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$g^{-1}(g(z)), z \in T_0$ , is mapped injectively into  $Y$  by  $\hat{f}$ , hence  $\dim(g^{-1}(g(z))) \leq \dim Y$ . Thus we obtain the inequalities

$$(*) \quad \dim T_0 \leq \dim A + \dim Y \leq \dim X - 1.$$

Now we shall see that  $\dim T_0 = \dim X - 1$ . Therefore we have equality in (\*), hence  $\dim X - \dim Y = \dim A + 1$ . We obtain also  $\dim S_0 = \dim S = \dim A$ , hence  $S_0 = S = A$ , since  $A$  is irreducible; moreover,  $\dim(g^{-1}(a)) = \dim Y$  for every  $a \in A$ , consequently  $\bar{f}(a) = \hat{f}(g^{-1}(a)) = Y$ .

In order to show that  $\dim T_0 = \dim X - 1$ , we use the following theorem due to Grauert and Remmert [5] (a proof was also given by Kerner [7]):

Let  $X$  be a complex manifold,  $Z$  a normal complex space,  $K$  an analytic set in  $Z$  with  $\text{codim } K \geq 2$ ,  $\tau : Z \rightarrow X$  a holomorphic map such that  $\tau|_{Z-K}$  is locally biholomorphic. Then  $\tau$  is locally biholomorphic.

Now assume first that  $G_{\bar{f}}$  is a normal complex subspace of  $X \times Y$ . The holomorphic map  $\check{f} : G_{\bar{f}} \rightarrow X$  is locally biholomorphic in a point  $\zeta \in G_{\bar{f}}$  if and only if  $\zeta \in T = \check{f}^{-1}(S)$ . Hence, by the theorem of Grauert and Remmert,  $T$  is pure-dimensional and  $\dim T = \dim X - 1$ . If  $G_{\bar{f}}$  is not normal, we take a normalization  $(\tilde{G}, \nu)$  of  $G_{\bar{f}}$  and look at  $\check{f} \circ \nu : \tilde{G} \rightarrow X$  and  $\tilde{T} = (\check{f} \circ \nu)^{-1}(S)$  instead of  $\bar{f}$  and  $T$ . We see then that  $\tilde{T}$  is pure-dimensional with  $\dim \tilde{T} = \dim X - 1$ , but then it follows that  $\nu(\tilde{T}) = T$  has the same properties.

*Remark.* If  $Y$  is not compact, then  $\bar{f}$  is always a holomorphic map under the hypothesis of Theorem 3 since  $\bar{f}(a)$  is compact for  $a \in A$ . If the assumption that  $X$  be a complex manifold is dropped, then both assertions of Theorem 3 become false as can be shown by examples.

## 5. MAXIMAL MEROMORPHIC MAPPINGS

All complex spaces in this section are irreducible. Before we state the problem we give the necessary definitions.

Let  $f : X \xrightarrow{k} Y$  be weakly holomorphic and not empty. The *rank*  $\text{rk } f$  of  $f$  is by definition the global rank of the holomorphic mapping  $\hat{f} : G_f \rightarrow Y$ , i.e.,  $\text{rk } f = \sup_{z \in G_f} \text{codim}_z \hat{f}^{-1}(\hat{f}(z))$ .

For two meromorphic mappings  $f: X \xrightarrow{m} Y$  and  $f_0: X \xrightarrow{m} Y_0$  we always have  $\text{rk } [f, f_0] \geq \max \{ \text{rk } f, \text{rk } f_0 \}$ . We say that  $f_0$  *depends on*  $f$ , if  $\text{rk } f = \text{rk } [f, f_0]$ . If  $f_0$  depends on  $f$  and  $f$  depends on  $f_0$ , we say that  $f_0$  is *related* to  $f$ . Then clearly  $\text{rk } f = \text{rk } f_0$ .

