Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 14 (1968)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: MEROMORPHIC MAPPINGS

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**Kapitel:** 4. Extension of meromorphic mappings

**DOI:** https://doi.org/10.5169/seals-42342

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Proposition 10. Let  $f: X_{\stackrel{\rightarrow}{m}}Y$ ,  $f_1': X_{\stackrel{\rightarrow}{m}}Y_1$ ,  $g: Y_{\stackrel{\rightarrow}{m}}Z$  be meromorphic mappings, assume that  $g \triangle f$  exists. Then we have:

- 1) If f is proper,  $[f, f'_1]$  is proper,
- 2) If f and g are proper,  $g \triangle f$  is proper,
- 3) If  $g \triangle f$  is proper, f is proper,
- 4) If  $g \triangle f$  is proper and f surjective, g is proper.

## 4. EXTENSION OF MEROMORPHIC MAPPINGS

We start with some classical results. Let D be a domain in  $\mathbb{C}^n$  and  $A \neq D$  an irreducible analytic set in D. Let  $\varphi : D - A \to \mathbb{C}$  be a holomorphic mapping and  $f : D - A \to \mathbb{P}_1$  a meromorphic mapping. Then we have (see [2], [8], [14] and the references given there):

- 1) If codim A > 1, then  $\varphi$  and f have extensions over A.
- 2) Assume codim A = 1. Then
- a)  $\varphi$  has an extension over A if for some  $z_0 \in A$  there is a neighborhood U of  $z_0$  such that  $\varphi$  is bounded in  $U-(A \cap U)$ ,
- b) f has an extension over A if for some  $z_0 \in A$  f has an extension into a neighborhood of  $z_0$ .

We shall see that these statements can be generalized in some respects.<sup>2</sup> Throughout this section, X and Y are irreducible complex spaces,  $A \neq X$  is an irreducible analytic set in X,  $f: X - A \to Y$  a meromorphic mapping. We shall study conditions under which f has an extension over A, which means that there exists a meromorphic mapping  $g: X \to Y$  such that  $g \mid X - A = f$ .

The meromorphic mapping f can always be extended topologically to a correspondence  $\bar{f}: X_{\to} Y$  by setting  $G_{\bar{f}} = \overline{G_f}$  where the closure is with respect to  $X \times Y$ . On the other hand, if  $f: X_{\to} Y$  is an extension of f, then

<sup>1)</sup> The generalization 2a) of Riemann's classical theorem on removable singularities is due to Kistler and Hartogs. 2b) is due to Hartogs and E. E. Levi. 1) follows easily from 2); the statement 1) for holomorphic functions  $\varphi$  is sometimes called "the second Riemann theorem on removable singularities" (2. Riemannscher Hebbarkeitssatz).

<sup>2)</sup> The extension problem for holomorphic maps is also treated in [1] and [6].

 $\tilde{f} = \bar{f}$ . We are thus led to study the properties of  $\bar{f}$ . Of essential use is the following extension theorem for analytic sets.

Theorem 1. Let Z be a complex space and M an irreducible analytic set in Z. Let further N be a pure dimensional (all irreducible components have the same dimension) analytic set in Z-M such that dim  $N=\dim M$ . Then the closure  $\overline{N}$  of N with respect to Z is an analytic set in Z if it is analytic in at least one point of M.

This theorem was proved by Thullen [21] in the case where Z is a domain in  $\mathbb{C}^n$  and where dim  $M = \dim N = n-1$ . In [13] the theorem is stated without restriction on the dimension of M but likewise for a domain Z in  $\mathbb{C}^n$  (the special case treated by Thullen is used here in the proof). From this one can obtain the theorem in the form above by using imbeddings of open sets of Z into domains of number space.

Corollary 1. If dim  $N > \dim M$ , then  $\overline{N}$  is analytic in Z.

This can be deduced from Theorem 1 by imbedding arguments in an obvious manner. A direct proof is contained in [8].

Corollary 2. Let Z and M be as in the theorem and  $\{N_i\}$  a set of mutually different irreducible analytic sets in Z-M for which dim  $N_i \geqslant$  dim M, and  $\bigcup N_i$  is analytic in Z-M. If every neighborhood of a point  $z_0 \in M$  intersects an infinite number of sets  $N_i$ , then every point of M has this property.

This is a simple consequence of Theorem 1 and Corollary 1.

Proposition 11. Let D be a domain in  $\mathbb{C}^n$ , M an irreducible analytic set in D, N a pure dimensional analytic set in D-M such that dim  $N=\dim M$ . Suppose there exists an analytic plane  $E_0$  through a point  $z_0 \in M$  such that the following conditions hold:

- 1)  $E_0$  is in general position with respect to M, i.e., dim  $(E_0 \cap M) = \dim E_0 + \dim M \dim D$ ,
- 2) There exists a neighborhood U of  $z_0$  such that for every analytic plane E with dim  $E = \dim E_0$  which is parallel to  $E_0$  and which intersects  $U, \overline{N} \cap E$  is analytic in D ( $\overline{N}$  is the closure of N with respect to D).

