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FLATNESS AND PRIVILEGE

by A. Douady

I. FLAT MORPHISMS

§ 1. Analytic subspaces of an analytic space

Let Y_1 and Y_2 be closed analytic subspaces of an analytic space X, and let them be defined by the \mathcal{O}_X ideals J_1 , J_2 .

Definition 1: We say that Y_1 is analytically included in Y_2 , and we write $Y_1 \subset Y_2$, when $J_1 \supset J_2$.

Remark: The analytic inclusion implies the set theoretic inclusion, but the converse is not true.

Example: $X = (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$; $J_1 = (x)$, $J_2 = (x^2)$. The space Y_1 is a simple point, Y_2 is a double point, $Y_1 \Rightarrow Y_2$, while they have the same underlying set.

Definition 2: The subspace $Y_1 \cup Y_2$ is the smallest subspace of X containing Y_1 and Y_2 , and it is defined by $J_1 \cap J_2$. The subspace $Y_1 \cap Y_2$ is the biggest subspace of X contained in both Y_1 and Y_2 , and it is defined by $J_1 + J_2$.

Remark: The underlying set of $Y_1 \cup Y_2$ (Resp. $Y_1 \cap Y_2$) is the union (Resp. intersection) of the underlying sets of Y_1 and Y_2 . However \cup and \cap of analytic spaces do not satisfy the distributivity laws which hold in settheory: $(Y_1 \cup Y_2) \cap Y_3$ contains $Y_1 \cap Y_3$ and $Y_2 \cap Y_3$, and therefore their union; similarly $(Y_1 \cap Y_2) \cup Y_3 \subset (Y_1 \cup Y_3) \cap (Y_2 \cup Y_3)$. In general the converse inclusions do not hold.

Example: Let $X = \mathbb{C}^2$ and Y_1 , Y_2 , Z be given by ideals (x-y), (x+y) and (x) respectively.

 $(Y_1 \cup Y_2) \cap Z$ is $\{0\}$ provided with $\mathbb{C}\{y\}/(y^2)$, while $(Y_1 \cap Z) \cup (Y_2 \cap Z)$ is the reduced space $\{0\}$. On the other hand: $Y_1 \cap Y_2 \subset Z$, $(Y_1 \cap Y_2) \cup Z = Z$, while $(Y_1 \cup Z) \cap (Y_2 \cup Z)$ is the space defined by the ideal (x^2, xy) . Its local ring at the origin is $\mathbb{C}\{x, y\}/(x^2, xy)$ in which x is nilpotent.

Definition 3: Let X', X be analytic spaces, Y a closed analytic subspace of X defined by J, and $f = (f_0, f^1) : X' \rightarrow X$ a morphism.

The inverse image of Y by f, $f^{-1}(Y)$, is the analytic subspace Y' of X' defined by the ideal $J' = f^1(J) \mathcal{O}_{X'}$.

The inverse image of a simple point x in X is called the f-fiber over x, and is denoted by $f^{-1}(x)$ or X'(x).

Proposition 1: If $f = (f_0, f^1): X' \to X$ is a morphism of analytic spaces, and Y is a subspace of X, then $f^{-1}(Y) \simeq Y \times X'$.

Proof: Let T be any analytic space, and $g: T \to X'$ a morphism. Then g can be considered as a morphism from T to $f^{-1}(Y)$ if and only if $f \circ g$ can be considered as a morphism from T to Y. Thus $f^{-1}(Y)$ and $X' \times X$ are solutions of the same universal problem.

§ 2. Analytic pull-back

In the following we want to generalize the notion of inverse image of a subspace.

We shall first recall the basic properties of the tensor product $E \otimes F$, where A is a commutative ring and E, F are two A-modules.

- $(1^o) \quad E \otimes A^n = E^n \ (n \in N)$
- (2°) If the sequence of A-modules $F' \rightarrow F \rightarrow F'' \rightarrow 0$ is exact, then also the sequence $E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$ is exact. (Right exactness of the tensor product)
- (3°) If $(F_i)_{i \in I}$; $f_{ij}: F_j \to F_i$ is an inductive system, then

$$E \otimes \lim_{\longrightarrow} F_i = \lim_{\longrightarrow} (E \otimes F_i).$$

On the other hand these properties characterize completely the functor \otimes .

Definition $I: \operatorname{Let} f = (f_0 \ f^1): X' \to X$ be a morphism of analytic spaces, and $\mathscr E$ an $\mathscr O_X$ -module. Then $f \ _0^* \mathscr E$ is an $f \ _0^* \mathscr O_X$ -module and $\mathscr O_{X'}$ is also an $f \ _0^* \mathscr O_X$ -module (by $f \ : f \ _0^* \mathscr O_X \to \mathscr O_{X'}$).

The analytic pull-back $f * \mathcal{E}$ of \mathcal{E} by f is defined by scalar extension:

$$f * \mathscr{E} = f_0^* \mathscr{E} \otimes \mathscr{O}_{X'}$$
$$f_0^* \mathscr{O}_X$$

Remark: The inverse image is a particular case of the analytic pullback.

In fact, if Y is a closed analytic subspace of X and $f: X' \rightarrow X$ is a morphism:

$$\begin{split} f * \mathscr{O}_{\mathbf{Y}} &= f_{0}^{*} \left(\mathscr{O}_{\mathbf{X}} / J_{\mathbf{Y}} \right) \otimes \mathscr{O}_{\mathbf{X}'} \simeq f_{0}^{*} \mathscr{O}_{\mathbf{X}} / f_{0}^{*} J_{\mathbf{Y}} \otimes f_{0}^{*} \mathscr{O}_{\mathbf{X}'} \\ & f_{0}^{*} \mathscr{O}_{\mathbf{X}} \end{split}$$
$$\simeq \mathscr{O}_{\mathbf{X}'} / f^{1} \left(J_{\mathbf{Y}} \right) \cdot \mathscr{O}_{\mathbf{X}'} \simeq \mathscr{O}_{f^{-1}(\mathbf{Y})} \end{split}$$

(The third isomorphism follows from the fact, that $A/I \otimes E \simeq E/IE$).

Elementary properties of the analytic pull-back:

- (a) $(f * \mathscr{E})_{x'} = (f_0^* \mathscr{E})_{x'} \otimes_{(f_0^* \mathscr{O}_X)_{x'}} \mathscr{O}_{X',x'} \simeq \mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X',x'}$ where $x = f_0(x')$ (since \otimes commutes with inductive limits).
- (b) $f^*(\mathscr{E} \otimes_{\mathscr{O} X} \mathscr{F}) = f^*\mathscr{E} \otimes_{\mathscr{O} X'} f^*\mathscr{F}$, where \mathscr{E} and \mathscr{F} are \mathscr{O}_X -modules.
- (c) If $\mathscr E$ is a coherent $\mathscr O_X$ -module, then $f * \mathscr E$ is a coherent $\mathscr O_{X'}$ -module.

In fact, \mathscr{E} has a locally finite presentation: $\mathscr{O}_X^q \to \mathscr{O}_X^p \to \mathscr{E} \to 0$, and f^* is compatible with cokernels, $f^*(\mathscr{O}_X^r) = \mathscr{O}_X^r$.

Special case: The pull-back of vector bundle. Let (E, π) be an analytic

vector bundle over the analytic space X, and $f: X' \rightarrow X$ a morphism of analytic spaces. The fiber product carries a unique structure of vector bundle over X', such that \overline{f} is a bundle morphism. We call this bundle E'.

Proposition 1: Let \mathscr{E} (Resp. \mathscr{E}') be the sheaf of analytic sections of E (Resp. E'). Then $\mathscr{E}' = f * \mathscr{E}$.

Proof (Sketch): We have a $f_0^* \mathcal{O}_X$ linear morphism $f_0^* \mathcal{E} \to \mathcal{E}'$, which extends to a morphism $f^* \mathcal{E} \to \mathcal{E}'$. We can prove that this is an isomorphism. Since the question is local with respect to X', we can suppose that E is a trivial bundle over X with fiber \mathbf{C}^r , then $\mathcal{E} = \mathcal{O}_X^r$. Also $\mathcal{O}_{X'}^r = f^* \mathcal{O}_X^r$. Therefore $f^* \mathcal{E} = \mathcal{E}'$.

§ 3. Introduction to flatness by examples

Let S be an analytic space. By analytic space over s we mean an analytic space X provided with a morphism $\pi: X \to S$. Let S be a simple point in S, and consider $X(s) = f^{-1}(s)$.

The main purpose of these lectures is to give a precise meaning to the expression:

"X(s) depends nicely on s", and to give a criterion for the "nice" behaviour.

We begin with some examples.

Example 1: X is the closed subspace on \mathbb{C}^2 defined by (y^2-x) , $S=\mathbb{C}$ and $\pi = 1$ st projection.

$$X(s) = \begin{cases} 2 \text{ simple points if } s \neq 0 \\ \text{double point if } s = 0. \end{cases}$$

Here the behaviour of X(s) is nice.

