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$g^{-1}(g(z))$, $z \in T_0$, is mapped injectively into Y by $\hat{f}$, hence $\dim(g^{-1}(g(z)) \leqslant \dim Y$. Thus we obtain the inequalities

$$(*) \quad \dim T_0 \leqslant \dim A + \dim Y \leqslant \dim X - 1 .$$

Now we shall see that $\dim T_0 = \dim X - 1$. Therefore we have equality in (*), hence $\dim X - \dim Y = \dim A + 1$. We obtain also $\dim S_0 = \dim S = \dim A$, hence $S_0 = S = A$, since A is irreducible; moreover, $\dim(g^{-1}(a)) = \dim Y$ for every $a \in A$, consequently $\bar{f}(a) = \hat{f}(g^{-1}(a)) = Y$.

In order to show that $\dim T_0 = \dim X - 1$, we use the following theorem due to Grauert and Remmert [5] (a proof was also given by Kerner [7]):

Let X be a complex manifold, Z a normal complex space, K an analytic set in Z with codim $K \geqslant 2$, $\tau : Z \to X$ a holomorphic map such that $\tau \mid Z - K$ is locally biholomorphic. Then τ is locally biholomorphic.

Now assume first that $G_{\bar{f}}$ is a normal complex subspace of $X \times Y$. The holomorphic map $\check{f}: G_{\bar{f}} \to X$ is locally biholomorphic in a point $\zeta \in G_{\bar{f}}$ if and only if $\zeta \in T = \check{f}^{-1}(S)$. Hence, by the theorem of Grauert and Remmert, T is puredimensional and $\dim T = \dim X - 1$. If $G_{\bar{f}}$ is not normal, we take a normalization $(\tilde{G}, \nu)$ of $G_{\bar{f}}$ and look at $\check{f} \circ \nu : \tilde{G} \to X$ and $\tilde{T} = (\check{f} \circ \nu)^{-1}(S)$ instead of $\bar{f}$ and T. We see then that $\tilde{T}$ is puredimensional with $\dim \tilde{T} = \dim X - 1$, but then it follows that $\nu(\tilde{T}) = T$ has the same properties.

Remark. If Y is not compact, then $\bar{f}$ is always a holomorphic map under the hypothesis of Theorem 3 since $\bar{f}(a)$ is compact for $a \in A$. If the assumption that X be a complex manifold is dropped, then both assertions of Theorem 3 become false as can be shown by examples.

5. Maximal meromorphic mappings

All complex spaces in this section are irreducible. Before we state the problem we give the necessary definitions.

Let $f : X \underset{k}{\to} Y$ be weakly holomorphic and not empty. The *rank* rk f of f is by definition the global rank of the holomorphic mapping $\hat{f} : G_f \to Y$, i.e.,

$$\operatorname{rk} f = \sup_{z \in G_f} \operatorname{codim}_z \hat{f}^{-1}(\hat{f}(z)).$$

For two meromorphic mappings $f: X \underset{m}{\rightarrow} Y$ and $f_0: X \underset{m}{\rightarrow} Y_0$ we always have $\mathrm{rk}\,[f, f_0] \geqslant \max\{\mathrm{rk} f, \mathrm{rk} f_0\}$. We say that f_0 *depends on* f, if $\mathrm{rk} f = rk\,[f, f_0]$. If f_0 depends on f and f depends on f_0, we say that f_0 is *related* to f. Then clearly $\mathrm{rk} f = \mathrm{rk} f_0$.

Let $f: X \underset{m}{\rightarrow} Y$ and $f_0: X \underset{m}{\rightarrow} Y_0$ be given. Suppose that there exists a meromorphic mapping $\alpha: Y \underset{m}{\rightarrow} Y_0$ such that the meromorphic product $\alpha \triangle f$ is defined and $f_0 = \alpha \triangle f$. Then we say that f *majorizes* f_0. If this is the case, f_0 depends on f ([15]).

If $f: X \underset{m}{\rightarrow} Y$ is surjective and if f majorizes every meromorphic mapping g dependent on f, f is called meromorphically maximal or $m-$*maximal.*

Let us now consider the following problem:

Given $f_0: X \underset{m}{\rightarrow} Y_0$, is it possible to find a meromorphic mapping $f_s: X \underset{m}{\rightarrow} Y_s$ such that f_s is related to f_0 and f_s is m-maximal? If possible, the pair (f_s, Y_s) is called a meromorphic base or an $m-$*base* with respect to f_0.

