuristic remark : The formula $\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ expresses the fact that X and Y are “in general position”. If for example X and Y are two linear subspaces in \mathbb{C}^n of dimensions p and q , we have $\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ if $\dim(X \cap Y) = p + q - n$, and $\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) \neq 0$ otherwise.

Next we shall prove an elementary flatness criterion.

Proposition 5 : Let E be an A -module. The following conditions are equivalent:

- (a) E is flat.
- (b) For all finitely generated ideals I of A , $\text{Tor}_1(E, A/I) = 0$.
- (c) For all monogenous A -modules F , $\text{Tor}_1(E, F) = 0$.

Proof : (a) \Rightarrow (b), by prop. 3.

(b) \Rightarrow (c): Because Tor is compatible with inductive limit, we can suppose, that $\text{Tor}_1(E, A/I) = 0$ for an arbitrary ideal I of A . But every monogenous A -module F can be represented by A/I .

(c) \Rightarrow (a). By prop. 3 it is sufficient to prove that $\text{Tor}_1(E, F) = 0$ for any A -module F .

First consider the case, where F is finitely generated. We use induction, supposing that $\text{Tor}_1(E, F) = 0$, when F has n generators. Let F have $(n+1)$ generators x_1, \dots, x_n, x_{n+1} . If F' is the submodule generated by $\{x_1, \dots, x_n\}$, then $F' \subset F$ and $F'' = F/F'$ is monogenous. The exact sequence $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ gives the exact sequence $\text{Tor}_1(E, F') \rightarrow \text{Tor}_1(E, F) \rightarrow \text{Tor}_1(E, F'')$. Now $\text{Tor}_1(E, F') = \text{Tor}_1(E, F'') = 0$, thus $\text{Tor}_1(E, F) = 0$. In the general case, F can be considered as an inductive limit of finitely generated modules, and because Tor is compatible with inductive limits, $\text{Tor}_1(E, F) = 0$.

Proposition 6 : Let A be an integral domain, and E an A -module. Then E is torsionfree if and only if $\text{Tor}_1(E, A/(a)) = 0$, for any element $a \in A$.

Proof : If E is A -module, $a \in A$, then the exact sequence $0 \rightarrow A \xrightarrow{aI} A \rightarrow A/(a) \rightarrow 0$ gives the exact sequence $0 \rightarrow \text{Tor}_1(E, A/(a)) \rightarrow E \xrightarrow{aI} E$. In other words $\text{Tor}_1(E, A/(a)) = \{x \in E \mid ax = 0\}$, from which the result follows.

Corollary: Let A be a principal ideal domain. E is flat if and only if E is torsionfree.

Proof: We have already proved that, if E is flat, then it is torsion free. The converse follows from prop. 6 and prop. 5.

The first flatness criterion for noetherian local rings is the following:

Theorem 2: Let A be a noetherian local ring with maximal ideal m ; $k = A/m$, and E a finitely generated A -module. The following conditions are equivalent:

- (a) E is free.
- (b) E is flat.
- (c) $\text{Tor}_1^A(E, k) = 0$.

Proof: We have already proved $(a) \Rightarrow (b) \Rightarrow (c)$.

$(c) \Rightarrow (a)$: We recall first Nakayma's lemma. If A is a local ring with maximal ideal m ; $k = A/m$, and E is a finitely generated A -module, such that $k \otimes_A E = E/mE = 0$, then $E = 0$.

The module $\bar{E} = k \otimes_A E = E/mE$ is a finitely generated vector space over k . Let $\{\bar{x}_1, \dots, \bar{x}_r\}$ be a base of \bar{E} (over k), and $\{x_1, \dots, x_r\}$ E representatives of \bar{x}_i : s . Consider the homomorphism $\phi : A^r \rightarrow E$, $\phi(a_1, \dots, a_r) = \sum a_i x_i$. Denoting by R and Q the kernel and the cokernel of ϕ , we get an exact sequence:

$$(*) \quad 0 \rightarrow R \rightarrow A^r \rightarrow E \rightarrow Q \rightarrow 0$$

and R, Q are finitely generated A -modules. From $(*)$ we get the exact sequence

$$A^r \otimes_A k \rightarrow E \otimes_A k \rightarrow Q \otimes_A k \rightarrow 0.$$

But $\bar{E} = E \otimes_A k \simeq k^r = A^r \otimes_A k$, so $Q \otimes_A k = 0$, and by Nakayama's lemma $Q = 0$.