Example 2: X is the closed subspace of \mathbb{C}^2 defined by (xy), $S = \mathbb{C}$ and $\pi = 1$ st projection.

$$X(s) \text{ is given by } (x-s, xy), \text{ and}$$

$$(x-s, xy) = \begin{cases} (x-s, y) & \text{if } s \neq 0 \\ (x) & \text{if } s = 0. \end{cases}$$
The first case is a simple point, the y-axis.

$$X(s)$$
 is given by $(x-s, xy)$, and

$$(x-s, xy) = \begin{cases} (x-s, y) & \text{if } s \neq 0 \\ (x) & \text{if } s = 0 \end{cases}.$$

The first case is a simple point, the second one the,

A similar example is the map of a point into C.

In both of these examples the dimension of the fiber makes a jump at one point. We notice, however, that the exceptional point corresponds to an irreductible component of X, and after removing this component π behaves nicely.

This kind of removing is not possible in general, as the following example shows:

Example 3: X is given in \mathbb{C}^3 by (xz-y), and π is the projection on the (x, y)-plane.

If $s = (x_0, y_0)$, then the fiber X(s) is defined by

$$(x-x_0, y-y_0, xz-y) = \begin{cases} \left(x-x_0, y-y_0, z-\frac{y_0}{x_0}\right) & \text{if } x_0 \neq 0 \\ (x, y) & \text{if } x_0 = y_0 = 0 \\ (1) & \text{if } x_0 = 0 \ y_0 \neq 0 \ . \end{cases}$$

The set of "nice" fibers is dense in X, so we cannot remove the z-axis and still get a closed subspace of \mathbb{C}_3 .

§ 4. Algebraic study of flatness

In the following all rings are commutative, with 1, and all modules are unitary.

Definition 1: An A-module E is flat, if for every exact sequence of A-modules

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$$
,

the sequence $0 \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$ is also exact. We can also say, because \otimes is right exact, that E is flat, if for every injective homomorphism $F' \to F$, $E \otimes F' \to E \otimes F$ is also injective.

Examples of modules which are not flat:

- (1) if $A = \mathbb{Z}$, $E = \mathbb{Z}_2 = \mathbb{Z}/2 \mathbb{Z}$, $F = F' = \mathbb{Z}$; then the sequence $0 \to \mathbb{Z} \to \mathbb{Z}$ ($2I : x' \to 2x$) is exact. But now $\mathbb{Z}_2 \otimes \mathbb{Z} = \mathbb{Z}_2$, and the homomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$ is the zero homomorphism, which is not injective. So \mathbb{Z}_2 is not a flat \mathbb{Z} module.
- (2) If $A = \mathbb{C}\{x\}$, $E = \mathbb{C} = \mathbb{C}\{x\}/(x)$, $F = F' = \mathbb{C}\{x\}$, then the sequence $0 \to F \xrightarrow{xI} F' (xI : p(x) \to xp(x))$ is exact. But the homomorphism $E \xrightarrow{xI} E$ is not injective.

Proposition 1: If A is an integral domain and E a flat A-module, then E is torsion-free.

Proof: Let $a \in A$, $a \neq 0$. Because A is an integral domain, the sequence $0 \rightarrow AA \xrightarrow{aI} (aI : x \rightarrow ax)$ is exact. Since E is flat, the sequence $0 \rightarrow E \xrightarrow{aI} E$ is also exact. In other words E has no torsion elements.

Proposition 2: If A is a principal-ideal domain, then E is flat if and only if E is torsionfree.

Proof: See corollary of prop. 6.

Examples of flat modules:

(1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.

(2) Every free module is flat. In fact, if E is free and finite type, then $E = A^n$ and $E \otimes F = F^n$. If $F' \to F$ is injective, so is $F'^n \to F^n$ too.

If E is an arbitrary free module, then it is an inductive limit of free modules of finite type, and the flatness of E follows from (1).

(3) Let S be a multiplicative system in A. Then the ring of fractions $S^{-1}A$ is a flat A-module. In fact the ring $S^{-1}A$ can be identified with an inductive limit of free modules, so it is flat ((1)(2)). We assume for simplicity that S has only regular elements. We can define in the set S a partial order in the following way:

$$s' \ge s \Leftrightarrow \exists t \in A$$
, $ts = s'$ (such a t is then unique).

Let $E_s = A$ for every $s \in S$, and if $s' \ge s$ (i.e. s' = ts) then let $f_s^{s'}$ be the homomorphism t. $I_A : E_s \to E_{s'}$. The family $(E_s)_{s \in S}$ with the homomorphisms $(f_s^{s'})$ is an inductive system.

Let $E = \lim_{s \to \infty} E_s$ be the inductive limit of this system, and φ_s the canonical homomorphism $E_s \to E$. We shall define an isomorphism $\psi : E \to S^{-1}A$.

We first define for every s a homomorphism $\psi_s: E_s = A \rightarrow S^{-1}A$; $x \rightarrow x/s$. Now if $s' \geq s$, then

$$(\psi_{s'} \circ f_{s'}^{s'})(x) = \psi_{s'}(tx) = \frac{tx}{s'} = \frac{tx}{ts} = \frac{x}{s} = \psi_{s}(x).$$

Therefore there exists a homomorphism $\psi: E \to S^{-1}A$, satisfying $\psi_s = \psi \circ \varphi_s$ for every $s \in S$.

Because every element of $S^{-1}A$ has the form a/s, ψ is surjective. On the other hand if ψ ($\phi_s(x)$) = 0, then $\psi_s(x) = x/s = 0$. Thus x = 0, and ψ is also injective.

The above proof can be extended to the general case, not assuming that the elements of S are regular. The extended proof involves the notion of inductive limit of an inductive system indexed by a category instead of an ordered set.

From (1) and (2) above, any module which is the inductive limit of free modules, is flat. Conversely:

Theorem 1: (Daniel, Lazard)

Any flat module is a inductive limit of free modules.

For the proof: See C.R. Acad. Sci. Paris, 258 (1964), pp. 6313-6316.

Some elementary properties of flat modules:

- (1) If E and F are flat A-modules, then $E \otimes F$ is also flat. In fact, if $G' \to G$ is injective, then $F \otimes G' \to F \otimes G$ is injective, and also $E \otimes (F \otimes G') \to E \otimes (F \otimes G)$ is injective. The result follows from the assosiativity of the tensor product.
- (2) Let $\phi: A \rightarrow B$ be a ring homomorphism, and E a flat A-module. The module $B \otimes E$ is a flat B-module.

If F is a B-module, then $F \otimes (B \otimes E) = (F \otimes B) \otimes E = F \otimes E$ further if F' and F are B-modules, and $F' \rightarrow F$ an injective homomorphism of B-modules, we can consider this homomorphism as an injective homomorphism of A-modules. Because E is A-flat,

$$F' \otimes_A E \rightarrow F \otimes_A E$$
 is injective.

(3) Let $\phi: A \to B$ be a ring homomorphism, such that B is a flat A-module. If F is a flat B-module, then F is a flat A-module. In fact: if $E' \to E$ is injective, then $E' \otimes B \to E \otimes B$ is injective, and also $(E \otimes B) \otimes F' \to (E \otimes B) \otimes F$ is injective. But $(E' \otimes_A B)_B \otimes F' = E' \otimes_A F$; $(E \otimes_A B) \otimes_B F = E \otimes_A F$.

If an A-module E is not flat, we want to measure how far it is from being flat. For this purpose we introduce the functor Tor.

Definition 2: A free resolution of E is an exact sequence: $... \rightarrow L_n \rightarrow L_{n-1} \rightarrow ... \rightarrow L_1 \rightarrow L_0 \rightarrow E \rightarrow 0$, where all L_i are free A-modules.

The complex of the resolution is the sequence

(L.) ...
$$\rightarrow L_n \rightarrow L_{n-1} \rightarrow ... \rightarrow L_1 \rightarrow L_0 \rightarrow 0$$
.

Every module has a free resolution. Two resolutions are algebraically homotopy-equivalent. Forming the tensorproducts $L_i \otimes F$, we get

$$(L.\otimes F) \dots \to L_n \otimes F \to L_{n-1} \otimes F \to \dots \to L_1 \otimes F \to L_0 \otimes F \to 0 \ .$$

Definition 3:

$$\operatorname{Tor}_{n}^{A}(E, F) = H_{n}(L \otimes F) = \frac{\operatorname{Ker}(L_{n} \otimes F \to L_{n-1} \otimes F)}{\operatorname{Im}(L_{n+1} \otimes F \to L_{n} \otimes F)}$$

if
$$n \ge 1$$
, and $\operatorname{Tor}_0^A(E, F) = \operatorname{Coker}(L_1 \otimes F \to L_0 \otimes F) = E \otimes F$.