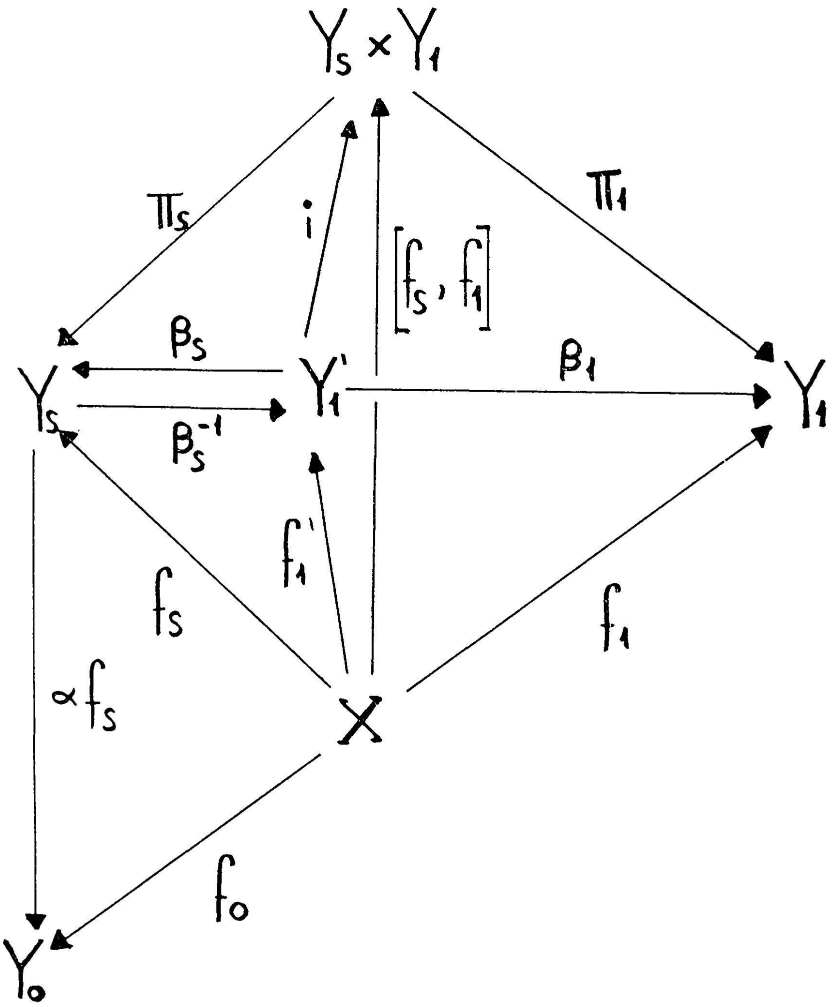
Proposition 14. If $f_0: X \underset{m}{\rightarrow} Y_0$ is proper, then an m-base with respect to f_0 exists.

We give a sketch of the proof (compare [15]).

Since f_0 is proper, $f_0(X) = Y_0'$ is an irreducible $\mathrm{rk} f_0$ — dimensional analytic set in Y_0; there is a surjective meromorphic mapping $f_0': X \underset{m}{\rightarrow} Y_0'$ such that $f_0 = I_{Y_0}^{Y_0'} \circ f_0' \left(I_{Y_0}^{Y_0'} \text{ is the inclusion map } Y_0' \rightarrow Y_0 \right)$. f_0' is proper by Proposition 10, moreover it is surjective and related to f_0. Now, a complex m-base with respect to f_0' is also a complex m-base with respect to f_0. Therefore we can suppose that f_0 is surjective.

We consider the class $\mathfrak{F}$ of those surjective meromorphic mappings of X which are dependent on f_0 and majorize f_0. If $(f: X \underset{m}{\rightarrow} Y) \in \mathfrak{F}$, there exists a unique surjective meromorphic mapping $\alpha_f: Y \underset{m}{\rightarrow} Y_0$ such that $f_0 = \alpha_f \triangle f$. This implies that f is related to f_0 and, by Proposition 10, that f and α_f are proper. We have $\mathrm{rk} f = \dim Y$, $\mathrm{rk}\,\alpha_f = \dim Y_0 = \mathrm{rk} f_0$, $\mathrm{rk} f = \mathrm{rk} f_0$, hence $\dim Y = \dim Y_0 = \mathrm{rk}\,\alpha_f$. Thus (Y, α_f, Y_0) is a " meromorphic covering " of Y_0 with a well defined number $n(f)$ of sheets. The $n(f)$, $f \in \mathfrak{F}$, have a finite upper bound: If not, one can show that there exists a point $y_0 \in Y_0$ such that $f_0^{-1}(y_0)$ has infinitely many connected components, but this is impossible since f_0 is proper.

