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It should be noted that a morphism may be bijective and bicontinuous and still fail to be an isomorphism. As an example we consider the map $t \to (t^2, t^3)$ of $X = \mathbf{C}$ into the space Y of all pairs (x,y) satisfying $x^3 - y^2 = 0$. This is a bijective and bicontinuous morphism, but its inverse ψ is no morphism since $\psi^* f_0 \notin \mathcal{O}_{Y,0}$ if $f(t) = t$.

Real analytic sets are not as well behaved as complex ones. To illustrate this we consider " Cartan's umbrella " which is the subset of $\mathbf{R}^3$ defined by the equation $z(x^2 + y^2) - x^3 = 0$. Its intersection with the plane $z = 1$ has an isolated double point at (0, 0, 1) and so it has a stick (the z-axis) joining the rest of the " umbrella " at the origin. Here the Oka-Cartan theorem fails. Indeed, suppose that the sheaf $\mathscr{I}$ of germs of real-analytic functions vanishing on the umbrella were generated by sections $s_1, \ldots, s_n \in \Gamma(U, \mathscr{I})$ over some neighborhood U of the origin. Then, denoting by $f_1, \ldots, f_n$ the corresponding real-analytic functions in U, we find (using a complexification and the Nullstellensatz for principal ideals) that every f_j is a multiple of $z(x^2 + y^2) - x^3$ for it can easily be seen that this polynomial defines in the complex domain an irreducible germ at the origin. Hence the germ in $\mathscr{I}$ defined by the coordinate function x at a point $(0, 0, z)$, $z \neq 0$, cannot be a linear combination of $S_1, \ldots, S_n$ which is a contradiction.

1.2. Definition of general analytic spaces.

Let U be an open subset of $\mathbf{C}^n$ (or $\mathbf{R}^n$) and let $\mathscr{I}$ be an arbitrary coherent sheaf of ideals in $\mathcal{O}_U$, the sheaf on U of germs of holomorphic (or real-analytic) functions. Then $V = \operatorname{supp} \mathcal{O}_U/\mathscr{I}$ is an analytic subset of U. The restriction of $\mathcal{O}_U/\mathscr{I}$ to V will be denoted by $\mathcal{O}_V$. It is, in general, not a subsheaf of $\mathscr{C}_V$. The definition of a general analytic space will be based on *local models* $(V, \mathcal{O}_V)$ of the type just constructed. Note that a model $(V, \mathcal{O}_V)$ is of the previously considered reduced type if and only if $\mathscr{I}$ is the sheaf of *all* germs of holomorphic functions vanishing on V. In the general case the set V does not determine the local model; one has to specify the structure sheaf.

Before proceeding to the formal definitions we shall look at a few examples.

Example 1. Let $U = \mathbf{C}$, $\mathscr{I}$ the sheaf of ideals generated by x^2. Here $V = \{0\}$ and $\mathcal{O}_{V,0} = \mathbf{C}\{x\}/(x^2)$ ($\mathbf{C}\{x\}$ denotes the space of converging power series in the variable x). Thus $\mathcal{O}_{V,0}$ is the space of " dual numbers " representable as $a + b\,\varepsilon$ where $a, b \in \mathbf{C}$ and $\varepsilon^2 = 0$, ε being the class of x. Evidently $\mathcal{O}_{V,0}$ cannot be a subring of the continuous functions on $\{0\}$. The only prime ideal of $\mathcal{O}_{V,0}$ is that generated by ε, hence the Krull dimension of $\mathcal{O}_{V,0}$ is 0. (Recall that the Krull dimension of a commutative ring A is the supremum of all numbers k such that there exists a strictly increasing chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_k$$

of prime ideals $\mathfrak{p}_j$.)

Example 2. Let V be the subspace of $\mathbf{C}^4$ defined by the requirement that $M(x) = \begin{pmatrix} x_1, & x_2 \\ x_3, & x_4 \end{pmatrix}$ be nilpotent. It can easily be seen that V can be defined by

$$\text{(1)} \qquad \det M(x) = tr\, M(x) = 0$$

and as well by

$$\text{(2)} \qquad M(x)^2 = 0\,.$$

Let $\mathscr{I}$ and $\mathscr{I}'$ denote the sheaves of ideals defined by (1) and (2), respectively. Explicitly this means that $\mathscr{I}$ is generated by $x_1 + x_4$, $x_1 x_4 - x_2 x_3$ and $\mathscr{I}'$ by $x_1^2 + x_2 x_3$, $x_2(x_1 + x_4)$, $x_3(x_1 + x_4)$, $x_2 x_3 + x_4^2$. It can be seen easily that $\mathscr{I}' \subset \mathscr{I}$ but this inclusion is strict since the generators of $\mathscr{I}'$ are all of the second degree. Thus the two ideals provide two different structure sheaves on the same set V.

