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Autor: Malgrange, Bernard
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for some coherent sheaf \mathcal{I} of ideals of \mathcal{O}_X . An *open analytic subspace* of (X, \mathcal{O}_X) is just a restriction $(U, \mathcal{O}_X|_U)$, U open in X . An *analytic subspace* of an analytic space (X, \mathcal{O}_X) is a closed analytic subspace (Y, \mathcal{O}_Y) of the open analytic subspace $(\mathbb{C} \bar{Y} \cup Y, \mathcal{O}_{\mathbb{C} \bar{Y} \cup Y})$ of (X, \mathcal{O}_X) , provided $\mathbb{C} \bar{Y} \cup Y$ is indeed open in X , i.e. Y is locally closed in X .

Examples. The “single point” $(0, \mathbb{C})$ is an analytic subspace of the “double point” $(0, \mathbb{C} \{x\}/(x^2))$, but not conversely. The double point is, however, a closed analytic subspace of, e.g., $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$. A “point” of an analytic space will always mean a single point embedded in (X, \mathcal{O}_X) by means of a map $(0, \mathbb{C}) \rightarrow (X, \mathcal{O}_X)$.

1.3. Operations on analytic spaces.

In this section we shall write X for the analytic space (X, \mathcal{O}_X) .

a) *Product.* By a general definition in the theory of categories, a product of two analytic spaces X, X' is a triple (Z, π, π') where Z is an analytic space and $\pi : Z \rightarrow X, \pi' : Z \rightarrow X'$ are two morphisms with the following property:

Given any analytic space Y and any pair $f : Y \rightarrow X, f' : Y \rightarrow X'$ of morphisms there exists a unique morphism $g : Y \rightarrow Z$ such that $f = \pi \circ g, f' = \pi' \circ g$.

For example, the product of \mathbb{C}^p and \mathbb{C}^q is \mathbb{C}^{p+q} , according to proposition 1.2.4.

We shall see that a product of analytic spaces always exists. The uniqueness of g clearly implies the uniqueness of the product (Z, π, π') up to isomorphism; we denote one such Z by $X \times X'$.

To prove that the product always exists, let us suppose first that X and X' are special models, i.e. X is defined by a triple (U, f, F) where U is open in \mathbb{C}^n , F is a finite-dimensional complex linear space, and $f : U \rightarrow F$ is an analytic map; similarly for X' . We claim that the special model Z defined by $(U \times U', f \times f', F \times F')$ is a product. Indeed, from the description of the morphisms into a special model provided by Proposition 1.2.5. it follows that we have natural maps $\pi : Z \rightarrow X, \pi' : Z \rightarrow X'$ induced by the projections $U \times U' \rightarrow U, U \times U' \rightarrow U'$. Also, if $f : Y \rightarrow X$ and $f' : Y \rightarrow X'$ are given, $g : Y \rightarrow Z$ is determined by

$$\begin{array}{ccccc} Y & \xrightarrow{f} & X & \rightarrow & U \\ & & & \searrow & \\ & & & & U \times U' \\ & \searrow & X' & \rightarrow & U' \\ & & & \nearrow & \end{array}$$

In the general case we take $X \times X'$ as the ringed space whose topological underlying space is the cartesian product of the underlying space of X and X' , and whose structure sheaf is given locally by the product of local models for X and X' . (From the uniqueness "up to isomorphism" of the product results that these sheaves stick together in a well-determined way).

b) *Kernel of a double arrow.* If $X \xrightarrow[u]{u} Y$ is a double arrow, i.e. a pair of morphisms, a kernel X' of (u, v) is an analytic subspace of X such that the morphisms of an arbitrary analytic space Z into X' are exactly the morphisms h of Z into X such that $u \circ h = v \circ h$. In other words, if $i : X' \rightarrow X$ is the natural map of X' into X , the morphisms $h : Z \rightarrow X'$ satisfy $u \circ i \circ h = v \circ i \circ h$ and if a morphism $g : Z \rightarrow X$ satisfies $u \circ g = v \circ g$, then $g = i \circ h$ for some $h : Z \rightarrow X'$. To prove the existence of the kernel it suffices, again, to do this locally, i.e. for special models. If X is defined by (U, f, F) and Y by (V, g, G) we may (perhaps, after restricting U) extend u and v to maps $\bar{u}, \bar{v} : U \rightarrow E$ where E denotes the complex linear space of which V is an open subset. The kernel is then defined by the triple

$$(U, f \times (\bar{u} - \bar{v}), F \times E).$$

It follows from the Proposition 1.2.5. that this special model satisfies the universal property of kernels.