Basic properties of Tor:

(1) $\operatorname{Tor}_n(E, F)$ is independent of the choice of the resolution (up to a canonical isomorphism).

- (2) If we take a free resolution of F, we get $\operatorname{Tor}_n(F, E) = \operatorname{Tor}_n(E, F)$ (Symmetry of the Tor). We can also define $\operatorname{Tor}_n(E, F)$ by taking two free resolutions, one of E and one of F.
- (3) If $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ is a short exact sequence, then we get a long exact sequence:

$$\operatorname{Tor}_{n}(E',F) \to \operatorname{Tor}_{n}(E,F) \to \operatorname{Tor}_{n}(E'',F) \to \\ \to \operatorname{Tor}_{n-1}(E',F) \to \operatorname{Tor}_{n-1}(E,F) \to \operatorname{Tor}_{n-1}(E'',F) \to \\ \to - - - - - - - - - - - - - - - - \to \\ \to \operatorname{Tor}_{1}(E',F) \to \operatorname{Tor}_{1}(E,F) \to \operatorname{Tor}_{1}(E'',F) \to \\ \to E' \otimes F \to E \otimes F \to E'' \otimes F \to 0.$$

- (4) Tor is compatible with inductive limit, i.e. if $E = \lim_{\longrightarrow} (E_i)$, then $Tor_n(\lim_{\longrightarrow} E_i, F) = \lim_{\longrightarrow} (Tor_n(E_i, F))$.
- (5) We can define $\operatorname{Tor}_n(E, F)$ by taking a flat resolution of E.

 Proposition 3: Let E be an A-module. Then the following conditions are equivalent:
- (a) E is flat.
- (b) For all A-modules F, and for all $n \ge 1$, $\operatorname{Tor}_n(E, F) = 0$.
- (c) For all A-modules F, $Tor_1(E, F) = 0$.

Proof: (a) \Rightarrow (b). If ... $\rightarrow L_n \rightarrow L_{n-1} \rightarrow ... \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0$ is a free resolution of F, then the sequence

$$\dots \to E \otimes L_n \to E \otimes L_{n-1} \to \dots \to E \otimes L_1 \to E \otimes L_0 \to E \otimes F \to 0$$

is exact, thus $\operatorname{Tor}_n(E, F) = 0$ for all $n \ge 1$.

 $(b)\Rightarrow (c)$ clear. $(c)\Rightarrow (a)$: If the sequence $0\to F'\to F\to F''\to 0$ is exact, so is also (by (3) above) $\operatorname{Tor}_1(E,F'')\to E\otimes F'\to E\otimes F\to E\otimes F''\to 0$. Now $\operatorname{Tor}_1(E,F'')=0$, thus E is flat.

Proposition 4: If I and J are two ideals in A, then $\operatorname{Tor}_1^A(A/I,A/J) = I \cap J/I$. J.

Proof: From the exact sequence $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$, we get the exact sequence:

 $\operatorname{Tor}_1(A,A/J) \to \operatorname{Tor}_1(A/I,A/J) \to I \otimes A/J \to A \otimes A/J \to A/I \otimes A/J \to 0.$ But now $\operatorname{Tor}_1(A,A/J) = 0 \quad (A \text{ beeing } A\text{-free}), \text{ and } I \otimes A/J = I/I \cdot J;$ $A \otimes A/J = A/J. \text{ Therefore the sequence } 0 \to \operatorname{Tor}_1(A/I,A/J) \to I/I \cdot J \to A/J \text{ is exact, and } \operatorname{Tor}_1(A/I,A/J) = \operatorname{Ker}(I/I \cdot J \to A/J) = I \cap J/I \cdot J.$

Example: Let U be an open set in \mathbb{C}^n , and $x \in U$. Further let $X, Y \subset U$ be two hypersurfaces, defined by I = (f) and J = (g). Supposing that f and g do not have common factors: $I_x \cap J_x = I_x J_x$, and

$$\operatorname{Tor}_{1}\left(\mathcal{O}_{X,x},\mathcal{O}_{Y,x}\right) = \operatorname{Tor}_{1}\left(\mathcal{O}_{U,x}/I_{x}, \quad \mathcal{O}_{U,x}/J_{x}\right) = \frac{I_{x} \cap J_{x}}{I_{x} \cdot J_{x}} = 0.$$

Heuristic remark: The formula $\operatorname{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_Y, x) = 0$ expresses the fact that X and Y are "in general position". If for example X and Y are two linears subspaces in \mathbb{C}^n of dimensions p and q, we have $\operatorname{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$ if $\dim(X \cap Y) = p + q - n$, and $\operatorname{Tor}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) \neq 0$ otherwise.

Next we shall prove an elementary flatness criterion.

Proposition 5: Let E be an A-module. The following conditions are equivalent:

- (a) E is flat.
- (b) For all finitely generated ideals I of A, $Tor_1(E, A/I) = 0$.
- (c) For all monogenous A-modules F, $Tor_1(E, F) = 0$.

Proof: $(a) \Rightarrow (b)$, by prop. 3.

- $(b) \Rightarrow (c)$: Because Tor is compatible with inductive limit, we can suppose, that $\text{Tor}_1(E, A/I) = 0$ for an arbitrary ideal I of A. But every monogenous A-module F can be represented by A/I.
- $(c) \Rightarrow (a)$. By prop. 3 it is sufficient to prove that $Tor_1(E, F) = 0$ for any A-module F.

First consider the case, where F is finitely generated. We use induction, supposing that $\operatorname{Tor}_1(E,F)=0$, when F has n generators. Let F have (n+1) generators $x_1, ..., x_n, x_{n+1}$. If F' is the submodule generated by $\{x_1, ..., x_n\}$, then $F' \subset F$ and F'' = F/F' is monogenous. The exact sequence $0 \to F' \to F \to F'' \to 0$ gives the exact sequence $\operatorname{Tor}_1(E, F') \to \operatorname{Tor}_1(E, F) \to \operatorname{Tor}_1(E, F'')$. Now $\operatorname{Tor}_1(E, F') = \operatorname{Tor}_1(E, F'') = 0$, thus $\operatorname{Tor}_1(E, F) = 0$. In the general case, F can be considered as an inductive limit of finitely generated modules, and because $\operatorname{Tor}_1(E, F) = 0$. (E, F) = 0.

Proposition 6: Let A be an integral domain, and E an A-module. Then E is torsionfree if and only if $Tor_1(E, A/(a)) = 0$, for any element $a \in A$.

Proof: If E is A-module, $a \in A$, then the exact sequence $0 \to A \to A \to A \to A \to A/(a) \to 0$ gives the exact sequence $0 \to Tor_1(E, A/(a)) \to E \to E$. In other words $Tor_1(E, A/(a)) = \{x \in E \mid ax = 0\}$, from which the result follows.

Corollary: Let A be a principal ideal domain. E is flat if and only if E is torsionfree.

Proof: We have already proved that, if E is flat, then it is torsion free. The converse follows from prop. 6 and prop. 5.

The first flatness criterion for noetherian local rings is the following:

Theorem 2: Let A be a noetherian local ring with maximal ideal m; k = A/m, and E a finitely generated A-module. The following conditions are equivalent:

- (a) E is free.
- (b) E is flat.
- (c) $\operatorname{Tor}_{1}^{A}(E, k) = 0$.

Proof: We have already proved $(a) \Rightarrow (b) \Rightarrow (c)$.

 $(c) \Rightarrow (a)$: We recall first Nakayma's lemma. If A is a local ring with maximal ideal m; k=A/m, and E is a finitely generated A-module, such that $k\otimes E=E/mE=0$, then E=0.

The module $\overline{E} = k \otimes E = E/mE$ is a finitely generated vector space over k. Let $\{\overline{x}_1, ..., \overline{x}_r\}$ be a base of \overline{E} (over k), and $\{x_1, ..., x_r\}$ E representatives of \overline{x}_i : s. Consider the homomorphism $\phi: A^r \to E$, $\phi(a_1, ..., a_r) = \sum a_i x_i$. Denoting by R and Q the kernel and the cokernel of ϕ , we get an exact sequence:

$$(*) 0 \to R \to A^r \to E \to Q \to 0$$

and R, Q are finitely generated A-modules. From (*) we get the exact sequence

$$A^r \underset{A}{\otimes} k \to E \underset{A}{\otimes} k \to Q \underset{A}{\otimes} k \to 0$$
.

But $\overline{E} = E \otimes k \simeq k^r = A^r \otimes k$, so $Q \otimes k = 0$, and by Nakayama's lemma Q = 0.

Therefore ge have an exact sequence

$$0 \rightarrow R \rightarrow A^r \rightarrow E \rightarrow 0$$
.

