**Zeitschrift:** L'Enseignement Mathématique

**Herausgeber:** Commission Internationale de l'Enseignement Mathématique

**Band:** 14 (1968)

**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** MEROMORPHIC MAPPINGS

**Autor:** Stein, K.

**Kapitel:** 3. Meromorphic mappings

**DOI:** https://doi.org/10.5169/seals-42342

is analytic in $Y$. If $f$ is holomorphic and $A' \subset Y$ analytic in $Y$, then, since $\hat{f}^{-1}(A')$ is analytic in $G_f$ and $\check{f}$ is proper, $f^{-1}(A') = \check{f}(\hat{f}^{-1}(A'))$ is analytic in $X$ by Remmert's mapping theorem [11] (see also [8], p. 129).

The correspondences $f \times f_1$, $(f, f_1')$, and $g \circ f$ are holomorphic if the correspondences $f, f_1, f_1'$, and $g$ are holomorphic.

A weakly holomorphic correspondence $f : X \underset{k}{\rightarrow} Y$ is called *reducible* resp. *irreducible* if $G_f$ is reducible resp. irreductible. $G_f$ is always a union of irreducible components $G^{(i)}$; let $f_i : X \underset{k}{\rightarrow} Y$ be the (weakly holomorphic) correspondence whose graph is $G^{(i)}$. Then the correspondences $f_i$ are called the irreducible components of $f$ and we write $f = \cup f_i$.

## 3. Meromorphic mappings

Let $f : X \underset{k}{\rightarrow} Y$ be a correspondence where $X$ is a topological space A point $x \in X$ is called a *distinguished point of f* if there is a neighborhood $U$ of $x$ such that the restriction $f \mid U$ is a mapping (in the usual sense).

*Definition 4.* A holomorphic correspondence $f : X \underset{k}{\rightarrow} Y$ is called a *meromorphic mapping* if the following holds. If $X$ is irreducible, then

1) $f$ is irreducible,

2) There exists a distinguished point $x_0 \in X$ of $f$.

In the general case, if $X = \cup X^{(i)}$ is the decomposition of $X$ into irreducible components, then there exist holomorphic correspondences $f_i : X \underset{k}{\rightarrow} Y$ such that

1) $f_i \mid X^{(i)}$ is a meromorphic mapping and $f_i \mid X - X^{(i)}$ is empty,

2) $f = \cup f_i$.

A meromorphic mapping $f$ is *bimeromorphic* if $f^{-1}$ is meromorphic.

We use the notation $f : X \underset{m}{\rightarrow} Y$ for a meromorphic mapping. Note that a meromorphic mapping is in general not a mapping in the strong sense.

An example of a meromorphic mapping is the correspondence $f$ of $\mathbf{C}^2$ onto the extended complex plane $\mathbf{P}_1$ defined by $f(z_1, z_2) = \dfrac{z_1}{z_2}$ if $(z_1, z_2) \neq (0, 0)$, and $f(0, 0) = \mathbf{P}_1$.

*Definition 5.* A proper holomorphic mapping $\varphi : X' \to X$ is called *a proper modification map* if there exists an open subset $U \subset X$ such that

1) $U \cap X^{(i)} \neq \varnothing$ and $\varphi^{-1}(U) \cap X'^{(j)} \neq \varnothing$ for all irreducible components $X^{(i)} \subset X$ and $X'^{(j)} \subset X'$,

2) $\varphi^{-1} \mid U : U \underset{k}{\to} X'$ is a holomorphic mapping.

It follows that a correspondence $f$ is a meromorphic mapping if an only if $\check{f}$ is a proper modification map.

A proper modification map $\varphi : X' \to X$ is always surjective. The inverse correspondence $\varphi^{-1} : X \underset{k}{\to} X'$ is always a meromorphic mapping.

