

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 14 (1968)  
**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** MEROMORPHIC MAPPINGS  
**Autor:** Stein, K.  
**Kapitel:** 4. Extension of meromorphic mappings  
**DOI:** <https://doi.org/10.5169/seals-42342>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

*Proposition 10.* Let  $f: X \xrightarrow{m} Y$ ,  $f'_1: X \xrightarrow{m} Y_1$ ,  $g: Y \xrightarrow{m} Z$  be meromorphic mappings, assume that  $g \triangle f$  exists. Then we have:

- 1) If  $f$  is proper,  $[f, f'_1]$  is proper,
- 2) If  $f$  and  $g$  are proper,  $g \triangle f$  is proper,
- 3) If  $g \triangle f$  is proper,  $f$  is proper,
- 4) If  $g \triangle f$  is proper and  $f$  surjective,  $g$  is proper.

#### 4. EXTENSION OF MEROMORPHIC MAPPINGS

We start with some classical results. Let  $D$  be a domain in  $\mathbb{C}^n$  and  $A \neq D$  an irreducible analytic set in  $D$ . Let  $\varphi: D - A \rightarrow \mathbb{C}$  be a holomorphic mapping and  $f: D - A \xrightarrow{m} \mathbb{P}_1$  a meromorphic mapping. Then we have (see [2], [8], [14] and the references given there):

- 1) If  $\text{codim } A > 1$ , then  $\varphi$  and  $f$  have extensions over  $A$ .
- 2) Assume  $\text{codim } A = 1$ . Then
  - a)  $\varphi$  has an extension over  $A$  if for some  $z_0 \in A$  there is a neighborhood  $U$  of  $z_0$  such that  $\varphi$  is bounded in  $U - (A \cap U)$ ,
  - b)  $f$  has an extension over  $A$  if for some  $z_0 \in A$   $f$  has an extension into a neighborhood of  $z_0$ .<sup>1</sup>

We shall see that these statements can be generalized in some respects.<sup>2</sup>

Throughout this section,  $X$  and  $Y$  are irreducible complex spaces,  $A \neq X$  is an irreducible analytic set in  $X$ ,  $f: X - A \xrightarrow{m} Y$  a meromorphic mapping. We shall study conditions under which  $f$  has an extension over  $A$ , which means that there exists a meromorphic mapping  $g: X \xrightarrow{m} Y$  such that  $g|_{X-A} = f$ .

The meromorphic mapping  $f$  can always be extended topologically to a correspondence  $\bar{f}: X \xrightarrow{k} Y$  by setting  $G_{\bar{f}} = \overline{G_f}$  where the closure is with respect to  $X \times Y$ . On the other hand, if  $\tilde{f}: X \xrightarrow{m} Y$  is an extension of  $f$ , then

<sup>1</sup>) The generalization 2a) of Riemann's classical theorem on removable singularities is due to Kistler and Hartogs. 2b) is due to Hartogs and E. E. Levi. 1) follows easily from 2); the statement 1) for holomorphic functions  $\varphi$  is sometimes called "the second Riemann theorem on removable singularities" (2. Riemannscher Hebbarkeitssatz).

<sup>2</sup>) The extension problem for holomorphic maps is also treated in [1] and [6].

$\tilde{f} = \bar{f}$ . We are thus led to study the properties of  $\bar{f}$ . Of essential use is the following extension theorem for analytic sets.

*Theorem 1.* Let  $Z$  be a complex space and  $M$  an irreducible analytic set in  $Z$ . Let further  $N$  be a pure dimensional (all irreducible components have the same dimension) analytic set in  $Z - M$  such that  $\dim N = \dim M$ . Then the closure  $\bar{N}$  of  $N$  with respect to  $Z$  is an analytic set in  $Z$  if it is analytic in at least one point of  $M$ .

This theorem was proved by Thullen [21] in the case where  $Z$  is a domain in  $\mathbb{C}^n$  and where  $\dim M = \dim N = n - 1$ . In [13] the theorem is stated without restriction on the dimension of  $M$  but likewise for a domain  $Z$  in  $\mathbb{C}^n$  (the special case treated by Thullen is used here in the proof). From this one can obtain the theorem in the form above by using imbeddings of open sets of  $Z$  into domains of number space.