Then  $\overline{N}$  is analytic in  $z_0$  and hence in D by Theorem 1.

As to the proof we refer to [13], p. 301.1

<sup>1)</sup> The statement actually proved in [13] is a little more special than Proposition 11, but by suitable supplementary arguments one can obtain the proposition in the form above.

We turn now to the study of two problems:

- 1) When is  $\bar{f}$  weakly holomorphic?
- 2) When is  $\bar{f}$  continuous?

If  $\bar{f}$  is weakly holomorphic, then  $\bar{f}$  is irreducible, because the irreducibility of  $G_f$  implies that of  $G_{\bar{f}}$ . Hence  $\bar{f}$  is a meromorphic mapping if it is weakly holomorphic and continuous.

Moreover, if  $\bar{f}$  is weakly holomorphic, then the closure  $\overline{f^{-1}(y)}$  of  $f^{-1}(y)$  with respect to X is analytic in X for every  $y \in Y$ :  $f^{-1}(y)$  is analytic in X - A and  $\bar{f}^{-1}(y)$  is analytic in X; since  $\overline{f^{-1}(y)} \subset \bar{f}^{-1}(y)$  and  $\overline{f^{-1}(y)} \cap (X - A) = \bar{f}^{-1}(y) \cap (X - A) = f^{-1}(y)$ , it follows that  $\overline{f^{-1}(y)}$  is analytic in X.

We assume now, in the rest of this section, that dim X-dim  $Y \geqslant \dim A$ . We set  $Z = X \times Y$ ,  $M = A \times Y$ ,  $N = G_f$ . Then dim  $M = \dim A + \dim Y$ , dim  $N = \dim G_f = \dim X$  and, by our assumption, dim  $N \geqslant \dim M$ . If dim X-dim Y>dim A, i.e., if dim N>dim M, Corollary 1 of Theorem 1 implies that  $\overline{f}$  is weakly holomorphic. Furthermore, we have

Proposition 12. Assume dim X-dim Y == dim A. Then the correspondence  $\bar{f}$  is weakly holomorphic if there exists a non-empty open set  $V \subset Y$  such that the closure  $\bar{f}^{-1}(v)$  of  $f^{-1}(v)$  with respect to X is analytic in X for all  $v \in V$ .

The condition dim X-dim Y = dim A implies that dim N = dim M. Hence, by Theorem 1,  $\overline{N} = G_{\overline{I}}$  is analytic in  $Z = X \times Y$ , i.e.,  $\overline{f}$  is weakly holomorphic, if there is a point of  $M = A \times Y$  in which  $\overline{N}$  is analytic. We show that this is the case for points of  $A \times V$ . Choose a point  $(a_0, v_0) \in A \times V$  such that A is irreducible in  $a_0$  and such that  $v_0$  is an ordinary point of Y. There are open neighborhoods  $U_1 \subset X$  of  $a_0$  and  $U_2 \subset V$  of  $v_0$  with the following properties:  $A' = A \cap U_1$  is an irreducible analytic set in  $U_1$ ;  $U_1$  can be mapped biholomorphically onto an analytic set X' in a domain  $D_1$  of a number space  $\mathbb{C}^{n_1}$ ;  $U_2$  can be mapped biholomorphically onto a domain  $D_2$  of a number space  $\mathbb{C}^{n_2}$   $(n_2 = \dim Y)$ . It is enough to show that the closure  $\overline{N'}$  of  $N'=G_f\cap (U_1\times U_2)$  with respect to  $U_1\times U_2$ is analytic in  $U_1 \times U_2$ . Set  $D = D_1 \times D_2$ ,  $M' = A' \times D_2$  and, for  $w \in D_2$ ,  $E_w = \mathbb{C}^{n_1} \times \{ w \}$ . Then we have  $\dim (E_w \cap M') = \dim (A' \times \{ w \}) =$  $\dim A' = \dim A$ , on the other hand  $\dim E_w + \dim M' - \dim D = n_1 +$  $(\dim A' + n_2) - (n_1 + n_2) = \dim A$ . The hypothesis on the analyticity of  $\overline{f^{-1}(v)}$  for all  $v \in V$  implies that  $\overline{N'} \cap E_w$  is analytic in D for every  $w \in D_2$ . Hence, by Proposition 11,  $\overline{N}'$  is analytic in D; then  $\overline{N}'$  is, in particular, analytic in  $X' \times D_2 = U_1 \times U_2$ .