From this we get: $\operatorname{Tor}_1(E, k) \to k \otimes R \to k^r \to \overline{E} \to 0$ (exact). Now: $\overline{E} \simeq k^r$, $\operatorname{Tor}_1(E, k) = 0$ (by assumption). Therefore $k \otimes R = 0$, and once more by Nakayama's lemma R = 0, thus $E \simeq A^r$, i.e. E is free.

Proposition 7: Let $\phi: A \to B$ be a ring homomorphism, and let B be A-flat. If I is an ideal of A, we write $\overline{A} = A/I$, $\overline{B} = B/IB = \overline{A} \otimes B$. Let F be a B-module, then: $\operatorname{Tor}_{i}^{A}(\overline{A}, F) = \operatorname{Tor}_{i}^{B}(\overline{B}, F)$ $(i \ge 0)$.

Proof: We choose first a B-free resolution of F

$$\rightarrow L_{n+1} \rightarrow L_n \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0$$
.

If L. is the respective complex of resolution, then

$$\overline{B} \underset{B}{\otimes} L. = B/IB \underset{B}{\otimes} L. = \overline{A} \underset{A}{\otimes} (B \underset{B}{\otimes} L.) = \overline{A} \underset{A}{\otimes} L.$$

Because every L_i is B-free, and B is A-flat, every L_i is A-flat (Property 3 after Th. 1). Thus L. is a flat A-resolution, and

$$\operatorname{Tor}_{i}^{A}(\overline{A}, F) = H_{i}(\overline{A} \otimes L.) = H_{i}(\overline{B} \otimes L.) = \operatorname{Tor}_{i}^{B}(\overline{B}, F).$$

We shall next state the second flatness criterion for noetherian local rings.

Theorem 3: Let A and B be two noetherian local rings, with maximal ideals \underline{m} , \underline{n} ; $k = A/\underline{m}$. If $\phi : A \rightarrow B$ is a local homomorphism (i.e. $\phi (\underline{m}) \subset \underline{n}$), and F finitely generated B module then

$$F$$
 is A-flat \Leftrightarrow Tor $_{1}^{A}(k, F) = 0$.

The proof of this theorem is much more difficult than that of th. 20 see for example:

Bourbaki: Algèbre commutative, Chapter III § 5, th1, $(i) \Leftrightarrow (iii)$, p. 98.

The conditions in Bourbaki's theorem are here fullfilled:

- 1° A finitely generated module F over a noetherian local ring B is idealwise separated for n. (Ibid., § 5. 1. Ex. 1, p. 97.)
- 2° If $\phi: A \to B$ is a local homomorphism, F is also idealwise separated for m. (*Ibid.*, § 5, prop. 2, p. 101.)
- 3° Also the flatness condition is fulfilled, because k is a field.

Remark: The main interest of the theorem lies in the fact, that it is true without any assumption of finitness on B.

Corollary: If the assumptions are the same as in the theorem 3, and if moreover B is A-flat, then

$$F ext{ is } A ext{-flat} \Leftrightarrow \operatorname{Tor}_1^B(\overline{B}, F) = 0$$

where $\overline{B} = B/mB$.

Proof: $\operatorname{Tor}_{1}^{A}(k, F) = \operatorname{Tor}_{1}^{B}(\overline{B}, F)$, by prop. 7.

§ 5. Geometric applications of the flatness criterions

A) Flatness for finite morphisms

Proposition 1: Let $\pi: X \to S$ be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then $\pi_*(\mathcal{O}_X)$ is a coherent analytic sheaf over S. The following conditions are equivalent:

- (a) π is flat (i.e. for every $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$ -module, $s = \pi(x)$).
- (b) For every s, $(\pi_* \mathcal{O}_X)_s$ is a flat $\mathcal{O}_{S,s}$ -module.
- (c) $\pi_* \mathcal{O}_X$ is a locally free sheaf.

Proof: Because π is finite $\pi_* (\mathcal{O}_X)_s = \bigoplus_{x \in \pi^{-1}(s)} \mathcal{O}_{X,x}$, thus the only point to prove is $(b) \Rightarrow (c)$.

Now if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$ -module, then (by theorem 2) $\mathcal{O}_{X,x}$ is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

Proposition 2: Let S be a reduced analytic space and $\mathscr E$ a coherent $\mathscr O_s$ -module. Let E(s) be the finite dimensional vector space (over C) $\mathscr E_s \otimes_{\mathscr O} C_s$. $\mathscr E$ is a locally free $\mathscr O_{S,s}$ -module if an only if $\dim_C E(s)$ is locally constant.

Proof: If \mathscr{E} is locally free, then $\dim_{\mathbb{C}} E(s)$ is locally constant. Suppose now that $\dim_{\mathbb{C}} E(s)$ is locally constant in an open set $U \subset S$, and that $\mathcal{O}_U^p \to \mathcal{O}_U^q \to \mathcal{E}_U \to 0$ is exact. d is determined by a $p \times q$ matrix of analytic functions on U, so it gives a morphism $\mathbf{C}_U^p \to \mathbf{C}_U^q$ of trivial vector bundles over U.

From the exact sequence $\mathcal{O}_s^p \to \mathcal{O}_s^q \to \mathcal{E}_s \to 0$, we get (by making tensor-products with C_s) the exact sequence:

$$\mathbf{C}_{s}^{p} \stackrel{d(s)}{\rightarrow} \mathbf{C}_{s}^{q} \rightarrow E(s) \rightarrow 0$$
,

which shows that d has constant rank in U. Thus Ker d and Im d are vector bundles, and we can write

$$\mathbf{C}_{U}^{p} = F_{1} \oplus G_{1}$$
, $\mathbf{C}_{U}^{q} = F_{0} \oplus G_{0}$,
$$d: \begin{cases} F_{1} \rightarrow 0 \\ G_{1} \simeq F_{0} \end{cases}$$

Now $\mathscr{E} \simeq$ the sheaf of analytic sections of G_0 , therefore \mathscr{E} is locally free.

Definition 1: Let $\pi: X \to S$ be a finite morphism of analytic spaces, and $s \in S$. For each $x \in X(s) = \pi^{-1}(s)$, $\mathcal{O}_{X(s),x} = \mathbb{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$ is finite dimensional vectorspace over \mathbb{C} . Denote its dimension by v(x). Then the degree v(s) of s is defined by $v(s) = \sum_{x \in X(s)} v(x)$.

Theorem 1: Let $\pi: X \to S$ be a finite morphism of analytic space and let S be a reduced space. Then X is flat over S if and only if v(s) is locally constant function of s.

$$Proof: v(s) = \sum_{x \in X(s)} \dim_{\mathbb{C}} \mathcal{O}_{X(s),x} = \dim_{\mathbb{C}} \left(\bigoplus_{x \in X(s)} \mathcal{O}_{X(s),x} \right)$$

$$= \dim_{\mathbb{C}} \left(\bigoplus_{x \in X(s)} \left(\mathbf{C} \bigotimes_{\mathscr{O}_{S,s}} \mathscr{O}_{X,x} \right) \right)$$

$$= \dim_{\mathbb{C}} \mathbf{C} \bigotimes_{\mathscr{O}_{S,s}} \pi_{*} (\mathscr{O}_{X})_{s} = \dim_{\mathbb{C}} E(s).$$

The theorem follows from propositions 1 and 2.

Examples of flat morphisms

Example 1: If $\pi: X \to S$ is a local isomorphism near x, then π is flat at x.

Example 2: Consider § 2, Ex. 1. Here v(x) = 1.

Examples of non-flat morphisms

Examples 1: If $X \subset S$ is a closed subspace, not open, v(s) is not locally constant.

Example 2: Let X be a subspace of \mathbb{C}^4 defined by the ideal intersection of (x_3, x_4) and $(x_1 - x_1, x_4 - x_2)$ (which is equal to the product ideal) and let π be the projection onto the (x_1, x_2) -plane \mathbb{C}^2 . Then X is a union of two 2-planes in \mathbb{C}^4 , whose intersection is (0). When $s \neq 0$. X(s) consists of two simple points, so v(s) = 2. X(0) is given by the ideal $(x_1, x_2, x_3^2, x_3x_4, x_4^2)$, thus v(0) = 3.

Example 3: Let $S = \{(u, v, w) \in \mathbb{C}^3 \mid v^2 = uw\}$ and $\pi : \mathbb{C}^2 \to S$ be the map $(x, y) \to (x^2, xy, y^2)$. This map identifies S with the quotient of \mathbb{C}^2 by the equivalence relation idenfying (x, y) with (-x, -y). However, π is not flat, since for $s \in S$, v(s) = 2 if $s \neq 0$ and v(s) = 3 if s = 0.

B) Projection of a product of analytic spaces

Theorem 2: Let S and X be analytic spaces. If $\pi: S \times X \to S$ is the projection morphism, then π is flat, i.e. $\mathcal{O}_{S \times X, s, x}$ is a flat $\mathcal{O}_{S, s}$ module for every $(s, x) \in S \times X$.