*Corollary 1.* If  $\dim N > \dim M$ , then  $\bar{N}$  is analytic in  $Z$ .

This can be deduced from Theorem 1 by imbedding arguments in an obvious manner. A direct proof is contained in [8].

*Corollary 2.* Let  $Z$  and  $M$  be as in the theorem and  $\{N_i\}$  a set of mutually different irreducible analytic sets in  $Z - M$  for which  $\dim N_i \geq \dim M$ , and  $\cup N_i$  is analytic in  $Z - M$ . If every neighborhood of a point  $z_0 \in M$  intersects an infinite number of sets  $N_i$ , then every point of  $M$  has this property.

This is a simple consequence of Theorem 1 and Corollary 1.

*Proposition 11.* Let  $D$  be a domain in  $\mathbb{C}^n$ ,  $M$  an irreducible analytic set in  $D$ ,  $N$  a pure dimensional analytic set in  $D - M$  such that  $\dim N = \dim M$ . Suppose there exists an analytic plane  $E_0$  through a point  $z_0 \in M$  such that the following conditions hold:

- 1)  $E_0$  is in general position with respect to  $M$ , i.e.,  $\dim(E_0 \cap M) = \dim E_0 + \dim M - \dim D$ ,
- 2) There exists a neighborhood  $U$  of  $z_0$  such that for every analytic plane  $E$  with  $\dim E = \dim E_0$  which is parallel to  $E_0$  and which intersects  $U$ ,  $\bar{N} \cap E$  is analytic in  $D$  ( $\bar{N}$  is the closure of  $N$  with respect to  $D$ ).

Then  $\bar{N}$  is analytic in  $z_0$  and hence in  $D$  by Theorem 1.

As to the proof we refer to [13], p. 301.<sup>1</sup>

<sup>1</sup> The statement actually proved in [13] is a little more special than Proposition 11, but by suitable supplementary arguments one can obtain the proposition in the form above.

We turn now to the study of two problems:

- 1) When is  $\bar{f}$  weakly holomorphic?
- 2) When is  $\bar{f}$  continuous?

If  $\bar{f}$  is weakly holomorphic, then  $\bar{f}$  is irreducible, because the irreducibility of  $G_f$  implies that of  $G_{\bar{f}}$ . Hence  $\bar{f}$  is a meromorphic mapping if it is weakly holomorphic and continuous.

Moreover, if  $\bar{f}$  is weakly holomorphic, then the closure  $\overline{f^{-1}(y)}$  of  $f^{-1}(y)$  with respect to  $X$  is analytic in  $X$  for every  $y \in Y$ :  $f^{-1}(y)$  is analytic in  $X - A$  and  $\bar{f}^{-1}(y)$  is analytic in  $X$ ; since  $\overline{f^{-1}(y)} \subset \bar{f}^{-1}(y)$  and  $\overline{f^{-1}(y)} \cap (X - A) = \bar{f}^{-1}(y) \cap (X - A) = f^{-1}(y)$ , it follows that  $\overline{f^{-1}(y)}$  is analytic in  $X$ .

We assume now, in the rest of this section, that  $\dim X - \dim Y \geq \dim A$ . We set  $Z = X \times Y$ ,  $M = A \times Y$ ,  $N = G_f$ . Then  $\dim M = \dim A + \dim Y$ ,  $\dim N = \dim G_f = \dim X$  and, by our assumption,  $\dim N \geq \dim M$ . If  $\dim X - \dim Y > \dim A$ , i.e., if  $\dim N > \dim M$ , Corollary 1 of Theorem 1 implies that  $\bar{f}$  is weakly holomorphic. Furthermore, we have

*Proposition 12.* Assume  $\dim X - \dim Y = \dim A$ . Then the correspondence  $\bar{f}$  is weakly holomorphic if there exists a non-empty open set  $V \subset Y$  such that the closure  $\overline{f^{-1}(v)}$  of  $f^{-1}(v)$  with respect to  $X$  is analytic in  $X$  for all  $v \in V$ .