Concerning the continuity of  $\bar{f}$  we have

Proposition 13. The correspondence  $\bar{f}$  is continuous if it is continuous at one point  $a_0 \in A$ .

Proof. We assume first that the topology of Y has a countable base. Then  $\overline{f}$  is continuous at  $a \in A$  if and only if the following condition holds: If  $(x_v)$  and  $(y_v)$ , v = 1, 2, ..., are sequences of points such that  $x_v \in X - A$ ,  $x_v \to a$ ,  $y_v \in f(x_v)$ , then the sequence  $(y_v)$  has a point of accumulation in Y. Suppose that  $\overline{f}$  is continuous at a point  $a_0 \in A$  and let  $(x_v)$ ,  $(y_v)$  be sequences as above. Then the fibres  $f^{-1}(y_v)$  are non-empty analytic sets in X - A, and the condition dim X-dim  $Y \geqslant \dim A$  implies dim  $F_v^{(\mu)} \geqslant \dim A$  for every irreducible component  $F_v^{(\mu)}$  of  $f^{-1}(y_v)$ . Suppose that  $L = \bigcup f^{-1}(y_v)$  is not analytic in X - A. Then there exists a subsequence  $(y_{v_i})$  such that one can find points  $x_i' \in f^{-1}(y_{v_i})$  which converge to a point  $x_0' \in X - A$ . By continuity at  $x_0'$  it follows that  $(y_{v_i})$  has a point of accumulation on  $f(x_0')$ . Let now L be analytic in X - A. Assume first:

( $\alpha$ ) There are infinitely many fibres  $f^{-1}(y_{v_i})$  which have a common irreducible component N.

In this case we take a point of N and use similarly the continuity of f at this point. Suppose now that  $(\alpha)$  is not satisfied. Then we apply Corollary 2 of Theorem 1 to the set of irreducible components  $F_{\nu}^{(\mu)}$  of the fibres  $f^{-1}(y_{\nu})$ . Since every neighborhood of a intersects infinitely many components  $F_{\nu}^{(\mu)}$  (this implies, in particular, that the closure  $\overline{L}$  of L with respect to X is not analytic in a), the same holds with respect to  $a_0$ . The  $y_{\nu}$  have then a point of accumulation on  $\overline{f}(a_0)$  because  $\overline{f}$  is continuous at  $a_0$ .

Now we drop the assumption that Y has countable topology. We remark first: To show that  $\bar{f}$  is continuous at  $a \in A$  we may replace X by any irreducible open subspace which contains the points a and  $a_0$ . Therefore we may assume that X has countable topology. Secondly: All points of Y used in the proof above belong to the topological subspace  $f(X-A) \cup \bar{f}(a_0) \subset Y$  which has countable topology since X has. If we now restrict Y to an irreducible open subspace with countable topology containing  $f(X-A) \cup \bar{f}(a_0)$ , the proof given above applies.

Corollary. If dim X-dim Y>dim A, then  $\overline{f}$  is always continuous.

In this case the hypothesis on the continuity of  $\bar{f}$  at a point  $a_0 \in A$  is not needed in the proof of Proposition 13: We have now dim  $F_v^{(\mu)} > \dim A$ . If L is analytic in X-A, Corollary 1 of Theorem 1 implies that  $\bar{L}$  is analytic in every point of A, and the condition  $(\alpha)$  is necessarily satisfied.

Combining the preceding statements we have the following result.

Theorem 2. Let  $f: X - A \to Y$  be a meromorphic mapping and dim  $X - \dim Y \gg \dim A$ . Then  $\bar{f}$  is a meromorphic mapping if and only if

- 1) there exists a non-empty open set  $V \subset Y$  such that  $\overline{f^{-1}(v)}$  is analytic in X for all  $v \in V$ , and
  - 2)  $\bar{f}$  is continuous at a point  $a_0 \in A$ .

If dim X-dim Y>dim A, then  $\overline{f}$  is always a meromorphic mapping.

Corollary. Assume there is an open subset  $U \subset X$  and a compact set  $K \subset Y$  different from Y such that  $U \cap A \neq \emptyset$  and  $f(U - (U \cap A)) \subset K$ . Then  $\overline{f}$  is a meromorphic mapping.