Let  $f: X \xrightarrow{m} Y$  and  $f_0: X \xrightarrow{m} Y_0$  be given. Suppose that there exists a meromorphic mapping  $\alpha: Y \xrightarrow{m} Y_0$  such that the meromorphic product  $\alpha \triangle f$  is defined and  $f_0 = \alpha \triangle f$ . Then we say that  $f$  *majorizes*  $f_0$ . If this is the case,  $f_0$  depends on  $f$  ([15]).

If  $f: X \xrightarrow{m} Y$  is surjective and if  $f$  majorizes every meromorphic mapping  $g$  dependent on  $f$ ,  $f$  is called *meromorphically maximal* or  *$m$ -maximal*.

Let us now consider the following problem:

Given  $f_0: X \xrightarrow{m} Y_0$ , is it possible to find a meromorphic mapping  $f_s: X \xrightarrow{m} Y_s$  such that  $f_s$  is related to  $f_0$  and  $f_s$  is  $m$ -maximal? If possible, the pair  $(f_s, Y_s)$  is called a *meromorphic base* or an  *$m$ -base* with respect to  $f_0$ .

*Proposition 14.* If  $f_0: X \xrightarrow{m} Y_0$  is proper, then an  $m$ -base with respect to  $f_0$  exists.

We give a sketch of the proof (compare [15]).

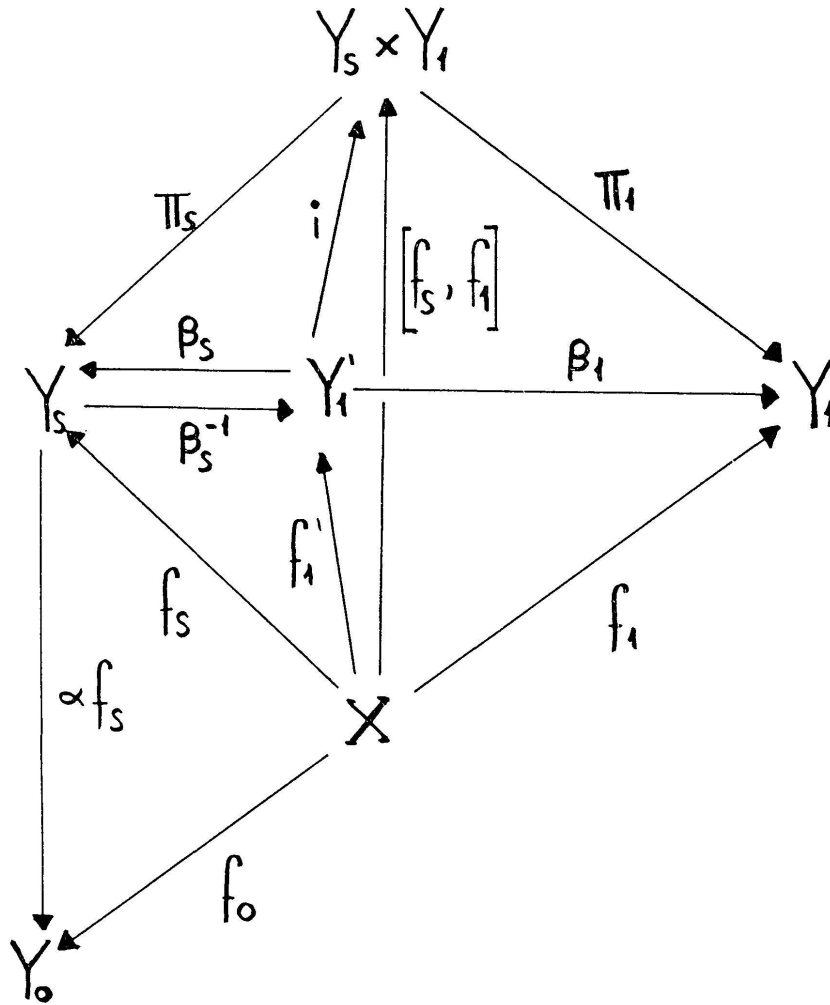
Since  $f_0$  is proper,  $f_0(X) = Y'_0$  is an irreducible  $\text{rk } f_0$  — dimensional analytic set in  $Y_0$ ; there is a surjective meromorphic mapping  $f'_0: X \xrightarrow{m} Y'_0$