Therefore we have an exact sequence

$$0 \rightarrow R \rightarrow A^r \rightarrow E \rightarrow 0.$$

From this we get: $\text{Tor}_1(E, k) \rightarrow k \otimes_A R \rightarrow k^r \rightarrow \bar{E} \rightarrow 0$ (exact). Now: $\bar{E} \simeq k^r$, $\text{Tor}_1(E, k) = 0$ (by assumption). Therefore $k \otimes_A R = 0$, and once more by Nakayama's lemma $R = 0$, thus $E \simeq A^r$, i.e. E is free.

Proposition 7: Let $\phi : A \rightarrow B$ be a ring homomorphism, and let B be A -flat. If I is an ideal of A , we write $\bar{A} = A/I$, $\bar{B} = B/IB = \bar{A} \otimes_A B$. Let F be a B -module, then: $\text{Tor}_i^A(\bar{A}, F) = \text{Tor}_i^B(\bar{B}, F)$ ($i \geq 0$).

Proof: We choose first a B -free resolution of F

$$\rightarrow L_{n+1} \rightarrow L_n \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0.$$

If L is the respective complex of resolution, then

$$\bar{B} \otimes_B L = B/IB \otimes_B L = \bar{A} \otimes_A (B \otimes_B L) = \bar{A} \otimes_A L.$$

Because every L_i is B -free, and B is A -flat, every L_i is A -flat (Property 3 after Th. 1). Thus L is a flat A -resolution, and

$$\text{Tor}_i^A(\bar{A}, F) = H_i(\bar{A} \otimes_A L) = H_i(\bar{B} \otimes_B L) = \text{Tor}_i^B(\bar{B}, F).$$

We shall next state the second flatness criterion for noetherian local rings.

Theorem 3: Let A and B be two noetherian local rings, with maximal ideals $\underline{m}, \underline{n}$; $k = A/\underline{m}$. If $\phi : A \rightarrow B$ is a local homomorphism (i.e. $\phi(\underline{m}) \subset \underline{n}$), and F finitely generated B module then

$$F \text{ is } A\text{-flat} \Leftrightarrow \text{Tor}_1^A(k, F) = 0.$$

The proof of this theorem is much more difficult than that of th. 20 see for example:

Bourbaki: *Algèbre commutative*, Chapter III § 5, th1, (i) \Leftrightarrow (iii), p. 98.

The conditions in Bourbaki's theorem are here fulfilled:

- 1° A finitely generated module F over a noetherian local ring B is idealwise separated for \underline{n} . (*Ibid.*, § 5. 1. Ex. 1, p. 97.)
- 2° If $\phi : A \rightarrow B$ is a local homomorphism, F is also idealwise separated for \underline{m} . (*Ibid.*, § 5, prop. 2, p. 101.)
- 3° Also the flatness condition is fulfilled, because k is a field.

Remark: The main interest of the theorem lies in the fact, that it is true without any assumption of finiteness on B .

Corollary: If the assumptions are the same as in the theorem 3, and if moreover B is A -flat, then

$$F \text{ is } A\text{-flat} \Leftrightarrow \text{Tor}_1^B(\bar{B}, F) = 0,$$

where $\bar{B} = B/\underline{m}B$.

Proof: $\text{Tor}_1^A(k, F) = \text{Tor}_1^B(\bar{B}, F)$, by prop. 7.

§ 5. Geometric applications of the flatness criterions

A) Flatness for finite morphisms

Proposition 1: Let $\pi: X \rightarrow S$ be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then $\pi_*(\mathcal{O}_X)$ is a coherent analytic sheaf over S . The following conditions are equivalent:

- (a) π is flat (i.e. for every $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$ -module, $s = \pi(x)$).
- (b) For every s , $(\pi_* \mathcal{O}_X)_s$ is a flat $\mathcal{O}_{S,s}$ -module.
- (c) $\pi_* \mathcal{O}_X$ is a locally free sheaf.