To conclude this from Theorem 2 we remark first that the set V = Y - K satisfies the above condition 1): If  $v \in V$ , then  $f^{-1}(v)$  does not intersect U, hence  $\overline{f^{-1}(v)}$  is analytic in every point of  $U \cap A$  and therefore, by Theorem 1, analytic in X. On the other hand,  $\overline{f}$  is continuous at every point  $a_0 \in U \cap A$ . For  $\overline{f}(a_0)$  is compact since it is a closed subset of K. Moreover, let  $V_0$  be a neighborhood of  $\overline{f}(a_0)$ ; we assert that there is a neighborhood  $U_0$  of  $u_0$  such that  $u_0 \in V_0$ . If this were false, then there would exist points  $u_0 \in U \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0$  such that  $u_0 \in V \cap A$  arbitrarily near  $u_0 \in V \cap A$  arbit

As to the extension of holomorphic maps we state:

Theorem 3. Let X be, in addition to the earlier assumptions, a complex manifold and  $f: X - A \rightarrow Y$  a holomorphic map. Then

- 1) If dim X-dim Y>dim A+1,  $\bar{f}$  is a holomorphic map,
- 2) If dim  $Y = \dim A + 1$ , then  $\overline{f}$  is either a holomorphic map or  $\overline{f}$  is a meromorphic mapping and  $\overline{f}(a) = Y$  for all  $a \in A$ .

*Proof.* Assume dim X-dim  $Y \geqslant$  dim A+1. Then, by Theorem 2,  $\overline{f}$  is a meromorphic mapping; if  $S = S(f) = \varnothing$ ,  $\overline{f}$  is even a holomorphic map. Suppose  $S \neq \varnothing$ , set  $T = \widecheck{f}^{-1}(S)$  and let  $T_0$  be an irreducible component of T. Set  $S_0 = \widecheck{f}(T_0)$ . By Remmert's mapping theorem  $S_0$  is an irreducible analytic set in X. We have

$$\dim T_0 = \dim S_0 + \inf_{z \in D_0} \dim_z (g^{-1}(g(z))) \text{ where } g = \widecheck{f} \mid T_0$$
 ,

furthermore dim  $S_0 \leqslant \dim S \leqslant \dim A$  because  $S \subset S_0 \subset A$ . Every fibre

 $g^{-1}(g(z)), z \in T_0$ , is mapped injectively into Y by  $\hat{f}$ , hence dim  $(g^{-1}(g(z)) \le dim Y$ . Thus we obtain the inequalities

(\*) dim 
$$T_0 \leq \dim A + \dim Y \leq \dim X - 1$$
.

Now we shall see that dim  $T_0 = \dim X - 1$ . Therefore we have equality in (\*), hence dim  $X - \dim Y = \dim A + 1$ . We obtain also dim  $S_0 = \dim S = \dim A$ , hence  $S_0 = S = A$ , since A is irreducible; moreover, dim  $(g^{-1}(a)) = \dim Y$  for every  $a \in A$ , consequently  $\bar{f}(a) = \hat{f}(g^{-1}(a)) = Y$ .

In order to show that dim  $T_0 = \dim X - 1$ , we use the following theorem due to Grauert and Remmert [5] (a proof was also given by Kerner [7]):

Let X be a complex manifold, Z a normal complex space, K an analytic set in Z with codim  $K \geqslant 2$ ,  $\tau : Z \rightarrow X$  a holomorphic map such that  $\tau \mid Z - K$  is locally biholomorphic. Then  $\tau$  is locally biholomorphic.

Now assume first that  $G_{\overline{f}}$  is a normal complex subspace of  $X \times Y$ . The holomorphic map  $\check{f} \colon G_{\overline{f}} \to X$  is locally biholomorphic in a point  $\zeta \in G_{\overline{f}}$  if and only if  $\zeta \in T = \check{f}^{-1}(S)$ . Hence, by the theorem of Grauert and Remmert, T is puredimensional and dim  $T = \dim X - 1$ . If  $G_{\overline{f}}$  is not normal, we take a normalization  $(\check{G}, v)$  of  $G_{\overline{f}}$  and look at  $\check{f} \circ v : \check{G} \to X$  and  $\check{T} = (\check{f} \circ v)^{-1}(S)$  instead of  $\bar{f}$  and T. We see then that T is puredimensional with dim  $T = \dim X - 1$ , but then it follows that V(T) = T has the same properties.

*Remark*. If Y is not compact, then  $\bar{f}$  is always a holomorphic map under the hypothesis of Theorem 3 since  $\bar{f}(a)$  is compact for  $a \in A$ . If the assumption that X be a complex manifold is dropped, then both assertions of Theorem 3 become false as can be shown by examples.

# 5. MAXIMAL MEROMORPHIC MAPPINGS

All complex spaces in this section are irreducible. Before we state the problem we give the necessary definitions.

Let  $f: X_{\xrightarrow{k}} Y$  be weakly holomorphic and not empty. The rank rk f of f is by definition the global rank of the holomorphic mapping  $\hat{f}: G_f \to Y$ , i.e., rk  $f = \sup_{z \in G_f} \operatorname{codim}_z \hat{f}^{-1} (\hat{f}(z))$ .