Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ANALYTIC SPACES
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Kapitel: 1.3. Operations on analytic spaces.
DOI: https://doi.org/10.5169/seals-42341

for some coherent sheaf $\mathscr{I}$ of ideals of $\mathcal{O}_X$. An *open analytic subspace* of $(X, \mathcal{O}_X)$ is just a restriction $(U, \mathcal{O}_X \mid U)$, U open in X. An *analytic subspace* of an analytic space $(X, \mathcal{O}_X)$ is a closed analytic subspace $(Y, \mathcal{O}_Y)$ of the open analytic subspace $(\complement \bar{Y} \cup Y, \mathcal{O}_{\complement\bar{Y} \cup Y})$ of $(X, \mathcal{O}_X)$, provided $\complement \bar{Y} \cup Y$ is indeed open in X, i.e. Y is locally closed in X.

Examples. The " single point " $(0, \mathbf{C})$ is an analytic subspace of the " double point " $(0, \mathbf{C}\{x\}/(x^2))$, but not conversely. The double point is, however, a closed analytic subspace of, e.g., $(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. A " point " of an analytic space will always mean a single point embedded in $(X, \mathcal{O}_X)$ by means of a map $(0, \mathbf{C}) \to (X, \mathcal{O}_X)$.

1.3. *Operations on analytic spaces.*

In this section we shall write X for the analytic space $(X, \mathcal{O}_X)$.

a) *Product.* By a general definition in the theory of categories, a product of two analytic spaces X, X' is a triple (Z, π, π') where Z is an analytic space and $\pi : Z \to X$, $\pi' : Z \to X'$ are two morphisms with the following property:

Given any analytic space Y and any pair $f : Y \to X$, $f' : Y \to X'$ of morphisms there exists a unique morphism $g : Y \to Z$ such that $f = \pi \circ g$, $f' = \pi' \circ g$.

For example, the product of $\mathbf{C}^p$ and $\mathbf{C}^q$ is $\mathbf{C}^{p+q}$, according to proposition 1.2.4.

We shall see that a product of analytic spaces always exists. The uniqueness of g clearly implies the uniqueness of the product (Z, π, π') up to isomorphism; we denote one such Z by $X \times X'$.

To prove that the product always exists, let us suppose first that X and X' are special models, i.e. X is defined by a triple (U, f, F) where U is open in $\mathbf{C}^n$, F is a finite-dimensional complex linear space, and $f : U \to F$ is an analytic map; similarly for X'. We claim that the special model Z defined by $(U \times U', f \times f', F \times F')$ is a product. Indeed, from the description of the morphisms into a special model provided by Proposition 1.2.5. it follows that we have natural maps $\pi : Z \to X$, $\pi' : Z \to X'$ induced by the projections $U \times U' \to U$, $U \times U' \to U'$. Also, if $f: Y \to X$ and $f' : Y \to X'$ are given, $g : Y \to Z$ is determined by

$$\begin{tikzcd}[row sep=small] & X \arrow[r] & U \arrow[dr] & \\ Y \arrow[ur, "f"] \arrow[dr, "f'"'] & & & U \times U' . \\ & X' \arrow[r] & U' \arrow[ur] & \end{tikzcd}$$

In the general case we take $X \times X'$ as the ringed space whose topological underlying space in the cartesian product of the underlying space of X and X', and whose structure sheaf is given locally by the product of local models for X and X'. (From the uniqueness "up to isomorphism" of the product results that these sheaves stick together in a well-determined way).

b) *Kernel of a double arrow.* If $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ is a double arrow, i.e. a pair of morphisms, a kernel X' of (u, v) is an analytic subspace of X such that the morphisms of an arbitrary analytic space Z into X' are exactly the morphisms h of Z into X such that $u \circ h = v \circ h$. In other words, if $i : X' \to X$ is the natural map of X' into X, the morphisms $h : Z \to X'$ satisfy $u \circ i \circ h = v \circ i \circ h$ and if a morphism $g: Z \to X$ satisfies $u \circ g = v \circ g$, then $g = i \circ h$ for some $h : Z \to X'$. To prove the existence of the kernel it suffices, again, to do this locally, i.e. for special models. If X is defined by (U, f, F) and Y by (V, g, G) we may (perhaps, after restricting U) extend u and v to maps $\bar{u}, \bar{v} : U \to E$ where E denotes the complex linear space of which V is an open subset. The kernel is then defined by the triple

$$\left(U, f \times (\bar{u} - \bar{v}), \quad F \times E\right).$$

It follows from the Proposition 1.2.5. that this special model satisfies the universal property of kernels.