To prove this theorem we need first some homological algebra. Then we shall show it in the particular case, when S is a manifold, and finally in the general case.

(a) Koszul complex

Let A be a ring, M an A-module and $h_1, ..., h_n$ homomorphisms $M \rightarrow M$, which commute with each other, i.e. $h_i h_j = h_j h_i$ for every i, j.

If $1 \le k \le n$, set $Q_k = M/h_1(M) + ... + h_k(M)$, and $Q_0 = M$, thus, in particular, $Q_n = Q = M/\sum_{i=j}^n h_i(M)$,. Every h_k induces a map $h_k Q_{k-1} \to Q_{k-1}$.

Definition 2: The sequence $(h_1, ..., h_n)$ is called regular if each of the mappings h_k $(1 \le k \le n)$ is injective.

The Koszul complex of the module M and of the mappings h_k $(1 \le k \le n)$ K = K. $[M; h_1, ..., h_n]$ is defined in the following way:

$$K_i = \bigwedge^{n+i} A^n \otimes M \simeq M^{\binom{n}{i}}, \quad 0 \leqslant i \leqslant n.$$

We define the homorphisms $d_i: K_i \to K_{i-1}$ (i > 0) by $\lambda \otimes x \to \sum_i (e_i \wedge \lambda) \otimes \otimes h_i(x)$, where (e_i) is the natural base of A^n . We also define $\varepsilon: K_0 \to Q$ as the natural map $: K_0 = M \to M / \sum_{i=1}^n h_i(M) = Q$. Using the fact that $h_1, ..., h_n$ commute with each other, it is easy to verify that

$$(d_{i-1} \circ d_i)(\lambda \otimes x) = \sum_{i,j} (e_j \wedge e_i \wedge \lambda) \otimes h_j(h_i(x)) = 0;$$

also $\varepsilon d_1 = 0$. Thus K. is really a complex.

Theorem 3 (Poincaré-Koszul).

If $(h_1, ..., h_n)$ is a regular sequence, then

$$H_i(K.) = \begin{cases} Q & if & i = 0 \\ & & . \\ 0 & if & i > 0 \end{cases}$$

 h_iI

If $h_i \in A$, it defines the map: $A \rightarrow A$, which we denote also by h_i . We say that $(h_1, ..., h_n)$ is a regular sequence of elements if $(h_1, ..., h_n, I)$ is a regular sequence.

Corollary. If $(h_1, ..., h_n)$ is a regular sequence of elements, then the Koszul complex K = K. $[A; h_1, ..., h_n] = \{ \wedge^{n-1} A^n \simeq A^{\binom{n}{i}} \}$ is a free resolution of $Q = A/(h_i)$ ((h_i) is the ideal generated by $h_1, ..., h_n$)).

Example: If $A = \mathbb{C} \{x_1, ..., x_n\}$; $h_i = x_i$, then $Q_k = A/(x_1, ..., x_k) = \mathbb{C} \{x_{k+1}, ..., x_n\}$ and $Q = Q_n = \mathbb{C}$. The complex K = K. $[A; x_1, ..., x_n]$ is a free resolution of \mathbb{C} .

(b) Proof of theorem 2, when S is a complex manifold

In this case we can take $\mathcal{O}_{S,s} = \mathbb{C}\{t_1,...,t_m\} = A$ and if $\mathcal{O}_{X,x} = \mathbb{C}\{x_1,...,x_n\}/(f_1,...,f_p)$, then

$$\mathcal{O}_{S \times X,(s,x)} = \mathbb{C} \{t_1, ..., t_m, x_1, ..., x_n\} / (f_1, ..., f_p) = B.$$

B is an A-module in a natural way.

By the corollary of the Poincaré-Koszul theorem K = K. $[A; t_1, ..., t_m]$ in a free resolution of \mathbb{C} . We want to compute the modules $\operatorname{Tor}_i^A(\mathbb{C}, B) = H_i(K \otimes B)$ (i > 0).

It's easily seen, that we can consider the complex $K \cdot \otimes B$ as a Koszul

complex K' = K. $[B; t_1, ..., t_m]$ (where $t_i : B \rightarrow B$). But now the sequence $(t_1, ..., t_m)$ is regular, thus by the Poincaré-Koszul theorem $H_i[K'] = 0$ if i > 0.

In particular: $\operatorname{Tor}_{1}^{A}(\mathbb{C}, B) = H_{1}[K \otimes B] = H_{1}[K'] = 0$. By the second flatness criterion B is A-flat.

(c) The general case

The question being local, we can suppose that $S \subset W \subset \mathbb{C}^n$, where W is open, and S an analytic subspace of W. Let S be defined by $g_1, ..., g_r$. Then

 $S \times X \subset W \times X$ and $\mathscr{O}_S = \mathscr{O}_W/(g_1, ..., g_r)$. On the other hand $\mathscr{O}_{S \times X} = \mathscr{O}_{W \times X}/(g_1, ..., g_r) = \mathscr{O}_S \otimes \mathscr{O}_{W \times X}$. The last equality follows from

the fact, that if $\pi: X \to S$ is a morphism, and $S' \subset S$ a subspace, $X' = \pi^{-1}(S')$,

then
$$\mathscr{O}_{X'} = \mathscr{O}_{S'} \otimes_{\mathscr{O}_{S}} \mathscr{O}_{X}$$
.

Remark: This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$X \qquad Y \qquad \emptyset_{X \times Y} = \emptyset_X \otimes_{\emptyset_S} \emptyset_y.$$

We have proved that $\mathcal{O}_{W \times X}$ is \mathcal{O}_W -flat, so by scalar extension $\mathcal{O}_{S \times X}$ is \mathcal{O}_S flat.

Corollary: If X and S are two manifolds and $\pi: X \rightarrow S$ is a submersion, then π is flat.

III. PRIVILEGED POLYCYLINDERS

§ 1. Banach vector bundles over an analytic space

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X.

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E. If $U \subset \mathbb{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g: U \to E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X, i.e. X' is the support of the quotient sheaf \mathcal{O}_U/J , where $U \subset \mathbb{C}^n$ is open and J is a coherent sheaf of ideals of \mathcal{O}_U , then $\mathscr{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \to \mathscr{H}(V, E)/J_V \cdot \mathscr{H}(V, E)$ $(V \subset U, V\text{-open})$.

Remark: If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U.

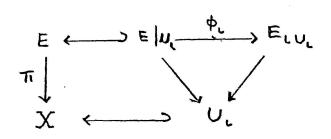
The sheaf $\mathscr{H}_X(E)$ is constructed with help of the local models X' of X, i.e. $\mathscr{H}_X(E)|X'=\mathscr{H}_{X'}(E)$, for every local model X'.

Definition 1: The set of analytic morphisms from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F.

Definition 2: An analytic vector bundle morphism from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.

Let E be a topological space, X an analytic space, and $\pi: E \rightarrow X$ a continuous projection.



Suppose that X has an open covering $(U_{\iota})_{\iota \in I}$, and that for every $\iota \in I$ there is given a trivial Banach space bundle $E_{\iota U_{\iota}}$ and a homeomosphism ϕ_{ι} , such that the following diagram is commutative:

We suppose further that for each pair ι , $\kappa \in I$ there is given an analytic vector bundle morphism $\gamma_{\iota\kappa}: E_{\kappa U_{\iota} \cap U_{\kappa}} \to E_{\iota U_{\iota} \cap U_{\kappa}}$, with the underlying mapping $\phi_{\iota} \circ \phi_{\kappa}^{-1}$, such that:

$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \, \gamma_{\kappa\lambda}; \quad \gamma \iota_{\iota} = I, \quad \text{ for all } \quad \iota, \kappa, \gamma \in I.$$

This data gives a Banach vector bundle atlas on E and provides E with the structure of a Banach vector bundle over X (two atlases are equivalent if there exists an atlas containing both).

Remark: If X is reduced, the $\gamma_{\iota\kappa}$ are determined by their underlying map and the condition $\gamma_{\iota\lambda} = \gamma_{\iota\kappa} \gamma_{\kappa\lambda}$ is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

Proposition 1: Let $\phi: E \to F$ be a morphism of two Banach vector

bundles E and F, and $x \in X$.

If $\phi_x \in \mathcal{L}(E(x), F(x))$ is an isomorphism, then there exists an open neighbourhood $U \subset X$ of x, such that $\phi | U : E | U \to F | U$ is a vector bundle isomorphism.

Proof: First we take a trivialisation $E|V = E_{0V}$, $F|V = F_{0V}$ at $x \in V \subset X$ (V-open).

The set Isom (E_0, F_0) of isomorphic mappings is an open subset of $\mathcal{L}(E_0, F_0)$ and the mapping $g \rightarrow q^{-1}$ is an analytic isomorphism:

Isom
$$(E_0, F_0) \simeq \text{Isom } (F_0, E_0)$$
.