such that  $f_0 = I_{Y_0}^{Y'_0} \circ f'_0$   $\left( I_{Y_0}^{Y'_0} \text{ is the inclusion map } Y'_0 \rightarrow Y_0 \right)$ .  $f'_0$  is proper by Proposition 10, moreover it is surjective and related to  $f_0$ . Now, a complex  $m$ -base with respect to  $f'_0$  is also a complex  $m$ -base with respect to  $f_0$ . Therefore we can suppose that  $f_0$  is surjective.

We consider the class  $\mathfrak{F}$  of those surjective meromorphic mappings of  $X$  which are dependent on  $f_0$  and majorize  $f_0$ . If  $(f: X \xrightarrow{m} Y) \in \mathfrak{F}$ , there exists a unique surjective meromorphic mapping  $\alpha_f: Y \xrightarrow{m} Y_0$  such that  $f_0 = \alpha_f \triangle f$ .

This implies that  $f$  is related to  $f_0$  and, by Proposition 10, that  $f$  and  $\alpha_f$  are proper. We have  $\text{rk } f = \dim Y$ ,  $\text{rk } \alpha_f = \dim Y_0 = \text{rk } f_0$ ,  $\text{rk } f = \text{rk } f_0$ , hence  $\dim Y = \dim Y_0 = \text{rk } \alpha_f$ . Thus  $(Y, \alpha_f, Y_0)$  is a “meromorphic covering” of  $Y_0$  with a well defined number  $n(f)$  of sheets. The  $n(f)$ ,  $f \in \mathfrak{F}$ , have a finite upper bound: If not, one can show that there exists a point  $y_0 \in Y_0$  such that  $f_0^{-1}(y_0)$  has infinitely many connected components, but this is impossible since  $f_0$  is proper.

Let  $(f_s : X \rightarrow Y_s) \in \mathfrak{F}$  be such that  $n(f_s)$  is maximal. We claim that  $(f_s, Y_s)$  is an  $m$ -base with respect to  $f_0$ . Suppose that  $f_1 : X \rightarrow Y_1$  depends on  $f_s$ , we have to show that  $f_s$  majorizes  $f_1$ . The meromorphic junction  $[f_s, f_1] : X \rightarrow Y_s \times Y_1$  is proper (Proposition 10) and  $\text{rk } [f_s, f_1] = \text{rk } f_s = \text{rk } f_0$ , therefore  $[f_s, f_1](X) = Y'_1$  is a  $\text{rk } f_0$  - dimensional analytic subset of  $Y_s \times Y_1$ . There is a meromorphic mapping  $f'_1 : X \rightarrow Y'_1$  such that  $[f_s, f_1] = i \circ f'_1$  where  $i : Y'_1 \rightarrow Y_s \times Y_1$ ;  $f'_1$  is surjective, proper and related