*Proof.* The condition  $\dim X - \dim Y = \dim A$  implies that  $\dim N = \dim M$ . Hence, by Theorem 1,  $\bar{N} = G_{\bar{f}}$  is analytic in  $Z = X \times Y$ , i.e.,  $\bar{f}$  is weakly holomorphic, if there is a point of  $M = A \times Y$  in which  $\bar{N}$  is analytic. We show that this is the case for points of  $A \times V$ . Choose a point  $(a_0, v_0) \in A \times V$  such that  $A$  is irreducible in  $a_0$  and such that  $v_0$  is an ordinary point of  $Y$ . There are open neighborhoods  $U_1 \subset X$  of  $a_0$  and  $U_2 \subset V$  of  $v_0$  with the following properties:  $A' = A \cap U_1$  is an irreducible analytic set in  $U_1$ ;  $U_1$  can be mapped biholomorphically onto an analytic set  $X'$  in a domain  $D_1$  of a number space  $\mathbb{C}^{n_1}$ ;  $U_2$  can be mapped biholomorphically onto a domain  $D_2$  of a number space  $\mathbb{C}^{n_2}$  ( $n_2 = \dim Y$ ). It is enough to show that the closure  $\bar{N}'$  of  $N' = G_f \cap (U_1 \times U_2)$  with respect to  $U_1 \times U_2$  is analytic in  $U_1 \times U_2$ . Set  $D = D_1 \times D_2$ ,  $M' = A' \times D_2$  and, for  $w \in D_2$ ,  $E_w = \mathbb{C}^{n_1} \times \{w\}$ . Then we have  $\dim(E_w \cap M') = \dim(A' \times \{w\}) = \dim A' = \dim A$ , on the other hand  $\dim E_w + \dim M' - \dim D = n_1 + (\dim A' + n_2) - (n_1 + n_2) = \dim A$ . The hypothesis on the analyticity of  $\overline{f^{-1}(v)}$  for all  $v \in V$  implies that  $\bar{N}' \cap E_w$  is analytic in  $D$  for every  $w \in D_2$ . Hence, by Proposition 11,  $\bar{N}'$  is analytic in  $D$ ; then  $\bar{N}'$  is, in particular, analytic in  $X' \times D_2 = U_1 \times U_2$ .

Concerning the continuity of  $\bar{f}$  we have

*Proposition 13.* The correspondence  $\bar{f}$  is continuous if it is continuous at one point  $a_0 \in A$ .

*Proof.* We assume first that the topology of  $Y$  has a countable base. Then  $\bar{f}$  is continuous at  $a \in A$  if and only if the following condition holds: If  $(x_v)$  and  $(y_v)$ ,  $v = 1, 2, \dots$ , are sequences of points such that  $x_v \in X - A$ ,  $x_v \rightarrow a$ ,  $y_v \in f(x_v)$ , then the sequence  $(y_v)$  has a point of accumulation in  $Y$ . Suppose that  $\bar{f}$  is continuous at a point  $a_0 \in A$  and let  $(x_v)$ ,  $(y_v)$  be sequences as above. Then the fibres  $f^{-1}(y_v)$  are non-empty analytic sets in  $X - A$ , and the condition  $\dim X - \dim Y \geq \dim A$  implies  $\dim F_v^{(\mu)} \geq \dim A$  for every irreducible component  $F_v^{(\mu)}$  of  $f^{-1}(y_v)$ . Suppose that  $L = \cup f^{-1}(y_v)$  is not analytic in  $X - A$ . Then there exists a subsequence  $(y_{v_i})$  such that one can find points  $x'_i \in f^{-1}(y_{v_i})$  which converge to a point  $x'_0 \in X - A$ . By continuity at  $x'_0$  it follows that  $(y_{v_i})$  has a point of accumulation on  $f(x'_0)$ . Let now  $L$  be analytic in  $X - A$ . Assume first:

( $\alpha$ ) There are infinitely many fibres  $f^{-1}(y_{v_i})$  which have a common irreducible component  $N$ .