A normalization $(\tilde{X}, \nu)$ of a complex space $X$ is a normal complex space $\tilde{X}$ ([8], p. 114) and a proper modification map $\nu : \tilde{X} \to X$, such that all fibres $\nu^{-1}(x)$, $x \in X$, are finite. To every complex space $X$ there exists a normalization (see [8]). Let $X_1$ and $X_2$ be complex spaces with normalizations $(\tilde{X}_1, \nu_1)$, $(\tilde{X}_2, \nu_2)$ where $\tilde{X}_1 = \tilde{X}_2$. Then it can easily be shown that $\nu_2 \circ \nu_1^{-1}$: $X_1 \underset{k}{\to} X_2$ is a bimeromorphic mapping.

*Definition 6.* Let $f$ be a meromorphic mapping of $X$. A point $x_0 \in X$ is called *non-singular with respect to $f$* if there exists an open neighborhood $U$ of $x_0$ such that $f \mid U$ is a holomorphic mapping. Otherwise $x_0$ is called *singular*. The set of singular points of $f$ is denoted by $S(f)$.

The meromorphic mapping in the example on *p.* 5 has the origin as a singular point.

*Proposition 8.* Let $f$ be a meromorphic mapping of $X$. Then

1) $S(f)$ is a nowhere dense analytic set in $X$,

2) If $X$ is locally irreducible at $x$, $f(x)$ is connected,

3) If $X$ is normal at $x$, then $x$ is singular if and only if $\dim f(x) > 0$.

For the proof we refer to [15].

The set of singular points is of importance in connection with the compositions of meromorphic mappings. Let $f$: $X \underset{m}{\to} Y$, $f_1$: $X_1 \underset{m}{\to} Y_1$, $f_1'$: $X \underset{m}{\to} Y_1$, $g : Y \underset{m}{\to} Z$ be meromorphic mappings where all the spaces are irreducible.[1] Then the correspondence $f \times f_1$ is easily seen to be meromorphic. The junc-

1) This restriction is introduced here for the sake of simplicity.

tion $(f, f_1')$ need not, however, be a meromorphic mapping. Let $f = f_1'$ be the meromorphic mapping in the example on p. 5. Then the graph $G_{(f,f_1')} \subset \mathbf{C}^2 \times (\mathbf{P}_1 \times \mathbf{P}_1)$ is not irreducible. The product $g \circ f$ too, may be reducible; moreover, it may happen that there is no distinguished point of $g \circ f$.

We can always define a " meromorphic junction " in the following way. There are distinguished points of $(f, f_1')$, for example, all points of $X - (S(f) \cup S(f_1')) \neq \varnothing$. Now it can easily be shown: If a holomorphic correspondence from an irreducible complex space into a complex space has a distinguished point, then the graph of the correspondence has exactly one irreducible component which is the graph of a meromorphic mapping. It follows that there exists a unique meromorphic mapping contained in $(f, f_1')$; this meromorphic mapping is called the *meromorphic junction* of $f$ and $f_1'$ and denoted by $[f, f_1'] : X \underset{m}{\rightarrow} Y \times Y_1$. The meromorphic junction is associative: $[[f_1, f_2], f_3] = [f_1 [f_2, f_3]]$, hence the meromorphic junction $[f_1, \ldots, f_n] : X \underset{m}{\rightarrow} Y_1 \times \ldots \times Y_n$ of $n$ meromorphic mappings $f_\nu : X \underset{m}{\rightarrow} Y_\nu$ is defined in a unique manner.

Furthermore we can define a " meromorphic product " of $f$ and $g$ if there is a distinguished point of $g \circ f$: There is then again a uniquely determined meromorphic mapping contained in $g \circ f$. This is called the *meromorphic product* of $f$ and $g$ and denoted by $g \triangle f : X \underset{m}{\rightarrow} Z$. A sufficient condition for the existence of a distinguished point of $g \circ f$ is that $f(X) \not\subset S(g)$. This condition is, in particular, fulfilled if $f$ is surjective or if $S(g)$ is empty (i.e., if $g$ is a holomorphic map; in this case we have $g \triangle f = g \circ f$). Note that the meromorphic product of bimeromorphic mappings always exists. The associative law $h \triangle (g \triangle f) = (h \triangle g) \triangle f$ holds if both sides exist.