Let $(f_s : X \underset{m}{\to} Y_s) \in \mathfrak{F}$ be such that $n(f_s)$ is maximal. We claim that (f_s, Y_s) is an m-base with respect to f_0. Suppose that $f_1 : X \underset{m}{\to} Y_1$ depends on f_s, we have to show that f_s majorizes f_1. The meromorphic junction $[f_s, f_1] : X \underset{m}{\to} Y_s \times Y_1$ is proper (Proposition 10) and $\mathrm{rk}\,[f_s, f_1] = \mathrm{rk}\, f_s = \mathrm{rk}\, f_0$, therefore $[f_s, f_1](X) = Y_1'$ is a $\mathrm{rk}\, f_0$ – dimensional analytic subset of $Y_s \times Y_1$. There is a meromorphic mapping $f_1' : X \underset{m}{\to} Y_1'$ such that $[f_s, f_1] = \mathrm{i} \circ \mathrm{f}_1'$ where $i: Y_1' \to Y_s \times Y_1$; f_1' is surjective, proper and related



to f_0. Let π_s and π_1 be the projections from $Y_s \times Y_1$ onto Y_s and Y_1, set $\beta_s = \pi_s \circ i$, $\beta_1 = \pi_1 \circ i$, respectively. We have $f_s = \beta_s \circ f_1'$, hence f_1' majorizes f_s. The holomorphic mapping $\pi_s \circ i = \beta_s$ is surjective and, by Proposition 10, proper. The meromorphic product $\alpha_{f_s} \triangle \beta_s$ is defined since β_s is surjective; we have $f_0 = (\alpha_{f_s} \triangle \beta_s) \triangle f_1'$, hence f_1' majorizes f_0 and, consequently, $f_1' \in \mathfrak{F}$. Then $n(f_1') \geqq n(f_s)$ since f_1' majorizes f_s, thus $n(f_1') = n(f_s)$ since $n(f_s)$ is maximal. It follows that the number of sheets of the covering (Y_1', β_s, Y_s) equals 1, and this implies that β_s is a bimeromorphic mapping. Now $f_1 = \beta_1 \circ f_1' = \beta_1 \circ (\beta_s^{-1} \triangle f_s) = (\beta_1 \circ \beta_s^{-1}) \triangle f_s$. Hence f_s majorizes f_1.

We give, without proof (see [15]) a more general result in this direction.

Theorem 4. Let $f_0 : X \underset{m}{\rightarrow} Y_0$ be a meromorphic mapping and A an irreducible analytic set in X such that the holomorphic correspondence

$$a_0 = f_0 \mid A : A \underset{k}{\rightarrow} Y_0$$

has at least one irreducible component $a_0' : A \underset{k}{\rightarrow} Y_0$ which is proper and satisfies $\operatorname{rk} a_0' = \operatorname{rk} f_0$. Then there exists $f_s : X \underset{m}{\rightarrow} Y_s$ such that (f_s, Y_s) is an m-base with respect to f_0.

By definition, for $f : X \underset{m}{\rightarrow} Y$ a point $x_0 \in X$ is *a point of indeterminacy of degree k*, if $\dim f(x_0) = k$, and a point of indeterminacy of *maximal degree*, if $\dim f(x_0) = \operatorname{rk} f$.

Let now the set A in Theorem 4 consist of one point x_0. Then $a_0 = f_0 \mid \{ x_0 \} : \{ x_0 \} \underset{k}{\rightarrow} Y_0$ is a proper holomorphic correspondence and $\operatorname{rk} f_0 \mid \{ x_0 \} = \operatorname{rk} a_0 = \dim f(x_0) \leqslant \operatorname{rk} f_0$. The hypothesis of the theorem means, in this case, that $\dim f_0(x_0) = \operatorname{rk} f_0$; this implies ([15]) that $f_0(x_0) = f_0(x)$. We obtain the following *specialization* of Theorem 4:

Let $f_0 : X \underset{m}{\rightarrow} Y_0$ be a meromorphic mapping with a point of indeterminacy of maximal degree. Then there exists an m-base with respect to f_0.

Finally we give applications of Proposition 14 and Theorem 4. We consider *meromorphic functions* defined on the complex space X. These are meromorphic mappings $\varphi : X \underset{m}{\rightarrow} \mathbf{P}_1$ such that $\varphi(X)$ does not reduce to the point ∞ of $\mathbf{P}_1$. The set of all meromorphic functions on X form a field $\mathfrak{M}(X)$. Let $\varphi_1, \dots \varphi_k$ be elements of $\mathfrak{M}(X)$. We say that $\varphi_1, \dots \varphi_k$ is a *system of independent meromorphic functions* if for the meromorphic mapping $\Phi = [\varphi_1, \dots, \varphi_k] : X \underset{m}{\rightarrow} \mathbf{P}_1 \times \dots \times \mathbf{P}_1 = \mathbf{P}_1^k$ we have $\operatorname{rk} \Phi = k$. There are always maximal systems of independent meromorphic functions on X; the length k of such a system is uniquely determined with $k \leq \dim X$.