Proof: Because π is finite $\pi_*(\mathcal{O}_X)_s = \bigoplus_{x \in \pi^{-1}(s)} \mathcal{O}_{X,x}$, thus the only point to prove is (b) \Rightarrow (c).

Now if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$ -module, then (by theorem 2) $\mathcal{O}_{X,x}$ is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

Proposition 2: Let S be a reduced analytic space and \mathcal{E} a coherent \mathcal{O}_S -module. Let $E(s)$ be the finite dimensional vector space (over \mathbb{C}) $\mathcal{E}_s \otimes_{\mathcal{O}_{S,s}} \mathbb{C}_s$. \mathcal{E} is a locally free $\mathcal{O}_{S,s}$ -module if and only if $\dim_{\mathbb{C}} E(s)$ is locally constant.

Proof: If \mathcal{E} is locally free, then $\dim_{\mathbb{C}} E(s)$ is locally constant. Suppose now that $\dim_{\mathbb{C}} E(s)$ is locally constant in an open set $U \subset S$, and that $\mathcal{O}_U^p \xrightarrow{d} \mathcal{O}_U^q \rightarrow \mathcal{E}_U \rightarrow 0$ is exact. d is determined by a $p \times q$ matrix of analytic functions on U , so it gives a morphism $\mathbb{C}_U^p \xrightarrow{d} \mathbb{C}_U^q$ of trivial vector bundles over U .

From the exact sequence $\mathcal{O}_S^p \xrightarrow{d_s} \mathcal{O}_S^q \rightarrow \mathcal{E}_S \rightarrow 0$, we get (by making tensor-products with \mathbb{C}_s) the exact sequence:

$$\mathbb{C}_s^p \xrightarrow{d(s)} \mathbb{C}_s^q \rightarrow E(s) \rightarrow 0,$$

which shows that d has constant rank in U . Thus $\text{Ker } d$ and $\text{Im } d$ are vector bundles, and we can write

$$\mathbb{C}_U^p = F_1 \oplus G_1, \quad \mathbb{C}_U^q = F_0 \oplus G_0,$$

$$d : \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_0 \end{cases}.$$

Now $\mathcal{E} \simeq$ the sheaf of analytic sections of G_0 , therefore \mathcal{E} is locally free.

Definition 1: Let $\pi : X \rightarrow S$ be a finite morphism of analytic spaces, and $s \in S$. For each $x \in X(s) = \pi^{-1}(s)$, $\mathcal{O}_{X(s),x} = \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$ is finite dimensional vectorspace over \mathbf{C} . Denote its dimension by $v(x)$. Then the degree $v(s)$ of s is defined by $v(s) = \sum_{x \in X(s)} v(x)$.

Theorem 1: Let $\pi : X \rightarrow S$ be a finite morphism of analytic space and let S be a reduced space. Then X is flat over S if and only if $v(s)$ is locally constant function of s .

$$\begin{aligned} \text{Proof: } v(s) &= \sum_{x \in X(s)} \dim_{\mathbf{C}} \mathcal{O}_{X(s),x} = \dim_{\mathbf{C}} \left(\bigoplus_{x \in X(s)} \mathcal{O}_{X(s),x} \right) \\ &= \dim_{\mathbf{C}} \left(\bigoplus_{x \in X(s)} (\mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}) \right) \\ &= \dim_{\mathbf{C}} \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \pi_* (\mathcal{O}_X)_s = \dim_{\mathbf{C}} E(s). \end{aligned}$$

The theorem follows from propositions 1 and 2.

Examples of flat morphisms

Example 1: If $\pi : X \rightarrow S$ is a local isomorphism near x , then π is flat at x .

Example 2: Consider § 2, Ex. 1. Here $v(x) = 1$.

Examples of non-flat morphisms

Examples 1: If $X \subset S$ is a closed subspace, not open, $v(s)$ is not locally constant.

