3.3 Finite morphisms

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Denote now by $\mathscr{I}$ (resp $\mathscr{J}$) the coherent sheaf of ideals generated in U (resp. V) by the $\bar{f}_i$'s | resp. the $\bar{g}'_j s$). We have $\Phi^*(\mathscr{J})_0 \subset \mathscr{I}_0$, hence, since $\mathscr{J}$ is finitely generated by restricting U and V if necessary, we have $\Phi^* (\mathscr{J}) \subset \mathscr{I}$. Finally we take $X = \operatorname{supp} \mathcal{O}_U/\mathscr{I}$, $\mathcal{O}_X = \mathcal{O}_U/\mathscr{I} \mid_X$ and the same for Y; it is clear that Φ induces the required morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$.

Finally, if two morphisms $\varphi, \psi : (X, 0) \to (Y, 0)$ induce the same homomorphism $\mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$, we have to prove that φ and ψ are equals. We may assume that Y is given by a local model $(Y, \mathcal{O}_V | \mathscr{J} | Y)$ for some coherent sheaf $\mathscr{J}$ of ideals on an open set $V \subset \mathbf{C}^m$; by composition with the injection $Y \to V$, we may restrict ourselves to the case where $Y = \mathbf{C}^m$; the morphisms φ and ψ are now given by sections $f, g \in \Gamma (X, \mathcal{O}_X{}^m)$, and the hypothesis means that the germs of f and g at 0 coincide; hence f and g coincide in a neighborhood of 0 in X, which proves the assertion.

3.3 *Finite morphisms*

Let $f : (X, 0) \to (Y, 0)$ be a morphism of germs of analytic spaces. Then f is called " finite " if the corresponding homomorphism $f^* : \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$ makes $\mathcal{O}_{X,0}$ finite over $\mathcal{O}_{Y,0}$. According to the preparation theorem 3.1.3. in order that f be finite, it is necessary and sufficient that $\mathcal{O}_{X,0}/\mathfrak{M}(\mathcal{O}_{Y,0})\, \mathcal{O}_{X,0}$ be finite over $\mathbf{C}$; in geometrical terms, this means that the germ of space $f^{-1}(0)$ is finite over the point 0 (see § 1.3, example 4).

In the global case (complex or real), we give the following definition:

Definition 3.3.1. A morphism of *separated* analytic spaces $f = (f_0, f^1)$: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is finite if the following properties hold:

1) f is proper (i.e. f_0 is proper).

2) For any point $x \in X$, the induced morphism of germs $f_x : (X, \mathcal{O}_X, x) \to (Y, \mathcal{O}_Y, f_0 (x))$ is finite.

In the *complex* case, we have the following results :

Proposition 3.3.2. f is finite if and only if f is proper and, for any $b \in Y$, the *set* $f_0{}^{-1} (b)$ is finite.

This proposition is more or less equivalent to the " Nullstellensatz "; for the proof see e.g. Houzel [6] or Narasimhan [9]. In the real case, the part " if " of this proposition is not even true when Y is a point : for instance the subspace of $\mathbf{R}^2$ defined by $\mathscr{I}$ = (coherent sheaf of ideals generated by $x_1^2 + x_2^2$) has support 0 ; but $\mathbf{R}\{ x_1, x_2 \}/(x_1^2 + x_2^2)$ is not finite over $\boldsymbol{R}$.

Proposition 3.3.2. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a finite morphism, then the direct image $f_* (\mathcal{O}_X)$ is a coherent analytic sheaf of $\mathcal{O}_Y$-modules ; conversely, let $\mathscr{A}$ be a sheaf of $\mathcal{O}_Y$-algebras, which is coherent as sheaf of $\mathcal{O}_Y$-modules. Then there exists an analytic space $(X, \mathcal{O}_X)$ and a finite morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f_* (\mathcal{O}_X)$ is isomorphic with $\mathscr{A}$ as sheaf of $\mathcal{O}_Y$-algebras ; the triple $(X, \mathcal{O}_X, f)$ is unique up to an isomorphism.

We do not prove this proposition here and refer to Houzel [6] or Narasimhan [9] for this proof. We note also that a proof of the direct part can be given along the same lines as theorem 3.1.3, combined with the fact that direct images under finite morphism preserve exact sequences of sheaves of $\mathcal{O}_X$-modules (in other words, that higher direct images are zero). We note also that, for *proper* morphisms (not necessarily finite), a much deeper result has been proved by Grauert [2], [3].

Finally, we remark that, in the real case, proposition 3.3.2. is false (take, for instance, X the submanifold of $\mathbf{R}^2$ defined by $x_2 - x_1^2 = 0$, $Y = \mathbf{R}$ and $f =$ the projection on the x_2-axis ; $f_* (\mathcal{O}_X)$ has support $x_2 \geqslant 0$, which is not an analytic subset of $\mathbf{R}$, hence $f_* (\mathcal{O}_X)$ cannot be coherent!)

CHAPTER 4.

THE FINITENESS THEOREM

In this chapter, we consider only *complex* analytic spaces, separated and having a countable basis of open sets.

4.1. *Stein spaces*

Let $(X, \mathcal{O}_X)$ be an analytic space, and K a subset of X ; we denote, as usual by $\hat{K}$ the set

$$\left\{ x \in X \mid \forall f \in \Gamma(X, \mathcal{O}_x) : |f(x)| \leqslant \sup_{y \in K} |f(y)| \right\}$$

Definition 4.1.1. a) $(X, \mathcal{O}_X)$ is called holomorphically convex if, for any K compact $\subset X$, $\hat{K}$ is compact ;
b) $(X, \mathcal{O}_X)$ is called a Stein space if it is holomorphically convex, and if, for any $x \in X$, there exist sections $f_1, \ldots, f_p \in \Gamma(X, \mathcal{O}_X)$ with $f_i(x) = 0$, such that x is an isolated point of the counter-image of 0 in the morphism $(X, \mathcal{O}_X) \to \mathbf{C}^p$ defined by $f_1, \ldots, f_p$, (This last property can also by expressed as the fact that the morphism of germs : $(X, \mathcal{O}_X, x) \to (\mathbf{C}^p, 0)$ defined by $f_1, \ldots, f_p$ is finite).