Example 3. Let us note here some less pleasant properties of real local models. Take, for example, $U = \mathbf{R}^2$, and let $\mathscr{I}$ be the sheaf of ideals generated by x^2+y^2. Then $V=\{0\}$ and $\mathcal{O}_{V,0} = \mathbf{R}\,\{x,y\}/(x^2 + y^2)$. Here $\{0\}$ and (x,y) are prime ideals so the Krull dimension of $\mathcal{O}_{V,0}$ is at least 1 (in fact it is 1) and therefore not equal to the geometric dimension of V as in the complex example above.

To give the definition of a general analytic space we first introduce that of a ringed space:

Definition 1.2.1. A $\mathbf{C}$-*ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of local $\mathbf{C}$-algebras. (This means that $\mathcal{O}_{X,x}$ are local algebras for $x \in X$ arbitrary; all algebras are assumed to be commutative and with units; furthermore $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is assumed to be isomorphic to $\mathbf{C}$ where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$.)

Definition 1.2.2. A *morphism*

$$\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

of one $\mathbf{C}$-ringed space into another is a pair $\varphi = (\varphi_0, \varphi^1)$ where $\varphi_0 : X \to Y$

is a continuous map, and $\varphi^1 : \varphi_0^* (\mathcal{O}_Y) \to \mathcal{O}_X$ is a morphism of sheaves of **C**-algebras (morphisms of algebras are always assumed to be unitary).

R-ringed spaces and their morphisms are of course defined similarly.

Let $f \in \Gamma(U, \mathcal{O}_X)$ be a section of a **C**-ringed space $(X, \mathcal{O}_X)$ over an open set $U \subset X$. We may then define the *value* $f(x)$ of f at a point $x \in U$ as $f_x \in \mathcal{O}_{X,x}$ taken modulo $\mathfrak{m}_x$. Since $\mathcal{O}_{X,x}/\mathfrak{M}_x \cong \mathbf{C}, f(x)$ is a complex number.

Example 4. The values $f(x)$ of f do not determine f completely. In the example

$$(\{0\}, \mathbf{C}\{x\}/(x^2))$$

we considered earlier, the sections are given by dual numbers $a + b\,\varepsilon$, and since $\mathfrak{m}_0 = (\varepsilon)$, we get $f(0) = a$. Hence one has to consider also " higher order terms " to determine f.

If $\varphi : A \to B$ is a unitary homomorphism of local **C**-algebras it follows that $\varphi(\mathfrak{m}(A)) \subset \mathfrak{m}(B)$, $\mathfrak{m}(A)$ denoting the maximal ideal of A; in other words, the homomorphism is local. To see this, let us note that $\varphi^{-1}(\mathfrak{m}(B))$ is an ideal of A and that φ induces an injective (in fact bijective) map of $A/\varphi^{-1}(\mathfrak{m}(B))$ into $B/\mathfrak{m}(B) \cong \mathbf{C}$, hence $\varphi^{-1}(\mathfrak{m}(B))$ is either all of A or a maximal ideal in A, but the first possibility is ruled out by the condition $\varphi(1) = 1$. It therefore follows that $\varphi^{-1}(\mathfrak{m}(B)) = \mathfrak{m}(A)$, hence $\mathfrak{m}(B) \supset \varphi(\mathfrak{m}(A))$. A consequence of this is that a morphism $(\varphi_0, \varphi^1) : (X, \mathcal{O}_X) \to$ $\to (Y, \mathcal{O}_Y)$ of ringed spaces preserves the values of the sections, in symbols

$$(*) \qquad \varphi^1(f)(x) = f(\varphi_0(x)),$$

if $x \in X$ and f is a section of $\mathcal{O}_Y$ over some open set containing $\varphi_0(x)$. Thus φ^1 and φ_0 are related, but our example " the double point " shows that φ^1 is not in general determined by φ_0:

Example 5. Let X be the **C**-ringed space $(\{0\}, \mathbf{C}\{x\}/(x^2))$, and let $Y = \mathbf{C}^n$ regarded as a **C**-ringed space (with the sheaf $\mathcal{O}_{\mathbf{C}^n}$ of germs of holomorphic functions). Let (φ_0, φ^1) be a morphism of X into Y with $\varphi(0) = 0$, say. Then φ^1 is a homomorphism.

$$\varphi^1 : \mathbf{C}\{y_1, \dots, y_n\} \to \mathbf{C}\{x\}/(x^2).$$

Let us express $\varphi^1(f)$ as $a(f) + \varepsilon b(f)$ (see the example[1]). Since the maximal ideal of $\mathbf{C}\{x\}/(x^2)$ is (ε), the value of $\varphi^1(f)$ is $a(f)$. From (*) it follows that

$$a(f) = \varphi^1(f)(0) = f(0) = \varphi_0^*(f).$$

Thus φ_0 determines the " zero order term " of $\varphi^1(f)(0)$. As to the properties of $b(f)$, it follows from the multiplication rule $\varepsilon^2 = 0$ that

$$b(fg) = f(0)\,b(g) + g(0)\,b(f),$$

hence that b is a tangent vector, or derivation, at $O \in \mathbf{C}^n$.

