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$$\Gamma\left(U,\mathcal{O}_{U}\right)^{q'} \xrightarrow{\alpha'} \Gamma\left(U,\mathcal{O}_{U}\right)^{p'} \xrightarrow{\beta'} \widetilde{\widetilde{F}} \to 0.$$

As  $\Gamma(U, \mathcal{O}_U)^p$  is free over  $\Gamma(U, \mathcal{O}_U)$ , we can find a  $\Gamma(U, \mathcal{O}_U)$ -linear map  $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$  such that  $\beta = \beta' \circ \gamma$ ; this induces a continuous map

$$\Gamma(U, \mathcal{O}_U)^p/\text{Im }\Gamma(U, \alpha) \to \Gamma(U, \mathcal{O}_U)^{p'}/\text{Im }\Gamma(U, \alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

## 2. General case

If X is an analytic space and F an analytic coherent sheaf on X, we can find a) a locally finite covering of X by open subspaces  $X_i$ , b) for each i, a morphism  $X_i o U_i$ ,  $U_i$  open polycylinder in  $\mathbb{C}^{n_i}$ , which identifies  $X_i$  with a closed subspace of  $U_i$  c) for each i, a coherent sheaf  $F_i$  on  $U_i$  admitting a finite presentation, such that  $F_i$  is the extension of  $F/X_i$ .

On  $\Gamma(X_i, F | X_i)$  we have already defined a topology; further, consider the natural injection

$$\Gamma(X,F) \to \prod_{i} \Gamma(X_{i},F|_{X_{i}})$$

We claim that its image is closed. For,  $(f_i)$  belongs to the image if and only if, for all  $x \in X_i \cap X_j$  ( $= X_i \times_X X_j$ ), we have  $(f_i)_x = (f_j)_x$ ; and the fact that this relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Frechet space on  $\Gamma(X, F)$ . It does not depend on the chosen covering (if one has tavo coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if X' is an open subspace of X, the restriction map  $\Gamma(X, F) \to \Gamma(X', F|_{X'})$  is continuous. If X' is relatively compact in X, then the restriction map is compact (this can be seen by choosing a covering  $X'_j$  of X' of the same type, such that, for any j, there exist i with  $X'_j \subset X_i$ ,  $X'_j$  relatively compact in  $X_i$ , and applying Ascoli's theorem).

# 4.3. Topology on $H^p(X, F)$

We consider a locally finite covering  $\mathcal{U} = \{X_i\}_{i \in I}$  by open subspaces of the preceding type. If we have  $i_0, ..., i_p \in I$ , we consider the natural morphisms

$$X_{i_0\dots i_p} \ = \ X_{i_0} \ \times \ \dots \ \times \ X_{i_p} \ \to \ X_{i_0} \ \times \ \dots \ \times \ X_{i_p} \ \to \ U_{i_0} \ \times \ \dots \ \times \ U_{i_p}$$

which makes  $X_{i_0}$ , ...,  $i_p$  isomorphic with a closed subspace of  $U_{i_0} \times ... \times U_{i_p}$  (the hypothesis that X is separated is essential here! See remark at the end of this paragraph), therefore,  $X_{i_0}$ , ...,  $i_p$  satisfies theorems A and B; more generally, if a finite number of open subspaces of X is Stein, their intersection is also Stein.

Introduce a total order on I. Given an analytic coherent sheaf on X, we can identify the alternating cochains of degree p of the covering  $\mathscr U$  with values in F with the space

$$C^{p}(\mathcal{U},F) = \prod_{i_{0} < i_{1} < \dots < i_{p}} \Gamma(X_{i_{0}\dots i_{p}}, F|_{X_{i_{0}\dots i_{p}}}).$$

This is a Frechet space, and the differential  $d: C^p(\mathcal{U}, F) \to C^{p+1}(\mathcal{U}, F)$  is clearly continuous. Therefore the kernel  $Z^p(\mathcal{U}, F)$  is a closed subspace of  $C^p(\mathcal{U}, F)$ . We denote  $B^p(\mathcal{U}, F)$  the image of  $C^{p-1}(\mathcal{U}, F)$  under d, and we consider on  $H^p(\mathcal{U}, F) = Z^p(\mathcal{U}, F)/B^p(\mathcal{U}, F)$  the quotient topology; according to Leray's theorem, there is a natural isomorphism  $H^p(X, F) \simeq H^p(\mathcal{U}, F)$ .

This gives a topology on  $H^p(X, F)$  of a quotient of a Frechet space. In general, this topology is *not separated*.

We prove now that this topology is independent of the covering  $\mathscr{U}$ ; to do that, it is sufficient to consider a refinement  $\mathscr{U}' = \{X'_j\}_{j \in J}$  of  $\mathscr{U}$  of the same type, a map  $\varphi: J \to I$  such that  $X'_j \subset X_{\varphi(j)}$  for any j to consider the map defined by  $\varphi: C^*(\mathscr{U},F) = \bigoplus_p C^p(\mathscr{U},F) \stackrel{p}{\to} C^*(\mathscr{U}',F)$  and to prove that the induced map  $\bar{\rho}: H^p(\mathscr{U},F) \to H^p(\mathscr{U}',F)$  is an isomorphism.