to  $f_0$ . Let  $\pi_s$  and  $\pi_1$  be the projections from  $Y_s \times Y_1$  onto  $Y_s$  and  $Y_1$ , set  $\beta_s = \pi_s \circ i$ ,  $\beta_1 = \pi_1 \circ i$ , respectively. We have  $f_s = \beta_s \circ f'_1$ , hence  $f'_1$  majorizes  $f_s$ . The holomorphic mapping  $\pi_s \circ i = \beta_s$  is surjective and, by Proposition 10, proper. The meromorphic product  $\alpha_{f_s} \triangle \beta_s$  is defined since  $\beta_s$  is surjective; we have  $f_0 = (\alpha_{f_s} \triangle \beta_s) \triangle f'_1$ , hence  $f'_1$  majorizes  $f_0$  and, consequently,  $f'_1 \in \mathfrak{F}$ . Then  $n(f'_1) \geq n(f_s)$  since  $f'_1$  majorizes  $f_s$ , thus  $n(f'_1) = n(f_s)$  since  $n(f_s)$  is maximal. It follows that the number of sheets of the covering  $(Y'_1, \beta_s, Y_s)$  equals 1, and this implies that  $\beta_s$  is a bimeromorphic mapping. Now  $f_1 = \beta_1 \circ f'_1 = \beta_1 \circ (\beta_s^{-1} \triangle f_s) = (\beta_1 \circ \beta_s^{-1}) \triangle f_s$ . Hence  $f_s$  majorizes  $f_1$ .

We give, without proof (see [15]) a more general result in this direction.

*Theorem 4.* Let  $f_0 : X \xrightarrow{m} Y_0$  be a meromorphic mapping and  $A$  an irreducible analytic set in  $X$  such that the holomorphic correspondence

$$a_0 = f_0 \mid A : A \xrightarrow{k} Y_0$$

has at least one irreducible component  $a'_0 : A \xrightarrow{k} Y_0$  which is proper and satisfies  $\text{rk } a'_0 = \text{rk } f_0$ . Then there exists  $f_s : X \xrightarrow{m} Y_s$  such that  $(f_s, Y_s)$  is an  $m$ -base with respect to  $f_0$ .

By definition, for  $f : X \xrightarrow{m} Y$  a point  $x_0 \in X$  is a point of indeterminacy of degree  $k$ , if  $\dim f(x_0) = k$ , and a point of indeterminacy of maximal degree, if  $\dim f(x_0) = \text{rk } f$ .

Let now the set  $A$  in Theorem 4 consist of one point  $x_0$ . Then  $a_0 = f_0 \mid \{x_0\} : \{x_0\} \xrightarrow{k} Y_0$  is a proper holomorphic correspondence and  $\text{rk } f_0 \mid \{x_0\} = \text{rk } a_0 = \dim f(x_0) \leq \text{rk } f_0$ . The hypothesis of the theorem means, in this case, that  $\dim f_0(x_0) = \text{rk } f_0$ ; this implies ([15]) that  $f_0(x_0) = f_0(x)$ . We obtain the following *specialization* of Theorem 4:

Let  $f_0 : X \xrightarrow{m} Y_0$  be a meromorphic mapping with a point of indeterminacy of maximal degree. Then there exists an  $m$ -base with respect to  $f_0$ .

Finally we give applications of Proposition 14 and Theorem 4. We consider meromorphic functions defined on the complex space  $X$ . These are meromorphic mappings  $\varphi : X \xrightarrow{m} \mathbf{P}_1$  such that  $\varphi(X)$  does not reduce to the point  $\infty$  of  $\mathbf{P}_1$ . The set of all meromorphic functions on  $X$  form a field  $\mathfrak{M}(X)$ . Let  $\varphi_1, \dots, \varphi_k$  be elements of  $\mathfrak{M}(X)$ . We say that  $\varphi_1, \dots, \varphi_k$  is a system of independent meromorphic functions if for the meromorphic mapping  $\Phi = [\varphi_1, \dots, \varphi_k] : X \xrightarrow{m} \mathbf{P}_1 \times \dots \times \mathbf{P}_1 = \mathbf{P}_1^k$  we have  $\text{rk } \Phi = k$ . There are always maximal systems of independent meromorphic functions on  $X$ ; the length  $k$  of such a system is uniquely determined with  $k \leq \dim X$ .

Let now  $X$  be a compact complex space. As a first application we obtain the theorem of Chow-Thimm [4], [20] (see also [10]):

The field  $\mathfrak{M}(X)$  of meromorphic functions on an irreducible compact complex space  $X$  is isomorphic to a finite algebraic extension of a field of rational functions.