As an example we consider the " meromorphic restriction " which is defined as follows. Let $A$ be an irreducible analytic subset of $X$. Then the correspondence $f \mid A : A \underset{k}{\rightarrow} Y$ need not be irreducible. But if $A \not\subset S(f)$, we can form the meromorphic product $f \triangle I_X^A$ where $I_X^A : A \rightarrow X$ is the inclusion map. We set $f \underset{m}{\mid} A = f \triangle I_X^A : A \underset{m}{\rightarrow} Y$ and call $f \underset{m}{\mid} A$ the *meromorphic restriction* of $f$ to $A$.

*Proposition 9.* Let $f : X \underset{m}{\rightarrow} Y$ and $g : Y \underset{m}{\rightarrow} Z$ be bimeromorphic. Then

1) $f^{-1} \triangle f = I_X$,

2) $g \triangle f$ is bimeromorphic and $(g \triangle f)^{-1} = f^{-1} \triangle g^{-1}$.

*Proposition 10.* Let $f : X \underset{m}{\rightarrow} Y$, $f_1' : X \underset{m}{\rightarrow} Y_1$, $g : Y \underset{m}{\rightarrow} Z$ be meromorphic mappings, assume that $g \triangle f$ exists. Then we have:

1) If $f$ is proper, $[f, f_1']$ is proper,
2) If $f$ and $g$ are proper, $g \triangle f$ is proper,
3) If $g \triangle f$ is proper, $f$ is proper,
4) If $g \triangle f$ is proper and $f$ surjective, $g$ is proper.

## 4. Extension of meromorphic mappings

We start with some classical results. Let $D$ be a domain in $\mathbf{C}^n$ and $A \neq D$ an irreducible analytic set in $D$. Let $\varphi : D - A \rightarrow \mathbf{C}$ be a holomorphic mapping and $f : D - A \underset{m}{\rightarrow} \mathbf{P}_1$ a meromorphic mapping. Then we have (see [2], [8], [14] and the references given there):

1) If codim $A > 1$, then $\varphi$ and $f$ have extensions over $A$.
2) Assume codim $A = 1$. Then

*a)* $\varphi$ has an extension over $A$ if for some $z_0 \in A$ there is a neighborhood $U$ of $z_0$ such that $\varphi$ is bounded in $U - (A \cap U)$,

*b)* $f$ has an extension over $A$ if for some $z_0 \in A$ $f$ has an extension into a neighborhood of $z_0$.[1]

We shall see that these statements can be generalized in some respects.[2]

Throughout this section, $X$ and $Y$ are irreducible complex spaces, $A \neq X$ is an irreducible analytic set in $X$, $f : X - A \underset{m}{\rightarrow} Y$ a meromorphic mapping. We shall study conditions under which $f$ has an extension over $A$, which means that there exists a meromorphic mapping $g : X \underset{m}{\rightarrow} Y$ such that $g \mid X - A = f$.

The meromorphic mapping $f$ can always be extended topologically to a correspondence $\bar{f} : X \underset{k}{\rightarrow} Y$ by setting $G_{\bar{f}} = \overline{G_f}$ where the closure is with respect to $X \times Y$. On the other hand, if $\tilde{f} : X \underset{m}{\rightarrow} Y$ is an extension of $f$, then

1) The generalization 2a) of Riemann's classical theorem on removable singularities is due to Kistler and Hartogs. 2b) is due to Hartogs and E. E. Levi. 1) follows easily from 2); the statement 1) for holomorphic functions $\varphi$ is sometimes called " the second Riemann theorem on removable singularities " (2. Riemannscher Hebbarkeitssatz).

2) The extension problem for holomorphic maps is also treated in [1] and [6].