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$$\Gamma(U, \mathcal{O}_U)^{q'} \xrightarrow{\alpha'} \Gamma(U, \mathcal{O}_U)^{p'} \xrightarrow{\beta'} \tilde{F} \to 0 .$$

As $\Gamma(U, \mathcal{O}_U)^p$ is free over $\Gamma(U, \mathcal{O}_U)$, we can find a $\Gamma(U, \mathcal{O}_U)$-linear map $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$ such that $\beta = \beta' \circ \gamma$; this induces a continuous map

$$\Gamma(U, \mathcal{O}_U)^p / \mathrm{Im}\, \Gamma(U, \alpha) \to \Gamma(U, \mathcal{O}_U)^{p'} / \mathrm{Im}\, \Gamma(U, \alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

2. *General case*

If X is an analytic space and F an analytic coherent sheaf on X, we can find a) a locally finite covering of X by open subspaces X_i, b) for each i, a morphism $X_i \to U_i$, U_i open polycylinder in $\mathbf{C}^{n_i}$, which identifies X_i with a closed subspace of U_i c) for each i, a coherent sheaf $\tilde{F}_i$ on U_i admitting a finite presentation, such that $\tilde{F}_i$ is the extension of $F/_{X_i}$.

On $\Gamma(X_i, F|_{X_i})$ we have already defined a topology ; further, consider the natural injection

$$\Gamma(X, F) \to \prod_i \Gamma(X_i, F|_{X_i}))$$

We claim that its image is closed. For, (f_i) belongs to the image if and only if, for all $x \in X_i \cap X_j$ $(= X_i \times_X X_j)$, we have $(f_i)_x = (f_j)_x$; and the fact that this relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Frechet space on $\Gamma(X, F)$. It does not depend on the chosen covering (if one has tavo coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if X' is an open subspace of X, the restriction map $\Gamma(X, F) \to \Gamma(X', F|_{X'})$ is continuous. If X' is relatively compact in X, then the restriction map is compact (this can be seen by choosing a covering X'_j of X' of the same type, such that, for any j, there exist i with $X'_j \subset X_i$, X'_j relatively compact in X_i, and applying Ascoli's theorem).

4.3. *Topology on* $H^p(X, F)$

We consider a locally finite covering $\mathcal{U} = \{X_i\}_{i \in I}$ by open subspaces of the preceding type. If we have $i_0, \ldots, i_p \in$ I, we consider the natural morphisms

$$X_{i_0 \ldots i_p} = X_{i_0} \underset{X}{\times} \ldots \underset{X}{\times} X_{i_p} \to X_{i_0} \times \ldots \times X_{i_p} \to U_{i_0} \times \ldots \times U_{i_p}$$

which makes $X_{i_0}, \ldots, {}_{i_p}$ isomorphic with a closed subspace of $U_{i_0} \times \ldots \times U_{i_p}$ (the hypothesis that X is separated is essential here! See remark at the end of this paragraph), therefore, $X_{i_0}, \ldots, {}_{i_p}$ satisfies theorems A and B ; more generally, if a finite number of open subspaces of X is Stein, their intersection is also Stein.

Introduce a total order on I. Given an analytic coherent sheaf on X, we can identify the alternating cochains of degree p of the covering $\mathcal{U}$ with values in F with the space

$$C^p(\mathcal{U},F) = \prod_{i_0 < i_1 < \ldots < i_p} \Gamma(X_{i_0 \ldots i_p}, F|x_{i_0 \ldots i_p}) .$$

This is a Frechet space, and the differential d: $C^p(\mathcal{U}, F) \to C^{p+1}(\mathcal{U}, F)$ is clearly continuous. Therefore the kernel $Z^p(\mathcal{U}, F)$ is a closed subspace of $C^p(\mathcal{U}, F)$. We denote $B^p(\mathcal{U},F)$ the image of $C^{p-1}(\mathcal{U},F)$ under d, and we consider on $H^p(\mathcal{U},F) = Z^p(\mathcal{U},F)/B^p(\mathcal{U}, F)$ the quotient topology; according to Leray's theorem, there is a natural isomorphism $H^p(X,F) \simeq H^p(\mathcal{U},F)$.

This gives a topology on $H^p(X, F)$ of a quotient of a Frechet space. In general, this topology is *not separated.*

We prove now that this topology is independent of the covering $\mathcal{U}$; to do that, it is sufficient to consider a refinement $\mathcal{U}' = \{ X'_j \}_{j \in J}$ of $\mathcal{U}$ of the same type, a map $\varphi : J \to I$ such that $X'_j \subset X_{\varphi(j)}$ for any j to consider the map defined by $\varphi : C^*(\mathcal{U},F) = \oplus_p C^p(\mathcal{U},F) \xrightarrow{p} C^*(\mathcal{U}',F)$ and to prove that the induced map $\bar{\rho} : H^p(\mathcal{U}, F) \to H^p(\mathcal{U}', F)$ is an isomorphism.