*Proof.* Choose a maximal system  $\varphi_1, \dots, \varphi_k$  of independent meromorphic functions on  $X$  and let  $\Phi$  be defined as above.  $\Phi$  is proper since  $X$

is compact, thus we can apply Proposition 14. Hence there exists an  $m$ -base  $(\Phi_s, Y_s)$  with respect to  $\Phi$  and there is a meromorphic mapping  $\alpha_s: Y_s \rightarrow \mathbf{P}_1^k$  such that  $\Phi = \alpha_s \triangle \Phi_s$ . If  $\varphi \in \mathfrak{M}(X)$ , we have  $\text{rk } \Phi = \text{rk } [\Phi, \varphi]$  since the

system  $\varphi_1, \dots, \varphi_k$  is maximal, therefore  $\varphi$  depends on  $\Phi$ . So  $\Phi_s$  majorizes every meromorphic function  $\varphi$  on  $X$ , i.e., there is a meromorphic function  $\alpha_\varphi: Y_s \rightarrow \mathbf{P}_1^k$  such

that  $\varphi = \alpha_\varphi \triangle \Phi_s$ . It is easily seen that the assignment  $\varphi \mapsto \alpha_\varphi$  gives an isomorphism from  $\mathfrak{M}(X)$  onto  $\mathfrak{M}(Y_s)$ . Now  $(Y_s, \alpha_s, \mathbf{P}_1^k)$  is a meromorphic covering of  $\mathbf{P}_1^k$ ; if  $n$  is its number of sheets, then every meromorphic function  $\alpha$  on  $Y_s$  satisfies an equation

$$\alpha^n + (b_1 \triangle \alpha_s) \cdot \alpha^{n-1} + \dots + (b_n \triangle \alpha_s) = 0,$$

where  $b_v \in \mathfrak{M}(\mathbf{P}_1^k)$  ( $v = 1, \dots, n$ ). This implies that  $\mathfrak{M}(Y_s)$  is isomorphic to a finite algebraic extension of  $\mathfrak{M}(\mathbf{P}_1^k)$ . But  $\mathfrak{M}(\mathbf{P}_1^k)$  is isomorphic to the field  $\mathbf{C}(z_1, \dots, z_k)$  of

the rational functions of  $k$  complex variables. Hence we obtain an isomorphism of  $\mathfrak{M}(X)$  with the desired properties.

As another application we sketch a proof of the following statement: Let  $\Phi: X \rightarrow_m Y$  be a meromorphic mapping with a point of indeterminacy  $x_0$  of maximal degree. Then the field  $\mathfrak{M}_\Phi(X)$  of meromorphic functions on  $X$  depending on  $\Phi$  is isomorphic to a finite algebraic extension of a field of rational functions.

By the special case of Theorem 4 there exists an  $m$ -base  $(\Phi_s, Y_s)$  with respect to  $\Phi$ . The meromorphic mapping  $\Phi_s: X \rightarrow_m Y_s$  majorizes every  $\varphi \in \mathfrak{M}_\Phi(X)$ ; if  $\varphi = \alpha_\varphi \triangle \Phi_s$ , then the assignment  $\varphi \mapsto \alpha_\varphi$  gives again an isomorphism  $\mathfrak{M}_\Phi(X) \cong \mathfrak{M}(Y_s)$ . The point  $x_0$  is also a point of indeterminacy of maximal degree for  $\Phi_s$  since  $\Phi_s$  depends on  $\Phi_0$  (see [15]), hence  $\Phi_s(x_0) = \Phi_s(X) = Y_s$  is compact. Now we can apply the theorem of Chow-Thimm, and we obtain the assertion.

*Remark.* In the case where  $Y = \mathbf{P}_1^k$  and  $\Phi$  is the junction of  $k$  meromorphic functions on  $X$ , the statement is a known theorem of Thimm [18], [19]. A proof of this theorem was also given by Remmert [12].