Example 2: Let X be a subspace of \mathbf{C}^4 defined by the ideal intersection of (x_3, x_4) and $(x_1 - x_1, x_4 - x_2)$ (which is equal to the product ideal) and let π be the projection onto the (x_1, x_2) -plane \mathbf{C}^2 . Then X is a union of two 2-planes in \mathbf{C}^4 , whose intersection is (0) . When $s \neq 0$, $X(s)$ consists of two simple points, so $v(s) = 2$. $X(0)$ is given by the ideal $(x_1, x_2, x_3^2, x_3x_4, x_4^2)$, thus $v(0) = 3$.

Example 3: Let $S = \{(u, v, w) \in \mathbf{C}^3 \mid v^2 = uw\}$ and $\pi : \mathbf{C}^2 \rightarrow S$ be the map $(x, y) \rightarrow (x^2, xy, y^2)$. This map identifies S with the quotient of \mathbf{C}^2 by the equivalence relation identifying (x, y) with $(-x, -y)$. However, π is not flat, since for $s \in S$, $v(s) = 2$ if $s \neq 0$ and $v(s) = 3$ if $s = 0$.

B) Projection of a product of analytic spaces

Theorem 2: Let S and X be analytic spaces. If $\pi : S \times X \rightarrow S$ is the projection morphism, then π is flat, i.e. $\mathcal{O}_{S \times X, (s, x)}$ is a flat $\mathcal{O}_{S, s}$ module for every $(s, x) \in S \times X$.

To prove this theorem we need first some homological algebra. Then we shall show it in the particular case, when S is a manifold, and finally in the general case.

(a) Koszul complex

Let A be a ring, M an A -module and h_1, \dots, h_n homomorphisms $M \rightarrow M$, which commute with each other, i.e. $h_i h_j = h_j h_i$ for every i, j .

If $1 \leq k \leq n$, set $Q_k = M/h_1(M) + \dots + h_k(M)$, and $Q_0 = M$, thus, in particular, $Q_n = Q = M/\sum_{i=1}^n h_i(M)$. Every h_k induces a map $\tilde{h}_k : Q_{k-1} \rightarrow Q_{k-1}$.

Definition 2: The sequence (h_1, \dots, h_n) is called regular if each of the mappings \tilde{h}_k ($1 \leq k \leq n$) is injective.

The Koszul complex of the module M and of the mappings h_k ($1 \leq k \leq n$) $K. = K. [M; h_1, \dots, h_n]$ is defined in the following way:

$$K_i = \wedge^{n+i} A^n \otimes M \simeq M^{(i)}, \quad 0 \leq i \leq n.$$

We define the homomorphisms $d_i : K_i \rightarrow K_{i-1}$ ($i > 0$) by $\lambda \otimes x \rightarrow \sum_i (e_i \wedge \lambda) \otimes \otimes h_i(x)$, where (e_i) is the natural base of A^n . We also define $\varepsilon : K_0 \rightarrow Q$ as the natural map $: K_0 = M \rightarrow M/\sum_{i=1}^n h_i(M) = Q$. Using the fact that h_1, \dots, h_n commute with each other, it is easy to verify that

$$(d_{i-1} \circ d_i)(\lambda \otimes x) = \sum_{i,j} (e_j \wedge e_i \wedge \lambda) \otimes h_j(h_i(x)) = 0;$$

also $\varepsilon d_1 = 0$. Thus $K.$ is really a complex.

Theorem 3 (Poincaré-Koszul).

If (h_1, \dots, h_n) is a regular sequence, then

$$H_i(K.) = \begin{cases} Q & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}.$$

If $h_i \in A$, it defines the map: $A \xrightarrow{h_i I} A$, which we denote also by h_i . We say that (h_1, \dots, h_n) is a regular sequence of elements if $(h_1 I, \dots, h_n I)$ is a regular sequence.