First, $\bar{\rho}$ is obviously continuous and bijective ; so, according to the closed graph theorem, all that we have to prove is that $\bar{\rho}$ maps the adherence of 0 onto the adherence of zero ; to do that, we consider $\bar{a}' \in H^p(\mathcal{U}, F)$, which is adherent to zero ; this means that $\bar{a}'$ is the class modulo $B^p(\mathcal{U}',F)$ of some $a' \in Z^p(\mathcal{U}' F)$ which is adherent to $B^p(\mathcal{U}', F)$; therefore, we have

$$a' = \lim_{n\to\infty} db'_n , \qquad b'_n \in C^{p-1}(\mathcal{U}', F) .$$

Now, the map

$$Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \xrightarrow{(\rho,d)} Z^p(\mathcal{U}', F)$$

is surjective hence, according to the closed graph theorem, we can find converging sequences $a_n \in Z^p(\mathcal{U}, F)$ and $b''_n \in C^{p-1}(\mathcal{U}', F)$ such that $d b'_n = \rho(a_n) + d b''_n$; but, $\bar{\rho}$ being an isomorphism, we have $a_n = d\alpha_n$, $\alpha_n \in C^{n-1}(\mathcal{U}, F)$; if we put $b = \lim\limits_{n\to\infty} b_n$, $a = \lim a_n$, we find that $a \in \overline{B^p(\mathcal{U}, F)}$ and that the class a of a is $H^p(\mathcal{U}, F)$ verifies $\bar{\rho}(\bar{a}) = \bar{a}'$; this proves the result.

Remark. If X is not separated, an intersection of two open Stein subspaces of X need not be Stein; take f.i. for X two copies of C^2, identified everywhere except at O; there is an obvious covering of X by two open subspaces, identicals with C^2; but their intersection is $C^2 - \{o\}$, and therefore is not Stein!

4.4. *The finiteness theorem*

Theorem 4.4.1. (Cartan — Serre). Let X be a compact analytic space, and F be a coherent analytic sheaf on X. Then, for every $p \geqslant 0$ $H^p(X, F)$ is separated and finite dimensional.

We shall give two proofs of this theorem ; both are interesting for further applications.

1st proof. Let $\{X_i\}$ and $\{X_i'\}$ be two finite coverings of X of the type considered in the previous articles, such that, for every i, X'_i is relatively compact in X_i. Then, if we denote by $\mathcal{U}$ (resp. $\mathcal{U}'$) the covering $\{X_i\}$ (resp. $\{X_i'\}$), the natural restriction map $C^p(\mathcal{U}, F) \to C^p(\mathcal{U}', F)$ is compact.

Consider now the map

$$(\rho, d) : Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \to Z^p(\mathcal{U}', F)$$

this map is surjective, and we have $(, d\rho) = (\rho, 0) + (0, d)$, $(\rho, 0)$ being compact ; then the following lemma proves that Im $(0, d)$ is closed and finite codimensional, q.e.d.

Lemma 4.4.2. Let E and F two Frechet spaces, u_1 and u_2 two linear continuous maps $E \to F$ such that $u_1 + u_2$ is surjective, and u_1 compact. Then Im (u_2) is closed and finite codimensional. For the proof, see e.g. [5].

2nd proof. Consider $\mathcal{U}$ and $\mathcal{U}'$ as above, and consider the map
(ρ, d) $C^{p-1}(\mathcal{U}, F)/Z^{p-1}(\mathcal{U}, F) \to [C^{p-1}(\mathcal{U}', F)/Z^{p-1}(\mathcal{U}', F)] \oplus Z^p(\mathcal{U}, F)$
(ρ, d) is clearly injective. I claim that its image is closed: In fact, since $\bar{\rho} : H^p(\mathcal{U}, F) \to H^p(\mathcal{U}', F)$ is injective, this image consists of the pairs $(\bar{a}', b)$, $a' \in C^{p-1}(\mathcal{U}', F)$, $b \in Z^p(\mathcal{U}, F)$ such that $da' = \rho\, b$, which proves the assertion.

Now we have $(\rho, d) = (\rho, 0) + (0, d)$ and $(\rho, 0)$ is compact. By a well-known lemma, it results that Im $(0, d)$ is closed, which means that $H^p(\mathcal{U}, F)$ is separated.

Finally, since $\bar{\rho}$ is compact, and is an isomorphism, it follows that the identity map of $H^p(\mathcal{U}, F)$ into itself is compact ; therefore this space is finite dimensional ; this proves the theorem.