So we have in an open neighbourhood $U \subset X$ of x an analytic morphism $y \to \phi_y^{-1} \in \mathcal{L}(F_0, E_0)$, which defines the inverse morphism $(\phi|U)^{-1} : F|U \to E|U$.

Definition 3: Let E and F be two Banach spaces and f a continuous linear mapping from E into F. f is a split mono-(epi) morphism, if there exists a mapping $g \in \mathcal{L}(F, E)$ such that $g \circ f = I_E$. (Resp. $f \circ g = I_F$.)

Definition 4: Let E_1 and E_2 be two Banach vector bundles over an analytic space X, and f a vector bundle morphism from E_1 into E_2 . f is a split mono (epi) morphism, if there exists a vector bundle morphism $g: E_2 \rightarrow E_1$ such that $g \circ f = I_{E_1}$. (Resp. $f \circ g = I_{E_2}$.)

Equivalently, $f: E_1 \to E_2$ is a split monomorphism if an only if E_2 can

be decomposed in a direct sum $E_2 = F_2 \oplus G_2$ such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \to G_2 \end{cases}.$$

and f is a split epimorphism if correspondingly

$$E_1 \,=\, F_1 \oplus G_1 \;, \quad \text{such that} \quad f\!:\! \left\{ \begin{array}{l} F_1 \,\to\, 0 \\ \\ G_1 \,\simeq\! E_2 \end{array} \right. .$$

Proposition 2: Let $E \stackrel{\varphi}{\to} F$ be a bundle morphism and $x \in X$.



If $\phi_x : E(x) \to F(x)$ is a split epi (mono) morphism, then the point x has an open neighbourhood $U \subset X$, such that $\phi | U : E | U \to F | U$ is a split vector bundle epi (mono) morphism.

Proof: Suppose that ϕ_x is a split epimorphism. We take first a trivilisation $E|V=E_{0V},F|V=F_{0V}$ at x, so that there exists a mapping $\sigma\in\mathcal{L}(F_0,E_0)$, $\phi_x\circ\sigma=I_{F_0}$. If we define a morphism $\psi:F_{0V}\to E_{0V}$ by $x\to\sigma\in\mathcal{L}(F_0,E_0)$, the morphism $\gamma=\phi\circ\psi:F_{0V}\to F_{0V}$ has an isomorphic fibre mapping $\gamma_x=I_{F_0}$ in x. By proposition 1 we have an isomorphic restriction $\gamma|U,\phi|U\circ(\psi|U\circ(\gamma|U)^{-1})=I_{F_{0U}}$.

When ϕ_x is a split monomorphism, the proof is similar.

Definition 5: Let B_1 , B_2 , B_3 be Banach spaces, and $j, k : B_1 \rightarrow B_2 \rightarrow B_3$ continuous linear mappings. This sequence forms a complex, if $k \circ j = 0$. This sequence is *split exact* if the space B_i can be decomposed in direct

sums $B_i = C_i \oplus D_i$ such that

$$j: \begin{cases} C_1 \to 0 \\ D_1 \simeq C_2 \end{cases} \qquad k: \begin{cases} C_2 \to 0 \\ D_2 \simeq C_3 \end{cases}.$$

Definition 6: A Banach vector bundle morphism sequence

$$E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$$
 is a complex if $g \circ f = 0$.

The sequence is *split exact*, if every E_i can be decomposed $E_i = F_i \oplus G_i$, such that:

$$f : \begin{cases} F_1 \to 0 \\ G_1 \simeq F_2 \end{cases} \qquad g : \begin{cases} F_2 \to 0 \\ G_2 \simeq F_3 \end{cases}$$

Theorem 1: Let $E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$ be a complex of Banach vector

bundles and $x_0 \in X$.

If the sequence of Banach spaces $E_1(x_0) \stackrel{f_{x_0}}{\to} E_2(x_0) \stackrel{f_{x_0}}{\to} E_3(x_0)$ is split exact, then there exists an open neighbourhood $U \subset X$ of x_0 , such that $E_1 | U \to E_2 | U \to E_3 | U$ is a split exact sequence of Banach vector bundles.

Proof: We take a neighbourhood V of x, such that we have a complex $f \mid V g \mid V$ $E_{1V} \rightarrow E_{2V} \rightarrow E_{3V}$ of trivial bundles. By assumption we have the decompositions $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$ with

$$f_{x_0} : \begin{cases} F_1(x_0) \to 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \qquad g_{x_0} : \begin{cases} F_2(x_0) \to 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases}.$$

By proposition $2, f|V: G_{1V} \to E_{2V}, g|V: G_{2V} \to E_{3V}$ are both split monomorphisms in a neighbourhood $W \subset V$ of x_0 and the images $F_2 = f(G_{1W})$, $F_3 = g(G_{2W})$ are subbundles of E_{2W} esp. E_{3W} , such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W}.$$

By our construction

$$g \mid W : \begin{cases} F_2 & \to 0 \\ G_2 & W \simeq F_3 \end{cases}.$$

If $p: E_{2W} \to F_2$ is the projection with kernel G_{2W} , the map, $p \circ f: E_{1W} \to F_2$ is a split epimorphism in x_0 . Again by prop. 2 we have over an open eighbourhood $U \subset W$ of x_0 a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \text{Ker p} \circ f$)

$$(p \circ f) \mid U : \begin{cases} F_1 \to 0 \\ & \\ G_{1U} \to F_{2U} \end{cases}.$$

The image $f | U(F_1)$ is contained in G_{2U} . But $g | U \circ f | U = 0$ and $g | G_{2U}$ is a monomorphism hence $f | U : F_1 \rightarrow 0$. We get finally (restricting all our morphisms to U)

$$f \mid U : \begin{cases} F_{1U} \to 0 \\ G_{1U} \simeq F_{2U} \end{cases} \qquad g \mid U : \begin{cases} F_{2U} \to 0 \\ \tilde{G}_{2U} \to F_{3U} \end{cases}.$$

§ 2. Privileged polycylinders

Definition 1: A polycylinder in \mathbb{C}^n is a compact set K of the form $K = K_1 \times ... \times K_n$ where each K_i is a compact, convex subset of \mathbb{C} , with nonempty interior. If each K_i is a disc, then K is a polydisc. We first recall the following theorem of Cartan.

Theorem 1: Let K be a polycylinder contained in an open subset U of \mathbb{C}^n . Let \mathscr{F} be a coherent analytic sheaf on U.

(A) There exists an open neighbourhood of K over which \mathcal{F} admits a finite free resolution

$$0 \to \mathcal{L}_n \to \dots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0 \ .$$

- (B) $H^q(K, \mathcal{F}) = 0$ for q > 0. (Reference: For instance Gunning and Rossi.) We have the following consequences of this theorem:
- 1) Given a finite free resolution

$$0 \to \mathcal{L}_n \to \dots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

of a coherent sheaf \mathcal{F} , the sequence

$$0 \to \mathcal{L}_n(K) \to \dots \to \mathcal{L}_0(K) \to \mathcal{F}(K) \to 0$$

is an $\mathcal{O}_{U}(K)$ - free resolution of $\mathscr{F}(K)$.

2) Given a short exact sequence of coherent sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

then the sequence

$$0 \to \mathcal{F}_{L}(K) \to \mathcal{F}(K) \to \mathcal{F}''(K) \to 0$$
 is exact.

Let \mathscr{F} be a coherent analytic sheaf on U, and let $K \subset U$ be a polycylinder If V is an open neighbourhood of K, then $\mathscr{F}(V)$ can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give $\mathscr{F}(K)$ the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from $\mathscr{F}(K)$ and by choosing K in a "privileged" way.

Let $B(K) = \{f : K \rightarrow \mathbb{C} | f \text{ continuous on } K \text{ and analytic on } \mathring{K} \}$, then B(K) is Banach algebra and $B(K) \subset C(K)$. The sections of \mathcal{O}_U over K are elements of B(K), and B(K) is in fact the uniform closure of $\mathcal{O}_U(K)$ in C(K).

If $\mathcal{L} = \mathcal{O}_U^r$, we define $B(K, \mathcal{L}) = B(K)^r$. Then $B(K; \mathcal{L})$ is a free B(K)-module, and since $\mathcal{L}(K) = \mathcal{O}_U(K)^r$, we have $B(K; \mathcal{L}) = B(K) \otimes \mathcal{L}(K)$.

We now assume that \mathscr{F} is a coherent sheaf on U, where $U \subset \mathbb{C}^n$ is open. Consider a free resolution

$$(R) 0 \to \mathcal{L}_n \to \dots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0 \text{of } \mathcal{F}.$$

From (R) we get an $\mathcal{O}_U(K)$ -free resolution of $\mathscr{F}(K)$

$$(R') 0 \to \mathcal{L}_n(K) \to \dots \to \to_1(K) \to \mathcal{L}_0(K) \to \mathcal{F}(K) \to 0.$$

Taking the tensorproduct $B(K) \otimes_{\mathcal{O}_{I}(K)}$ we get the complex

$$B(K; \mathcal{L}_{\cdot}): 0 \rightarrow B(K; \mathcal{L}_{n}) \rightarrow \dots \rightarrow B(K; \mathcal{L}_{1}) \rightarrow B(K; \mathcal{L}_{0}).$$

Definition 2: The polycylinder K is called \mathscr{F} -privileged if the complex $B(K; \mathscr{L})$ is split-exact in every degree >0.