Example 1. The kernel of $\mathbf{C} \underset{-t}{\overset{t}{\rightrightarrows}} \mathbf{C}$ is the simple point $\{0\}$, t denoting the identity of $\mathbf{C}$.

Example 2. The kernel of $\mathbf{C} \underset{t+t^2}{\overset{t}{\rightrightarrows}} \mathbf{C}$ is $\{0\}$ counted as a double point.

c) *Fiber product.* If $u : X \to S$ and $v : Y \to S$ are given morphisms of analytic spaces, the fiber product $X \times_S Y$ of X and Y over S is the kernel of the double arrow

$$X \times Y \underset{v \circ \pi'}{\overset{u \circ \pi}{\rightrightarrows}} S$$

where $\pi : X \times Y \to X$ and $\pi' : X \times Y \to Y$ are the maps defined by the product. Note that when S is a simple point, $X \times_S Y = X \times Y$.

One may also introduce the category of analytic spaces over S. Its objects are morphisms $u : X \to S$ of an analytic space X onto S and its morphisms are morphisms $f : X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & & Y \arrow[dl, "v"] \\ & S & \end{tikzcd}$$

is commutative. The product in this category, i.e. the object satisfying the universal property given above for the product $X \times Y$, is then exactly the fiber product $X \times_S Y$. If S is a point, we have the category of analytic spaces.

Example 3. If U and V are open subspaces of an analytic space X, the open subspace $U \cap V$ is isomorphic to $U \times_X V$. We may thus define, in general, the intersection of two analytic subspaces $X' \to X$ and $X'' \to X$ of X to be the fiber product $X' \times_X X''$.

Example 4. If $\varphi : Y \to X$ is a morphism of analytic spaces and $a \in X$ a point, i.e. a map $a : (0, \mathbf{C}) \to X$ we may consider the space $Y(a) = Y \times_X a$. It is natural to call this the inverse image of a under φ and to denote it by $\varphi^{-1}(a)$; its underlying space is exactly $\varphi_0^{-1}(a)$.

If $\varphi_0(b) = a$, then $\mathcal{O}_{Y(a),b}$ is $\mathcal{O}_{Y,b}$ taken modulo the image under $\varphi^1 : \mathcal{O}_{X,a} \to \mathcal{O}_{Y,b}$ of the maximal ideal in $\mathcal{O}_{X,a}$.

Example 5. The pull-back of a linear bundle E over X by a map $Y \to X$ is exactly $Y \times_X E$.

1.4. *Relations between reduced and non-reduced spaces.*

We shall first characterize those analytic spaces which are reduced.

Proposition 1.4.1. A analytic space $(X, \mathcal{O}_X)$ is reduced if and only if $\mathcal{O}_{X,x}$ has no nilpotent element for x arbitrary in X.

Proof. The necessity of the condition is obvious for $\mathcal{O}_X$ can be considered as a submodule of $\mathcal{C}_X$ if $(X, \mathcal{O}_X)$ is reduced.

Conversely, if $\mathcal{O}_{X,x}$ has no nilpotent elements, we shall prove that in any local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$, a germ g at $a \in V$ which vanishes on V belongs to the ideal $\mathcal{I}$ defining $\mathcal{O}_V$. The Nullstellensatz implies that $g^k \in \mathcal{I}_a$ if k is large enough. But it is then clear that $g \in \mathcal{I}_a$ if $\mathcal{O}_{V,a}/\mathcal{I}_a$ is free from nilpolent elements.

Given an analytic space $(X, \mathcal{O}_X)$ we can associate to it a reduced space in the following way. Let $\mathcal{N}_x$ be the ideal in $\mathcal{O}_{X,x}$ consisting of all nilpotent elements (the nil-radical of 0). Then $\mathcal{N} = U\mathcal{N}_x$ is a coherent sheaf by the Oka-Cartan theorem, for in a local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$ we have $\mathcal{N}_X = (\mathcal{I}'/\mathcal{I})_X$ where $\mathcal{I}'$ is the sheaf of germs vanishing on V and $\mathcal{I}$ the