Let now X be a *compact* complex space. As a first application we obtain the theorem of *Chow-Thimm* [4], [20] (see also [10]):

The field $\mathfrak{M}(X)$ of meromorphic functions on an irreducible compact complex space X is isomorphic to a finite algebraic extension of a field of rational functions.

Proof. Choose a maximal system $\varphi_1, \dots, \varphi_k$ of independent meromorphic functions on X and let Φ be defined as above. Φ is proper since X

is compact, thus we can apply Proposition 14. Hence there exists an m-base (Φ_s, Y_s) with respect to Φ and there is a meromorphic mapping $\alpha_s \colon Y_s \underset{m}{\to} \mathbf{P}_1^k$ such that $\Phi = \alpha_s \triangle \Phi_s$. If $\varphi \in \mathfrak{M}(X)$, we have rk Φ = rk $[\Phi, \varphi]$ since the system $\varphi_1, \ldots, \varphi_k$ is maximal, therefore φ depends on Φ. So Φ_s majorizes every meromorphic function φ on X, i.e., there is a meromorphic function $\alpha_\varphi : Y_s \underset{m}{\to} \mathbf{P}_1$ such that $\varphi = \alpha_\varphi \triangle \Phi_s$. It is easily seen that the assignment $\varphi \mapsto \alpha_\varphi$ gives an isomorphism from $\mathfrak{M}(X)$ onto $\mathfrak{M}(Y_s)$. Now $(Y_s, \alpha_s, \mathbf{P}_1^k)$ is a meromorphic covering of $\mathbf{P}_1^k$; if n is its number of sheets, then every meromorphic function α on Y_s satisfies an equation

$$\alpha^n + (b_1 \triangle \alpha_s) \,.\, \alpha^{n-1} + \ldots + (b_n \triangle \alpha_s) = 0 \,,$$

where $b_\nu \in \mathfrak{M}(\mathbf{P}_1^k)$ $(\nu = 1, \ldots, n)$. This implies that $\mathfrak{M}(Y_s)$ is isomorphic to a finite algebraic extension of $\mathfrak{M}(\mathbf{P}_1^k)$. But $\mathfrak{M}(\mathbf{P}_1^k)$ is isomorphic to the field $\mathbf{C}(z_1, \ldots z_k)$ of he rational functions of k complex variables. Hence we obtain an isomorphism of $\mathfrak{M}(X)$ with the desired properties.

$$\begin{tikzcd}
& Y_s \arrow[dr, "\alpha_s"] \arrow[dd, "\alpha_\varphi"] & \\
X \arrow[ur, "\Phi_s"] \arrow[rr, "\Phi"] \arrow[dr, "\varphi"'] & & \mathbf{P}_1^k \\
& \mathbf{P}_1 &
\end{tikzcd}$$

As another application we sketch a proof of the following statement: Let $\Phi : X \underset{m}{\to} Y$ be a meromorphic mapping with a point of indeterminacy x_0 of maximal degree. Then the field $\mathfrak{M}_\Phi(X)$ of meromorphic functions on X depending on Φ is isomorphic to a finite algebraic extension of a field of rational functions.

By the special case of Theorem 4 there exists an m-base (Φ_s, Y_s) with respect to Φ. The meromorphic mapping $\Phi_s : X \underset{m}{\to} Y_s$ majorizes every $\varphi \in \mathfrak{M}_\Phi(X)$; if $\varphi = \alpha_\varphi \triangle \Phi_s$, then the assignment $\varphi \mapsto \alpha_\varphi$ gives again an isomorphism $\mathfrak{M}_\Phi(X) \cong \mathfrak{M}(Y_s)$. The point x_0 is also a point of indeterminacy of maximal degree for Φ_s since Φ_s depends on Φ_0 (see [15]), hence $\Phi_s(x_0) = \Phi_s(X) = Y_s$ is compact. Now we can apply the theorem of Chow-Thimm, and we obtain the assertion.

Remark. In the case where $Y = \mathbf{P}_1^k$ and Φ is the junction of k meromorphic functions on X, the statement is a known theorem of Thimm [18], [19]. A proof of this theorem was also given by Remmert [12].