Example 1. The kernel of $\mathbf{C} \xrightarrow[t]{t} \mathbf{C}$ is the simple point $\{0\}$, t denoting the identity of \mathbf{C} .

Example 2. The kernel of $\mathbf{C} \xrightarrow[t+t^2]{t} \mathbf{C}$ is $\{0\}$ counted as a double point.

c) *Fiber product.* If $u : X \rightarrow S$ and $v : Y \rightarrow S$ are given morphisms of analytic spaces, the fiber product $X \times_s Y$ of X and Y over S is the kernel of the double arrow

$$X \times Y \xrightarrow[v \circ \pi']{u \circ \pi} S$$

where $\pi : X \times Y \rightarrow X$ and $\pi' : X \times Y \rightarrow Y$ are the maps defined by the product. Note that when S is a simple point, $X \times_s Y = X \times Y$.

One may also introduce the category of analytic spaces over S . Its objects are morphisms $u : X \rightarrow S$ of an analytic space X onto S and its morphisms are morphisms $f : X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ u \searrow & & \swarrow v \\ & S & \end{array}$$

is commutative. The product in this category, i.e. the object satisfying the universal property given above for the product $X \times Y$, is then exactly the fiber product $X \times_S Y$. If S is a point, we have the category of analytic spaces.

Example 3. If U and V are open subspaces of an analytic space X , the open subspace $U \cap V$ is isomorphic to $U \times_X V$. We may thus define, in general, the intersection of two analytic subspaces $X' \rightarrow X$ and $X'' \rightarrow X$ of X to be the fiber product $X' \times_X X''$.

Example 4. If $\varphi : Y \rightarrow X$ is a morphism of analytic spaces and $a \in X$ a point, i.e. a map $a : (0, \mathbb{C}) \rightarrow X$ we may consider the space $Y(a) = Y \times_X a$. It is natural to call this the inverse image of a under φ and to denote it by $\varphi^{-1}(a)$; its underlying space is exactly $\varphi_0^{-1}(a)$.

If $\varphi_0(b) = a$, then $\mathcal{O}_{Y(a),b}$ is $\mathcal{O}_{Y,b}$ taken modulo the image under $\varphi^1 : \mathcal{O}_{X,a} \rightarrow \mathcal{O}_{Y,b}$ of the maximal ideal in $\mathcal{O}_{X,a}$.

Example 5. The pull-back of a linear bundle E over X by a map $Y \rightarrow X$ is exactly $Y \times_X E$.

1.4. Relations between reduced and non-reduced spaces.

We shall first characterize those analytic spaces which are reduced.

Proposition 1.4.1. A analytic space (X, \mathcal{O}_X) is reduced if and only if $\mathcal{O}_{X,x}$ has no nilpotent element for x arbitrary in X .

Proof. The necessity of the condition is obvious for \mathcal{O}_X can be considered as a submodule of \mathcal{C}_X if (X, \mathcal{O}_X) is reduced.

Conversely, if $\mathcal{O}_{X,x}$ has no nilpotent elements, we shall prove that in any local model (V, \mathcal{O}_V) for (X, \mathcal{O}_X) , a germ g at $a \in V$ which vanishes on V belongs to the ideal \mathcal{I} defining \mathcal{O}_V . The Nullstellensatz implies that $g^k \in \mathcal{I}_a$ if k is large enough. But it is then clear that $g \in \mathcal{I}_a$ if $\mathcal{O}_{V,a}/\mathcal{I}_a$ is free from nilpotent elements.

Given an analytic space (X, \mathcal{O}_X) we can associate to it a reduced space in the following way. Let \mathcal{N}_x be the ideal in $\mathcal{O}_{X,x}$ consisting of all nilpotent elements (the nil-radical of 0). Then $\mathcal{N} = U\mathcal{N}_x$ is a coherent sheaf by the Oka-Cartan theorem, for in a local model (V, \mathcal{O}_V) for (X, \mathcal{O}_X) we have $\mathcal{N}_X = (\mathcal{I}'/\mathcal{I})_X$ where \mathcal{I}' is the sheaf of germs vanishing on V and \mathcal{I} the