is analytic in Y. If f is holomorphic and $A' \subset Y$ analytic in Y, then, since $\hat{f}^{-1}(A')$ is analytic in G_f and $\check{f}$ is proper, $f^{-1}(A') = \check{f}(\hat{f}^{-1}(A'))$ is analytic in X by Remmert's mapping theorem [11] (see also [8], p. 129).

The correspondences $f \times f_1$, (f, f_1'), and $g \circ f$ are holomorphic if the correspondences f, f_1, f_1', and g are holomorphic.

A weakly holomorphic correspondence $f: X \underset{k}{\to} Y$ is called *reducible* resp. *irreducible* if G_f is reducible resp. irreducible. G_f is always a union of irreducible components $G^{(i)}$; let $f_i: X \underset{k}{\to} Y$ be the (weakly holomorphic) correspondence whose graph is $G^{(i)}$. Then the correspondences f_i are called the irreducible components of f and we write $f = \cup f_i$.

3. Meromorphic mappings

Let $f: X \underset{k}{\to} Y$ be a correspondence where X is a topological space A point $x \in X$ is called a *distinguished point of f* if there is a neighborhood U of x such that the restriction $f \mid U$ is a mapping (in the usual sense).

Definition 4. A holomorphic correspondence $f: X \underset{k}{\to} Y$ is called a *meromorphic mapping* if the following holds. If X is irreducible, then

1) f is irreducible,

2) There exists a distinguished point $x_0 \in X$ of f.

In the general case, if $X = \cup X^{(i)}$ is the decomposition of X into irreducible components, then there exist holomorphic correspondences $f_i: X \underset{k}{\to} Y$ such that

1) $f_i \mid X^{(i)}$ is a meromorphic mapping and $f_i \mid X - X^{(i)}$ is empty,

2) $f = \cup f_i$.

A meromorphic mapping f is *bimeromorphic* if f^{-1} is meromorphic.

We use the notation $f: X \underset{m}{\to} Y$ for a meromorphic mapping. Note that a meromorphic mapping is in general not a mapping in the strong sense.

An example of a meromorphic mapping is the correspondence f of $\mathbf{C}^2$ onto the extended complex plane $\mathbf{P}_1$ defined by $f(z_1, z_2) = \dfrac{z_1}{z_2}$ if $(z_1, z_2) \neq (0, 0)$, and $f(0, 0) = \mathbf{P}_1$.

Definition 5. A proper holomorphic mapping $\varphi : X' \to X$ is called *a proper modification map* if there exists an open subset $U \subset X$ such that

1) $U \cap X^{(i)} \neq \varnothing$ and $\varphi^{-1}(U) \cap X'^{(j)} \neq \varnothing$ for all irreducible components $X^{(i)} \subset X$ and $X'^{(j)} \subset X'$,

2) $\varphi^{-1} \mid U : U \underset{k}{\to} X'$ is a holomorphic mapping.

It follows that a correspondence f is a meromorphic mapping if an only if $\check{f}$ is a proper modification map.

A proper modification map $\varphi : X' \to X$ is always surjective. The inverse correspondence $\varphi^{-1} : X \underset{k}{\to} X'$ is always a meromorphic mapping.

A normalization $(\tilde{X}, v)$ of a complex space X is a normal complex space $\tilde{X}$ ([8], p. 114) and a proper modification map $v : \tilde{X} \to X$, such that all fibres $v^{-1}(x)$, $x \in X$, are finite. To every complex space X there exists a normalization (see [8]). Let X_1 and X_2 be complex spaces with normalizations $(\tilde{X}_1, v_1)$, $(\tilde{X}_2, v_2)$ where $\tilde{X}_1 = \tilde{X}_2$. Then it can easily be shown that $v_2 \circ v_1^{-1} : X_1 \underset{k}{\to} X_2$ is a bimeromorphic mapping.

Definition 6. Let f be a meromorphic mapping of X. A point $x_0 \in X$ is called *non-singular with respect to f* if there exists an open neighborhood U of x_0 such that $f \mid U$ is a holomorphic mapping. Otherwise x_0 is called *singular*. The set of singular points of f is denoted by $S(f)$.

The meromorphic mapping in the example on *p.* 5 has the origin as a singular point.

Proposition 8. Let f be a meromorphic mapping of X. Then

1) $S(f)$ is a nowhere dense analytic set in X,

2) If X is locally irreducible at x, $f(x)$ is connected,

3) If X is normal at x, then x is singular if and only if $\dim f(x) > 0$.

For the proof we refer to [15].

The set of singular points is of importance in connection with the compositions of meromorphic mappings. Let $f: X \underset{m}{\to} Y, f_1 : X_1 \underset{m}{\to} Y_1, f_1' : X \underset{m}{\to} Y_1$, $g : Y \underset{m}{\to} Z$ be meromorphic mappings where all the spaces are irreducible.[1] Then the correspondence $f \times f_1$ is easily seen to be meromorphic. The junc-

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1) This restriction is introduced here for the sake of simplicity.

tion (f, f_1') need not, however, be a meromorphic mapping. Let $f = f_1'$ be the meromorphic mapping in the example on p. 5. Then the graph $G_{(f, f_1')} \subset \mathbf{C}^2 \times (\mathbf{P}_1 \times \mathbf{P}_1)$ is not irreducible. The product $g \circ f$ too, may be reducible; moreover, it may happen that there is no distinguished point of $g \circ f$.