In this case we take a point of  $N$  and use similarly the continuity of  $\bar{f}$  at this point. Suppose now that ( $\alpha$ ) is not satisfied. Then we apply Corollary 2 of Theorem 1 to the set of irreducible components  $F_v^{(\mu)}$  of the fibres  $f^{-1}(y_v)$ . Since every neighborhood of  $a$  intersects infinitely many components  $F_v^{(\mu)}$  (this implies, in particular, that the closure  $\bar{L}$  of  $L$  with respect to  $X$  is not analytic in  $a$ ), the same holds with respect to  $a_0$ . The  $y_v$  have then a point of accumulation on  $\bar{f}(a_0)$  because  $\bar{f}$  is continuous at  $a_0$ .

Now we drop the assumption that  $Y$  has countable topology. We remark first: To show that  $\bar{f}$  is continuous at  $a \in A$  we may replace  $X$  by any irreducible open subspace which contains the points  $a$  and  $a_0$ . Therefore we may assume that  $X$  has countable topology. Secondly: All points of  $Y$  used in the proof above belong to the topological subspace  $f(X - A) \cup \bar{f}(a_0) \subset Y$  which has countable topology since  $X$  has. If we now restrict  $Y$  to an irreducible open subspace with countable topology containing  $f(X - A) \cup \bar{f}(a_0)$ , the proof given above applies.

*Corollary.* If  $\dim X - \dim Y > \dim A$ , then  $\bar{f}$  is always continuous.

In this case the hypothesis on the continuity of  $\bar{f}$  at a point  $a_0 \in A$  is not needed in the proof of Proposition 13: We have now  $\dim F_v^{(\mu)} > \dim A$ . If  $L$  is analytic in  $X - A$ , Corollary 1 of Theorem 1 implies that  $\bar{L}$  is analytic in every point of  $A$ , and the condition ( $\alpha$ ) is necessarily satisfied.

Combining the preceding statements we have the following result.

*Theorem 2.* Let  $f: X \dashrightarrow Y$  be a meromorphic mapping and  $\dim X - \dim Y \geq \dim A$ . Then  $\bar{f}$  is a meromorphic mapping if and only if

- 1) there exists a non-empty open set  $V \subset Y$  such that  $\overline{f^{-1}(v)}$  is analytic in  $X$  for all  $v \in V$ , and
- 2)  $\bar{f}$  is continuous at a point  $a_0 \in A$ .

If  $\dim X - \dim Y > \dim A$ , then  $\bar{f}$  is always a meromorphic mapping.

*Corollary.* Assume there is an open subset  $U \subset X$  and a compact set  $K \subset Y$  different from  $Y$  such that  $U \cap A \neq \emptyset$  and  $f(U - (U \cap A)) \subset K$ . Then  $\bar{f}$  is a meromorphic mapping.

To conclude this from Theorem 2 we remark first that the set  $V = Y - K$  satisfies the above condition 1): If  $v \in V$ , then  $f^{-1}(v)$  does not intersect  $U$ , hence  $\overline{f^{-1}(v)}$  is analytic in every point of  $U \cap A$  and therefore, by Theorem 1, analytic in  $X$ . On the other hand,  $\bar{f}$  is continuous at every point  $a_0 \in U \cap A$ . For  $\bar{f}(a_0)$  is compact since it is a closed subset of  $K$ . Moreover, let  $V_0$  be a neighborhood of  $\bar{f}(a_0)$ ; we assert that there is a neighborhood  $U_0$  of  $a_0$  such that  $\bar{f}(U_0) \subset V_0$ . If this were false, then there would exist points  $x$  in  $U - (U \cap A)$  arbitrarily near  $a_0$  such that  $f(x) \cap (K - (K \cap V_0)) \neq \emptyset$ . But then it follows that  $f(a_0) \cap (K - (K \cap V_0)) \neq \emptyset$ , which is a contradiction.

As to the extension of holomorphic maps we state:

*Theorem 3.* Let  $X$  be, in addition to the earlier assumptions, a complex manifold and  $f: X \dashrightarrow Y$  a holomorphic map. Then

- 1) If  $\dim X - \dim Y > \dim A + 1$ ,  $\bar{f}$  is a holomorphic map,
- 2) If  $\dim X - \dim Y = \dim A + 1$ , then  $\bar{f}$  is either a holomorphic map or  $\bar{f}$  is a meromorphic mapping and  $\bar{f}(a) = Y$  for all  $a \in A$ .