Remark: The property of being \mathcal{F} -privileged is independent of the resolution (R).

The exactnes of $B(K; \mathcal{L})$ can be expressed by $\operatorname{Tor}_{i}^{\mathfrak{O}(K)}(B(K), \mathcal{F}(K)) = 0$, for every i > 0, and Tor is independent of the resolution (R). It is a little

more complicated to show, that the splitting property is independent of (R), and this is omitted.

Since $B(K; \mathcal{L}_i)$ is a Banach space, the image and its complement are thus Banach spaces if K is \mathcal{F} -privileged. In this case we define $B(K; \mathcal{F}) = \operatorname{Coker}(B(K, \mathcal{L}_1) \to B(K; \mathcal{L}_0)) = B(K) \otimes_{\mathcal{O}} \mathcal{F}(K)$ and we get a B(K)-module, which is a Banach-space.

Warning: In the definition of split-exactnes, the subspaces are splitting vector spaces, but they are not splitting B(K)-modules in general.

We have the following important theorem about the existence of privileged polycylinders:

Theorem 2: Let U be an open subset of \mathbb{C}^n , and let \mathscr{F} be a coherent analytic sheaf on U. For any $x \in U$ there exists a fundamental system of neighbourhoods of x in U, which are \mathscr{F} -privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

Example: (Curves in \mathbb{C}^2) Let $U \subset \mathbb{C}^2$ be an open connected neighbour hood of the origin, and let $h: U \to \mathbb{C}$ be analytic and $h \neq 0$.

Let X be the curve given by h, that is $X = h^{-1}(0)$, $\mathcal{O}_X = \mathcal{O}_U/(h)$. We have an exact sequence $0 \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_U \rightarrow \mathcal{O}_X \rightarrow 0$. Consider a polycylinder $K = K_1 \times K_2 \subset U$. By definition K is \mathcal{O}_X -priviledged if and only if $h: B(K) \rightarrow B(K)$ is a split monomorphism.

Let K_j denote the boundary of K_j , and define $K = K_1 \times K_2$ (K is called the Silov Boundary of K).

Proposition 1: (a) The following conditions are equivalent:

- (i) $h: B(K) \rightarrow B(K)$ is a monomorphism.
- (i') $\exists a > 0$ such that $||hf|| \ge a||f||$, $\forall f \in B(K)$.
- (ii) $X \cap K = \emptyset$.
- (b) If $(K_1 \times K_2) \cap X = \emptyset$, then h is a split monomorphism (i.e. K is \mathcal{O}_X privileged).

Proof: (a) (i) \Leftrightarrow (i') is a well known fact from the theory of normed vector spaces.

(ii) \Rightarrow (i'). Assume $X \cap K = \emptyset$. If $f \in B(K)$, then it follows from the maximum principle that $||f|| = \sup_{K} |f(x)| = \sup_{K} |f(x)|$. Since $h(x) \neq 0$

whenever $x \in K$, we get $a = \inf_{K} |h(x)| > 0$. Hence $||hf|| = \sup_{K} |hf(x)| \ge 2$ $\ge a \sup_{K} |f(x)| = a ||f||$.

(i') \Rightarrow (ii). Suppose that $X \cap K \neq \emptyset$ and $x = (x_1, x_2) \in X \cap K$. We choose an analytic function $f_1 : U_1 \to \mathbb{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \to \mathbb{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \to f_1(z_1) f_2(z_2)$. Since h(x) = 0 it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in K.

Applying Dini's theorem we get $||hf^n|| \to 0$. From the inequality $a||f^n|| \le$ $\le ||hf^n||$ we get $||f^n|| \to 0$, which is a contradiction, because for every $n: f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h: B(K) \rightarrow B(K)$ is a split monomorphism?

IV. FLATNESS AND PRIVILEGE

§ 1. Morphisms from an analytic space into B(K)

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbb{C}^n$. We want to construct an \mathcal{O}_S -algebra homomorphism $\phi : \mathcal{O}_{S \times U} (S \times U) \to \mathcal{H} (S; B(K))$.

- (a) Consider first $S = U' \subset \mathbb{C}^m$, U'-open. If $h \in \mathcal{O}_{U' \times U}$ ($U' \times U$) and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s,x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand its obvious that ϕ is an $\mathcal{O}_{U'}$ -algebra homomorphism.
- (b) Let S have a special model in the polydisc Δ in \mathbb{C}^m , defined by a sheaf \mathscr{J} of ideals of \mathscr{O}_{Δ} , and let \mathscr{J} be generated by $f_1, ..., f_p$, V-a polycylinder neighbourhood of K in U. By Cartan's theorem B for a polycylinder,

the sequence $0 \rightarrow \mathcal{J}(\Delta \times V) \rightarrow \mathcal{O}(\Delta \times V) \rightarrow \mathcal{O}(S \times V) \rightarrow 0$ is exact. If we denote by π the projection $\mathcal{H}(\Delta, B(K)) \rightarrow \mathcal{H}(S, B(K)), (f_1, ..., f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset$

 \subset Ker π . Therefore, because π is surjection, there exists a unique

 $\phi: \mathcal{O}(S \times V) \rightarrow \mathcal{H}(S, B(K))$, such that the diagram

$$\mathcal{O}\left(\Delta \times V\right) \xrightarrow{\phi} \mathcal{H}\left(\Delta, B\left(K\right)\right) \\
\pi \downarrow \qquad \qquad \downarrow \widetilde{\pi} \\
\mathcal{O}\left(S \times V\right) \xrightarrow{\phi} \mathcal{H}\left(S, B\left(K\right)\right)$$

is commutative; ϕ is evidently an \mathcal{O}_S -algebra homomorphism.

§ 2. The flatness and privilege theorem

Notation

Let S be an analytic space, U an open set in \mathbb{C}^n , and $\pi: S \times U \rightarrow S$ the first projection.

If \mathscr{F} is an $\mathscr{O}_{S\times U}$ module, then for every $s\in S$ we denote by $\mathscr{F}(s)$ the \mathscr{O}_U -module $i_s^*\mathscr{F}$, where i_s is the injective morphism $x\to(s,x)$ from U into $S\times U$. If $x\in U$

$$(\mathscr{F}(s))_x \simeq \mathscr{F}_{(s,x)}/m_s \cdot \mathscr{F}_{(s,x)} \simeq \mathscr{F}_{(s,x)} \otimes_{\mathscr{O}_{S,s}} \mathbf{C}_s.$$

Theorem 1: Let $\mathscr E$ be a coherent and S-flat $\mathscr O_{S\times U}$ -module, and K a polycylinder in U.

- (a) When K is privileged for $\mathscr{E}(s_0)$, s_0 has a neighbourhood V such that K is $\mathscr{E}(s)$ -privileged for each $s \in V$. In other words: the set $S' = \{s \in S \mid K \text{ is } \mathscr{E}(s)\text{-privileged}\}$ is open in S.
- (b) It is possible to define a Banach vector bundle over S' whose fibre at any $s \in S'$ is $B(K, \mathscr{E}(s))$.

To prove the theorem we need:

Lemma 1: Under the conditions of the theorem, we can, for every $s \in S$, find a neighbourhood W of $\{s\} \times K$ and a free resolution of finite length

$$0 \! \to \! \mathcal{L}_p \! \xrightarrow{d_p} \! \dots \xrightarrow{d_2} \! \mathcal{L}_1 \! \xrightarrow{d_1} \! \mathcal{L}_0 \! \xrightarrow{\varepsilon} \! \mathscr{E} \! \to \! 0 \; \text{in} \; W.$$

Proof: Let (s, x) be a point of $S \times U$ and \mathcal{L}^0_* a finite resolution of $\mathcal{F}(x)$ in a neighbourhood of x (there exists such one, by the theorem of syzygies). We shall show that that there exists a resolutin \mathcal{L}^* of \mathcal{F} in a neighbourhood of (s, x) such that $\mathcal{L}^*(s) = \mathcal{L}^0_*$; if $\mathcal{L}^0_i = \mathcal{O}^{r_i}_x$ define

$$\mathcal{L}_i = \mathcal{O}_{S \times U}^{r_i}$$
 and $\mathcal{K}_i^0 = \operatorname{Ker} d_i^0 \colon \mathcal{L}_i^0 \to \mathcal{L}_{i-1}^0$.