First,  $\bar{\rho}$  is obviously continuous and bijective; so, according to the closed graph theorem, all that we have to prove is that  $\bar{\rho}$  maps the adherence of 0 onto the adherence of zero; to do that, we consider  $\bar{a}' \in H^p(\mathcal{U}, F)$ , which is adherent to zero; this means that  $\bar{a}'$  is the class modulo  $B^p(\mathcal{U}', F)$  of some  $a' \in Z^p(\mathcal{U}', F)$  which is adherent to  $B^p(\mathcal{U}', F)$ ; therefore, we have

$$a' = \lim_{n \to \infty} db'_n$$
,  $b'_n \in C^{p-1}(\mathcal{U}', F)$ .

Now, the map

$$Z^{p}(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \stackrel{(\rho,d)}{\rightarrow} Z^{p}(\mathcal{U}', F)$$

is surjective hence, according to the closed graph theorem, we can find converging sequences  $a_n \in Z^p(\mathcal{U}, F)$  and  $b''_n \in C^{p-1}(\mathcal{U}', F)$  such that  $d \, b'_n = \rho \, (a_n) + d \, b''_n$ ; but,  $\bar{\rho}$  being an isomorphism, we have  $a_n = d \, \alpha_n$ ,  $\alpha_n \in C^{n-1}(\mathcal{U}, F)$ ; if we put  $b = \lim_{n \to \infty} b_n$ ,  $a = \lim_{n \to \infty} a_n$ , we find that  $a \in \overline{B^p(\mathcal{U}, F)}$  and that the class a of a is  $H^p(\mathcal{U}, F)$  verifies  $\bar{\rho}(\bar{a}) = \bar{a}'$ ; this proves the result.

Remark. If X is not separated, an intersection of two open Stein subspaces of X need not be Stein; take f.i. for X two copies of  $C^2$ , identified everywhere except at O; there is an obvious covering of X by two open subspaces, identicals with  $C^2$ ; but their intersection is  $C^2 - \{0\}$ , and therefore is not Stein!

## 4.4. The finiteness theorem

Theorem 4.4.1. (Cartan — Serre). Let X be a compact analytic space, and F be a coherent analytic sheaf on X. Then, for every  $p \ge 0$   $H^p(X, F)$  is separated and finite dimensional.

We shall give two proofs of this theorem; both are interesting for further applications.

Ist proof. Let  $\{X_i\}$  and  $\{X_i'\}$  be two finite coverings of X of the type considered in the previous articles, such that, for every i,  $X_i'$  is relatively compact in  $X_i$ . Then, if we denote by  $\mathscr{U}$  (resp.  $\mathscr{U}'$ ) the covering  $\{X_i\}$  (resp.  $\{X_i'\}$ ), the natural restriction map  $C^p(\mathscr{U}, F) \to C^p(\mathscr{U}', F)$  is compact.

Consider now the map

$$(\rho, d): Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \to Z^p(\mathcal{U}', F)$$

this map is surjective, and we have  $(d\rho) = (\rho, 0) + (0, d), (\rho, 0)$  being compact; then the following lemma proves that Im (0, d) is closed and finite codimensional, q.e.d.

Lemma 4.4.2. Let E and F two Frechet spaces,  $u_1$  and  $u_2$  two linear continuous maps  $E \to F$  such that  $u_1 + u_2$  is surjective, and  $u_1$  compact. Then Im  $(u_2)$  is closed and finite codimensional. For the proof, see e.g. [5].

2nd proof. Consider  $\mathcal{U}$  and  $\mathcal{U}'$  as above, and consider the map  $(\rho,d)$   $C^{p-1}(\mathcal{U},F)/Z^{p-1}(\mathcal{U},F) \to \left[C^{p-1}(\mathcal{U}',F)/Z^{p-1}(\mathcal{U}',F)\right] \oplus Z^p(\mathcal{U},F)$   $(\rho,d)$  is clearly injective. I claim that its image is closed: In fact, since  $\bar{\rho}: H^p(\mathcal{U},F) \to H^p(\mathcal{U}',F)$  is injective, this image consists of the pairs  $(\bar{a}',b), a' \in C^{p-1}(\mathcal{U}',F), b \in Z^p(\mathcal{U},F)$  such that  $da' = \rho b$ , which proves the assertion.

Now we have  $(\rho, d) = (\rho, 0) + (0, d)$  and  $(\rho, 0)$  is compact. By a well-known lemma, it results that Im (0, d) is closed, which means that  $H^p(\mathcal{U}, F)$  is separated.

Finally, since  $\bar{\rho}$  is compact, and is an isomorphism, it follows that the identity map of  $H^p(\mathcal{U}, F)$  into itself is compact; therefore this space is finite dimensional; this proves the theorem.