Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ANALYTIC SPACES
Autor: Malgrange, Bernard
Kapitel: 1.3. Operations on analytic spaces.
DOI: https://doi.org/10.5169/seals-42341

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 27.04.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

for some coherent sheaf $\mathscr{I}$ of ideals of $\mathcal{O}_X$. An *open analytic subspace* of $(X, \mathcal{O}_X)$ is just a restriction $(U, \mathcal{O}_X \mid U)$, U open in X. An *analytic subspace* of an analytic space $(X, \mathcal{O}_X)$ is a closed analytic subspace $(Y, \mathcal{O}_Y)$ of the open analytic subspace $(\complement\, \overline{Y} \cup Y, \mathcal{O}_{\complement \overline{Y} \cup Y})$ of $(X, \mathcal{O}_X)$, provided $\complement\, \overline{Y} \cup Y$ is indeed open in X, i.e. Y is locally closed in X.

Examples. The " single point " $(0, \mathbf{C})$ is an analytic subspace of the " double point " $(0, \mathbf{C}\{x\}/(x^2))$, but not conversely. The double point is, however, a closed analytic subspace of, e.g., $(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. A " point " of an analytic space will always mean a single point embedded in $(X, \mathcal{O}_X)$ by means of a map $(0, \mathbf{C}) \to (X, \mathcal{O}_X)$.

1.3. Operations on analytic spaces.

In this section we shall write X for the analytic space $(X, \mathcal{O}_X)$.

a) *Product.* By a general definition in the theory of categories, a product of two analytic spaces X, X' is a triple (Z, π, π') where Z is an analytic space and $\pi : Z \to X$, $\pi' : Z \to X'$ are two morphisms with the following property:

Given any analytic space Y and any pair $f : Y \to X, f' : Y \to X'$ of morphisms there exists a unique morphism $g : Y \to Z$ such that $f = \pi \circ g$, $f' = \pi' \circ g$.

For example, the product of $\mathbf{C}^p$ and $\mathbf{C}^q$ is $\mathbf{C}^{p+q}$, according to proposition 1.2.4.

We shall see that a product of analytic spaces always exists. The uniqueness of g clearly implies the uniqueness of the product (Z, π, π') up to isomorphism; we denote one such Z by $X \times X'$.

To prove that the product always exists, let us suppose first that X and X' are special models, i.e. X is defined by a triple (U, f, F) where U is open in $\mathbf{C}^n$, F is a finite-dimensional complex linear space, and $f : U \to F$ is an analytic map; similarly for X'. We claim that the special model Z defined by $(U \times U', f \times f', F \times F')$ is a product. Indeed, from the description of the morphisms into a special model provided by Proposition 1.2.5. it follows that we have natural maps $\pi : Z \to X$, $\pi' : Z \to X'$ induced by the projections $U \times U' \to U$, $U \times U' \to U'$. Also, if $f : Y \to X$ and $f' : Y \to X'$ are given, $g : Y \to Z$ is determined by

$$\begin{array}{ccccccc} & & f & & & & \\ & \nearrow & X & \to & U & \searrow & \\ Y & & & & & & U \times U'. \\ & \searrow & X' & \to & U' & \nearrow & \\ & & f' & & & & \end{array}$$

In the general case we take $X \times X'$ as the ringed space whose topological underlying space in the cartesian product of the underlying space of X and X', and whose structure sheaf is given locally by the product of local models for X and X'. (From the uniqueness "up to isomorphism" of the product results that these sheaves stick together in a well-determined way).

b) *Kernel of a double arrow.* If $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ is a double arrow, i.e. a pair of morphisms, a kernel X' of (u, v) is an analytic subspace of X such that the morphisms of an arbitrary analytic space Z into X' are exactly the morphisms h of Z into X such that $u \circ h = v \circ h$. In other words, if $i : X' \to X$ is the natural map of X' into X, the morphisms $h : Z \to X'$ satisfy $u \circ i \circ h = v \circ i \circ h$ and if a morphism $g: Z \to X$ satisfies $u \circ g = v \circ g$, then $g = i \circ h$ for some $h : Z \to X'$. To prove the existence of the kernel it suffices, again, to do this locally, i.e. for special models. If X is defined by (U, f, F) and Y by (V, g, G) we may (perhaps, after restricting U) extend u and v to maps $\bar{u}, \bar{v} : U \to E$ where E denotes the complex linear space of which V is an open subset. The kernel is then defined by the triple

$$\left(U, f \times (\bar{u} - \bar{v}), \quad F \times E\right).$$

It follows from the Proposition 1.2.5. that this special model satisfies the universal property of kernels.