We shall construct by induction (with respect to i) $d_i: \mathcal{L}_1 \to \mathcal{L}_{i-1}$ in a neighbourhood of (s, x) such that $d_i(s) = d_i^0$, and prove that $\mathcal{K}_i = \operatorname{Ker} d_i$ is S-flat and that $\mathcal{K}_i(s) = \mathcal{K}_i^0$.

Nakayama's lemma shows that Im $d_{i+1} = \mathcal{K}_i$ at the point (s, x), therefore in a neighbourhood of that point.

The exact sequence

$$0 \rightarrow \mathcal{K}_{i+1} \rightarrow \mathcal{L}_{i+1} \rightarrow \mathcal{K}_i \rightarrow 0$$
,

where \mathcal{K}_i and \mathcal{L}_{i+1} are S-flat, shows that \mathcal{K}_{i+1} is S-flat, and that $\mathcal{K}_{i+1}(s) = \mathcal{K}_{i+1}^0$. The first step of the induction is analogous.

Proof of the theorem: Let $s_0 \in S$ and

$$\begin{array}{cc} d_p & d_1 \\ 0 \rightarrow \mathcal{L}_p \rightarrow \dots \rightarrow \mathcal{L}_0 \rightarrow \mathcal{E} | W \rightarrow 0 \end{array}$$

be a free $\mathscr{O}_{S\times U}$ resolution of \mathscr{E} in a neighbourhood $W=V_1\times V_2$ of $\{s_0\}\times K$. The sheaf \mathscr{E} is \mathscr{O}_S -flat, so for each $s\in V_1$, the sequence

$$0 \to \mathcal{L}_p \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \dots \to \mathcal{L}_1 \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \mathcal{L}_0 \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to \mathscr{E}_{|W} \otimes_{\mathscr{O}_{S|V_1}} \mathbf{C}_s \to 0$$

is exact. So the sequence

is exact when $s \in V_1$. Now $\mathcal{L}_i(s) \simeq \mathcal{O}_{V_2}^{r_i}$ $(0 \le i \le p)$ and every $d_i(s)$ induces a continuous linear map:

 $B(K, \mathcal{L}_i(s)) \rightarrow B(K, \mathcal{L}_{i-1}(s))$, which we also denote by $d_i(s)$. We can consider $d_i = (d_{ijk})$ as an $r_i \times r_{i-1}$ -matrix with entries from $\mathcal{O}_{S \times U}(W)$.

By § 1 we have a \mathcal{O}_S -algebra homomorphism

$$\mathcal{O}_{S\times W}(S\times W) \rightarrow \mathcal{H}(S, B(K))$$
.

From the matrix (d_{ijk}) we get by this homomorphism a morphism d_i :

$$V_0 \to \mathcal{L}\left(B(K)^{r_i}, B(K)^{r_{i-1}}\right) = \mathcal{L}\left(B(K, \mathcal{L}_i(s)), B(K, \mathcal{L}_{i-1}(s))\right).$$

(Here V_0 is some neighbourhood of s_0) such that $d_i(s) = d_i(s)$ for each $s \in V_0$. In other words we have a sequence of Banach vector bundle morphisms

$$\begin{array}{ccc} d_p & \widetilde{d}_1 \\ 0 \rightarrow B(K, \mathcal{L}_p) \rightarrow \dots \rightarrow B(K, \mathcal{L}_0). \end{array}$$

Using the fact that $\mathcal{O}_{S\times U}(S\times U)\to \mathcal{H}(S,B(K))$ is an \mathcal{O}_S -algebra homomorphism, it easily follows that (B) is complex of Banach vector bundles over S.

Now K is $\mathscr{E}(s_0)$ -privileged, thus

$$0 \to B\left(K, \, \mathcal{L}_p\left(s_0\right)\right) \stackrel{d_1(s_0)}{\to} \dots \stackrel{}{\to} B\left(K, \, \mathcal{L}_0\left(s_0\right)\right)$$

is split exact, so by theorem III.1

$$\tilde{d}_{p}|V \quad \tilde{d}_{i}|V
0 \to B(K, \mathcal{L}_{p})|V \to \dots \to B(K, \mathcal{L}_{0})|V$$

is split exact for some neighbourhood V of s_0 .

Because $d_i(s) = d_i(s)$ and the sequence (A) is exact part (a) of the theorem follows.

(b) $B(K, \mathcal{L}_0)|V$ splits as the direct sum of im d_1 and a bundle E_V , such that $E_{V,s} \simeq B(K, \mathcal{E}(s))$, for each $s \in V$. We must show that these bundle structures fit together globally.

Suppose therefore that V is open in S' and that

$$\begin{aligned} d_{p} & d_{2} & d_{1} \\ 0 \rightarrow \mathcal{L}_{p} \rightarrow \dots \rightarrow \mathcal{L}_{1} \rightarrow \mathcal{L}_{0} \rightarrow \mathcal{E}_{|V \times V_{2}} \rightarrow 0 \\ d_{p}^{'} & d_{2}^{'} & d_{1}^{'} \\ 0 \rightarrow \mathcal{L}_{p}^{'} \rightarrow \dots \rightarrow \mathcal{L}_{1}^{'} \rightarrow \mathcal{L}_{0}^{'} \rightarrow \mathcal{E}_{|V \times V_{2}} \rightarrow 0 \end{aligned}$$

are free resolutions of ξ over $V \times V_2$.

If V_1 , V_2 are open polycylinders, we can find an $\mathcal{O}_{S\times U}$ -homomorphism $\phi_0: \mathcal{L}_0 \to \mathcal{L}_0'$ such that

$$\mathcal{L}_{0}^{'} \xrightarrow{\varepsilon'} \mathcal{E}_{|V \times V_{2}} \to 0$$

$$\phi_{0} \uparrow \qquad ||$$

$$\mathcal{L}_{0} \xrightarrow{\varepsilon} \mathcal{E}_{|V \times V_{2}} \to 0$$

commutes. ϕ_0 determines a bundle morphism $\overset{\sim}{\phi}_0: B(K, \mathcal{L}_0) \to B(K, \mathcal{L}_0')$. $B(K, \mathcal{L}_0)$ (resp. $B(K, \mathcal{L}_0')$) splits as $(\text{im } \tilde{d}_1) \otimes E_V$ [Resp. $(\text{im } \tilde{d}_1') \otimes E_V'$].

Let p' be the projection morphism: $B(K, \mathcal{L}_0) \to E_V'$ with kernel im d_1' , and put $\phi = p' \circ \phi_0 | E_V$.

The commutative diagram

and the open mapping theorem shows that $\phi(s)$ is an isomorphism of Banach spaces for each $s \in V$, so $\widetilde{\phi}: E_V \to E_V'$ is a bundle isomorphism. We also notice that $\widetilde{\phi}$ depends only on the choice of splittings in $B(K, \mathcal{L}_0)$ and $B(K, \mathcal{L}_0')$, and not on the choice of $\widetilde{\phi}_0$. This ends the proof of the theorem.

Remark: Consider the general situation where X and S are analytic spaces, and $\pi: X \to S$ is a morphism, $\mathscr E$ an $\mathscr O_X$ -module. To study the local dependence of $\mathscr E$ on S, one can imbed an open set X' in X in the open set $U \subset \mathbb C^n$. The morphism $\phi: X' \to U$, $\pi: X' \to S$ determine the imbedding $\pi \times \phi: X' \to S \times U$ such that the diagram commutes. $\mathscr E$ can be extended by zero into a sheaf $\mathscr E'$ over $U \times S$. Obviously this sheaf $\mathscr E'$ is S-flat iff $\mathscr E$ is S-flat.

Therefore theorem 1 makes clear also this general situation.

Corollary: If $\pi: X \to S$ is a morphism and $\mathscr E$ a coherent $\mathscr O_X$ -module. Then $\pi \mid \operatorname{Supp}\,(\mathscr E)$ is an open map.

Proof: Suppose as above that X is imbedded in $S \times U$, and \mathscr{E} in extended by zero to $S \times U$. Let $x_0 \in \text{Supp } \mathscr{E}$, and V be a neighbourhood of x_0 in $S \times U$. Let $s_0 = \pi(x_0)$ and choose an $\mathscr{E}(s_0)$ -privileged polycylinder K in U, such that $\{s_0\} \times K \subset V$, over some neighbourhood W of s_0 . We have the Banach bundle $B(K, \mathscr{E}|\pi^{-1}(W))$, whose fiber over s is $B(K, \mathscr{E}(s))$. Since $x_0 \in \text{Supp } \mathscr{E}(s_0)$ and K is a neighbourhood of x_0 , $B(K; \mathscr{E}(s)) \neq 0$. As all the fibers are isomorphic, then for all $s \in U$, $B(K; \mathscr{E}(s)) \neq 0$ and therefore $\{s\} \times K \cap \text{Supp } \mathscr{E} \neq 0$, and $s \in \pi$ (Supp \mathscr{E}). This proves that π is open.

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