*Proof.* Assume  $\dim X - \dim Y \geq \dim A + 1$ . Then, by Theorem 2,  $\bar{f}$  is a meromorphic mapping; if  $S = S(f) = \emptyset$ ,  $\bar{f}$  is even a holomorphic map. Suppose  $S \neq \emptyset$ , set  $T = \bigvee f^{-1}(S)$  and let  $T_0$  be an irreducible component of  $T$ . Set  $S_0 = \bigvee \bar{f}(T_0)$ . By Remmert's mapping theorem  $S_0$  is an irreducible analytic set in  $Y$ . We have

$$\dim T_0 = \dim S_0 + \inf_{z \in D_0} \dim_z (g^{-1}(g(z))) \text{ where } g = \bar{f}|_{T_0},$$

furthermore  $\dim S_0 \leq \dim S \leq \dim A$  because  $S \subset S_0 \subset A$ . Every fibre

$g^{-1}(g(z)), z \in T_0$ , is mapped injectively into  $Y$  by  $\hat{f}$ , hence  $\dim(g^{-1}(g(z))) \leq \dim Y$ . Thus we obtain the inequalities

$$(*) \quad \dim T_0 \leq \dim A + \dim Y \leq \dim X - 1.$$

Now we shall see that  $\dim T_0 = \dim X - 1$ . Therefore we have equality in (\*), hence  $\dim X - \dim Y = \dim A + 1$ . We obtain also  $\dim S_0 = \dim S = \dim A$ , hence  $S_0 = S = A$ , since  $A$  is irreducible; moreover,  $\dim(g^{-1}(a)) = \dim Y$  for every  $a \in A$ , consequently  $\bar{f}(a) = \hat{f}(g^{-1}(a)) = Y$ .

In order to show that  $\dim T_0 = \dim X - 1$ , we use the following theorem due to Grauert and Remmert [5] (a proof was also given by Kerner [7]):

Let  $X$  be a complex manifold,  $Z$  a normal complex space,  $K$  an analytic set in  $Z$  with  $\text{codim } K \geq 2$ ,  $\tau : Z \rightarrow X$  a holomorphic map such that  $\tau|_{Z-K}$  is locally biholomorphic. Then  $\tau$  is locally biholomorphic.

Now assume first that  $G_{\bar{f}}$  is a normal complex subspace of  $X \times Y$ . The holomorphic map  $\check{f} : G_{\bar{f}} \rightarrow X$  is locally biholomorphic in a point  $\zeta \in G_{\bar{f}}$  if and only if  $\zeta \in T = \check{f}^{-1}(S)$ . Hence, by the theorem of Grauert and Remmert,  $T$  is pure-dimensional and  $\dim T = \dim X - 1$ . If  $G_{\bar{f}}$  is not normal, we take a normalization  $(\tilde{G}, \nu)$  of  $G_{\bar{f}}$  and look at  $\tilde{f} \circ \nu : \tilde{G} \rightarrow X$  and  $\tilde{T} = (\tilde{f} \circ \nu)^{-1}(S)$  instead of  $\bar{f}$  and  $T$ . We see then that  $\tilde{T}$  is pure-dimensional with  $\dim \tilde{T} = \dim X - 1$ , but then it follows that  $\nu(\tilde{T}) = T$  has the same properties.

*Remark.* If  $Y$  is not compact, then  $\bar{f}$  is always a holomorphic map under the hypothesis of Theorem 3 since  $\bar{f}(a)$  is compact for  $a \in A$ . If the assumption that  $X$  be a complex manifold is dropped, then both assertions of Theorem 3 become false as can be shown by examples.

## 5. MAXIMAL MEROMORPHIC MAPPINGS

All complex spaces in this section are irreducible. Before we state the problem we give the necessary definitions.

Let  $f : X \xrightarrow{k} Y$  be weakly holomorphic and not empty. The *rank*  $\text{rk } f$  of  $f$  is by definition the global rank of the holomorphic mapping  $\hat{f} : G_f \rightarrow Y$ , i.e.,  $\text{rk } f = \sup_{z \in G_f} \text{codim}_z \hat{f}^{-1}(\hat{f}(z))$ .