Example 1. The kernel of $\mathbf{C} \underset{-t}{\overset{t}{\rightrightarrows}} \mathbf{C}$ is the simple point $\{0\}$, t denoting the identity of $\mathbf{C}$.

Example 2. The kernel of $\mathbf{C} \underset{t+t^2}{\overset{t}{\rightrightarrows}} \mathbf{C}$ is $\{0\}$ counted as a double point.

c) *Fiber product.* If $u : X \to S$ and $v : Y \to S$ are given morphisms of analytic spaces, the fiber product $X \times_S Y$ of X and Y over S is the kernel of the double arrow

$$X \times Y \underset{v \circ \pi'}{\overset{u \circ \pi}{\rightrightarrows}} S$$

where $\pi : X \times Y \to X$ and $\pi' : X \times Y \to Y$ are the maps defined by the product. Note that when S is a simple point, $X \times_S Y = X \times Y$.

One may also introduce the category of analytic spaces over S. Its objects are morphisms $u : X \to S$ of an analytic space X onto S and its morphisms are morphisms $f : X \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\ & S & \end{array}$$

is commutative. The product in this category, i.e. the object satisfying the universal property given above for the product $X \times Y$, is then exactly the fiber product $X \times_S Y$. If S is a point, we have the category of analytic spaces.

Example 3. If U and V are open subspaces of an analytic space X, the open subspace $U \cap V$ is isomorphic to $U \times_X V$. We may thus define, in general, the intersection of two analytic subspaces $X' \to X$ and $X'' \to X$ of X to be the fiber product $X' \times_X X''$.

Example 4. If $\varphi : Y \to X$ is a morphism of analytic spaces and $a \in X$ a point, i.e. a map $a : (0, \mathbf{C}) \to X$ we may consider the space $Y(a) = Y \times_X a$. It is natural to call this the inverse image of a under φ and to denote it by $\varphi^{-1}(a)$; its underlying space is exactly $\varphi_0^{-1}(a)$.

If $\varphi_0(b) = a$, then $\mathcal{O}_{Y(a),b}$ is $\mathcal{O}_{Y,b}$ taken modulo the image under $\varphi^1 : \mathcal{O}_{X,a} \to \mathcal{O}_{Y,b}$ of the maximal ideal in $\mathcal{O}_{X,a}$.

Example 5. The pull-back of a linear bundle E over X by a map $Y \to X$ is exactly $Y \times_X E$.

1.4. *Relations between reduced and non-reduced spaces.*

We shall first characterize those analytic spaces which are reduced.

Proposition 1.4.1. A analytic space $(X, \mathcal{O}_X)$ is reduced if and only if $\mathcal{O}_{X,x}$ has no nilpotent element for x arbitrary in X.

Proof. The necessity of the condition is obvious for $\mathcal{O}_X$ can be considered as a submodule of $\mathcal{C}_X$ if $(X, \mathcal{O}_X)$ is reduced.

Conversely, if $\mathcal{O}_{X,x}$ has no nilpotent elements, we shall prove that in any local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$, a germ g at $a \in V$ which vanishes on V belongs to the ideal $\mathcal{I}$ defining $\mathcal{O}_V$. The Nullstellensatz implies that $g^k \in \mathcal{I}_a$ if k is large enough. But it is then clear that $g \in \mathcal{I}_a$ if $\mathcal{O}_{V,a}/\mathcal{I}_a$ is free from nilpolent elements.

Given an analytic space $(X, \mathcal{O}_X)$ we can associate to it a reduced space in the following way. Let $\mathcal{N}_x$ be the ideal in $\mathcal{O}_{X,x}$ consisting of all nilpotent elements (the nil-radical of 0). Then $\mathcal{N} = U\mathcal{N}_x$ is a coherent sheaf by the Oka-Cartan theorem, for in a local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$ we have $\mathcal{N}_X = (\mathcal{I}'/\mathcal{I})_X$ where $\mathcal{I}'$ is the sheaf of germs vanishing on V and $\mathcal{I}$ the