Corollary. If (h_1, \dots, h_n) is a regular sequence of elements, then the Koszul complex $K. = K. [A; h_1, \dots, h_n] = \{ \wedge^{n-1} A^n \simeq A^{(n)} \}$ is a free resolution of $Q = A/(h_i)$ ((h_i) is the ideal generated by h_1, \dots, h_n).

Example: If $A = \mathbb{C}\{x_1, \dots, x_n\}$; $h_i = x_i$, then $Q_k = A/(x_1, \dots, x_k) = \mathbb{C}\{x_{k+1}, \dots, x_n\}$ and $Q = Q_n = \mathbb{C}$. The complex $K. = K. [A; x_1, \dots, x_n]$ is a free resolution of \mathbb{C} .

(b) *Proof of theorem 2, when S is a complex manifold*

In this case we can take $\mathcal{O}_{S,s} = \mathbb{C}\{t_1, \dots, t_m\} = A$ and if $\mathcal{O}_{X,x} = \mathbb{C}\{x_1, \dots, x_n\}/(f_1, \dots, f_p)$, then

$$\mathcal{O}_{S \times X, (s,x)} = \mathbb{C}\{t_1, \dots, t_m, x_1, \dots, x_n\}/(f_1, \dots, f_p) = B.$$

B is an A -module in a natural way.

By the corollary of the Poincaré-Koszul theorem $K. = K. [A; t_1, \dots, t_m]$ in a free resolution of \mathbb{C} . We want to compute the modules $\text{Tor}_i^A(\mathbb{C}, B) = H_i(K. \otimes B)$ ($i > 0$).

It's easily seen, that we can consider the complex $K. \otimes B$ as a Koszul

complex $K'. = K. [B; t_1, \dots, t_m]$ (where $t_i : B \xrightarrow{t_i I} B$). But now the sequence (t_1, \dots, t_m) is regular, thus by the Poincaré-Koszul theorem $H_i[K'] = 0$ if $i > 0$.

In particular: $\text{Tor}_1^A(\mathbb{C}, B) = H_1[K. \otimes B] = H_1[K'] = 0$. By the second flatness criterion B is A -flat.

(c) *The general case*

The question being local, we can suppose that $S \subset W \subset \mathbb{C}^n$, where W is open, and S an analytic subspace of W . Let S be defined by g_1, \dots, g_r . Then $S \times X \subset W \times X$ and $\mathcal{O}_S = \mathcal{O}_W/(g_1, \dots, g_r)$. On the other hand $\mathcal{O}_{S \times X} = \mathcal{O}_{W \times X}/(g_1, \dots, g_r) = \mathcal{O}_S \otimes_{\mathcal{O}_W} \mathcal{O}_{W \times X}$. The last equality follows from

the fact, that if $\pi : X \rightarrow S$ is a morphism, and $S' \subset S$ a subspace, $X' = \pi^{-1}(S')$,

$$\text{then } \mathcal{O}_{X'} = \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X.$$

Remark : This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ \pi \searrow & & \swarrow \pi' \\ & S & \end{array} \quad \mathcal{O}_{X \times_S Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

We have proved that $\mathcal{O}_{W \times X}$ is \mathcal{O}_W -flat, so by scalar extension $\mathcal{O}_{S \times X}$ is \mathcal{O}_S flat.

Corollary : If X and S are two manifolds and $\pi : X \rightarrow S$ is a submersion, then π is flat.

III. PRIVILEGED POLYCYLINDERS

§ 1. Banach vector bundles over an analytic space

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X .

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E . If $U \subset \mathbb{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g : U \rightarrow E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X , i.e. X' is the support of the quotient sheaf \mathcal{O}_U/J , where $U \subset \mathbb{C}^n$ is open and J is a coherent sheaf of ideals of \mathcal{O}_U , then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \rightarrow \mathcal{H}(V, E)/J_V \cdot \mathcal{H}(V, E)$ ($V \subset U$, V -open).

Remark : If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U .

The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models X' of X , i.e. $\mathcal{H}_X(E)|_{X'} = \mathcal{H}_{X'}(E)$, for every local model X' .

Definition 1 : The set of *analytic morphisms* from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F .

Definition 2 : An *analytic vector bundle morphism* from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.