We can always define a " meromorphic junction " in the following way. There are distinguished points of (f, f_1'), for example, all points of $X - (S(f) \cup S(f_1')) \neq \emptyset$. Now it can easily be shown: If a holomorphic correspondence from an irreducible complex space into a complex space has a distinguished point, then the graph of the correspondence has exactly one irreducible component which is the graph of a meromorphic mapping. It follows that there exists a unique meromorphic mapping contained in (f, f_1'); this meromorphic mapping is called the *meromorphic junction* of f and f_1' and denoted by $[f, f_1'] : X \underset{m}{\to} Y \times Y_1$. The meromorphic junction is associative: $[[f_1, f_2], f_3] = [f_1 [f_2, f_3]]$, hence the meromorphic junction $[f_1, ..., f_n] : X \underset{m}{\to} Y_1 \times ... \times Y_n$ of n meromorphic mappings $f_v : X \underset{m}{\to} Y_v$ is defined in a unique manner.

Furthermore we can define a " meromorphic product " of f and g if there is a distinguished point of $g \circ f$: There is then again a uniquely determined meromorphic mapping contained in $g \circ f$. This is called the *meromorphic product* of f and g and denoted by $g \triangle f : X \underset{m}{\to} Z$. A sufficient condition for the existence of a distinguished point of $g \circ f$ is that $f(X) \not\subset S(g)$. This condition is, in particular, fulfilled if f is surjective or if $S(g)$ is empty (i.e., if g is a holomorphic map; in this case we have $g \triangle f = g \circ f$). Note that the meromorphic product of bimeromorphic mappings always exists. The associative law $h \triangle (g \triangle f) = (h \triangle g) \triangle f$ holds if both sides exist.

As an example we consider the " meromorphic restriction " which is defined as follows. Let A be an irreducible analytic subset of X. Then the correspondence $f \,|\, A : A \underset{k}{\to} Y$ need not be irreducible. But if $A \not\subset S(f)$, we can form the meromorphic product $f \triangle I_X^A$ where $I_X^A : A \to X$ is the inclusion map. We set $f \,|\, A = f \triangle I_X^A : A \underset{m}{\to} Y$ and call $f \,|\, A$ the *meromorphic restriction* of f to A.

Proposition 9. Let $f : X \underset{m}{\to} Y$ and $g : Y \underset{m}{\to} Z$ be bimeromorphic. Then

1) $f^{-1} \triangle f = I_X$,

2) $g \triangle f$ is bimeromorphic and $(g \triangle f)^{-1} = f^{-1} \triangle g^{-1}$.

Proposition 10. Let $f : X \underset{m}{\to} Y$, $f'_1 : X \underset{m}{\to} Y_1$, $g : Y \underset{m}{\to} Z$ be meromorphic mappings, assume that $g \triangle f$ exists. Then we have:

1) If f is proper, $[f, f'_1]$ is proper,

2) If f and g are proper, $g \triangle f$ is proper,

3) If $g \triangle f$ is proper, f is proper,

4) If $g \triangle f$ is proper and f surjective, g is proper.

4. EXTENSION OF MEROMORPHIC MAPPINGS

We start with some classical results. Let D be a domain in $\mathbf{C}^n$ and $A \neq D$ an irreducible analytic set in D. Let $\varphi : D - A \to \mathbf{C}$ be a holomorphic mapping and $f : D - A \underset{m}{\to} \mathbf{P}_1$ a meromorphic mapping. Then we have (see [2], [8], [14] and the references given there):

1) If codim $A > 1$, then φ and f have extensions over A.

2) Assume codim $A = 1$. Then

   a) φ has an extension over A if for some $z_0 \in A$ there is a neighborhood U of z_0 such that φ is bounded in $U - (A \cap U)$,

   b) f has an extension over A if for some $z_0 \in A$ f has an extension into a neighborhood of z_0.[1]

We shall see that these statements can be generalized in some respects.[2]

Throughout this section, X and Y are irreducible complex spaces, $A \neq X$ is an irreducible analytic set in X, $f : X - A \underset{m}{\to} Y$ a meromorphic mapping. We shall study conditions under which f has an extension over A, which means that there exists a meromorphic mapping $g : X \underset{m}{\to} Y$ such that $g \mid X - A = f$.

The meromorphic mapping f can always be extended topologically to a correspondence $\bar{f} : X \underset{k}{\to} Y$ by setting $G_{\bar{f}} = \overline{G_f}$ where the closure is with respect to $X \times Y$. On the other hand, if $\tilde{f} : X \underset{m}{\to} Y$ is an extension of f, then

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1) The generalization 2a) of Riemann's classical theorem on removable singularities is due to Kistler and Hartogs. 2b) is due to Hartogs and E. E. Levi. 1) follows easily from 2); the statement 1) for holomorphic functions φ is sometimes called " the second Riemann theorem on removable singularities " (2. Riemannscher Hebbarkeitssatz)·

2) The extension problem for holomorphic maps is also treated in [1] and [6].