It is clear what the restriction of a ringed space $(X, \mathcal{O}_X)$ to an open subset U of X should mean: it is the ringed space $(U, \mathcal{O}_X | U)$. The following definition therefore makes sense.

Defintion 1.2.3. (Grothendieck [4]). A **C**-*analytic space* is a **C**-ringed space $(X, \mathcal{O}_X)$ where every point $x \in X$ has an open neighborhood U such that the restriction of $(X, \mathcal{O}_X)$ to U is isomorphic (in the sense of **C**-ringed spaces) to a model (defined at the beginning of Section 1.2.). A morphism of analytic spaces is a morphism in the sense of ringed spaces.

We shall determine the morphisms of $(X, \mathcal{O}_X)$ into $(Y, \mathcal{O}_Y)$ in two important special cases, viz. when $(X, \mathcal{O}_X)$ is arbitrary and $(Y, \mathcal{O}_Y)$ is either $\mathbf{C}^n$ or defined by the vanishing of finitely many analytic functions in an open set in $\mathbf{C}^n$.

Proposition 1.2.4. The morphisms of a **C**-analytic space $(X, \mathcal{O}_X)$ into $\mathbf{C}^n$ can be identified in a natural way with $\Gamma(X, \mathcal{O}_X)^n$ (or $\Gamma(X, \mathcal{O}_X^n)$).

Proof. Given a morphism $\varphi = (\varphi_0, \varphi^1)$ of $(X, \mathcal{O}_X)$ into $\mathbf{C}^n$ we shall construct an n-tuple $T\varphi = (f_1, \ldots, f_n)$ of sections of $\mathcal{O}_X$.

To define T we proceed as follows. Let $x \in X$. Recall that φ^1 maps $\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)}$ into $\mathcal{O}_{X,x}$. Define $(f_j)_x \in \mathcal{O}_{X,x}$ as the image under φ^1 of the germ at $\varphi_0(x)$ of the coordinate function y_j in $\mathbf{C}^n$. Somewhat less precisely, $f_j = \varphi^1(y_j)$. This defines $f_j \in \Gamma(X, \mathcal{O}_X)$ and hence T.

T is injective. For $T\varphi = T\psi$ means that

$$\mathcal{O}_{\mathbf{C}^n,\varphi_0(x)} \xrightarrow{\varphi^1} \mathcal{O}_{X,x}$$

and

$$\mathcal{O}_{\mathbf{C}^n,\psi_0(x)} \xrightarrow{\psi^1} \mathcal{O}_{X,x}$$

agree on the germs of the coordinate functions. Since in particular the *values* of the sections are preserved, i.e. φ^1 and ψ^1 are the identities modulo the respective maximal ideals, the *values* of the coordinates at $\varphi_0(x)$ and $\psi_0(x)$ must agree, hence $\varphi_0 = \psi_0$. Furthermore, since φ^1 and ψ^1 are homomorphisms, they agree on all polynomials. But the polynomials form a dense set in $\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)}$ and $\mathcal{O}_{X,x}$ is separated (for the Krull topology) in virtue of the Krull theorem (see Appendix). Finally φ^1 and ψ^1 are continuous maps since $\varphi^1(\mathfrak{m}(\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)})) \subset \mathfrak{m}(\mathcal{O}_{X,x})$. Now if two continuous maps from a topological space to a separated topological space coincide on a dense subset, then they are equal. Hence T is injective.

T is surjective. For if $(f_1, \ldots, f_n) \in \Gamma(X, \mathcal{O}_X)^n$ is given we first define $\varphi_0 : X \to \mathbf{C}^n$ by $\varphi_0(x) = (f_1(x), \ldots, f_n(x))$ (recall that $f(x)$ is the equivalence class of f_x modulo $\mathfrak{m}(\mathcal{O}_{X,x})$). Then we may define

$$\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)} \xrightarrow{\varphi^1} \mathcal{O}_{X,x}$$

first on the constants by the requirement that $\varphi^1(1) = 1$; then on the germs of the coordinates by putting $\varphi^1(y_j) = f_j$; next on the polynomials by the multiplicative property of homomorphisms and finally, by uniform continuity, in all of $\mathcal{O}_{\mathbf{C}^n, \varphi_0(x)}$. (Note that we have again used the fact that $\mathcal{O}_{X,x}$ is separated in the last step).

Before the next proposition we introduce the notion of special model. A *special model* $(V, \mathcal{O}_V)$ is a model (see the beginning of this section) where the ideal $\mathcal{I}$ is generated by the components of a vector-valued analytic function $f : U \to F$ where U is open in $\mathbf{C}^n$ and F is a finite-dimensional complex linear space. Here V is the set of zeros of f and $\mathcal{O}_V$ is the restriction of $\mathcal{O}_U/\mathcal{I}$ to its own support.

Proposition 1.2.5. Let $(X, \mathcal{O}_X)$ be an arbitrary analytic space and $(Y, \mathcal{O}_Y)$ a special model defined by the vanishing of a vector-valued analytic function $g_0 : U \to G$. Then there is a bijection between the morphisms $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and those morphisms $\psi : (X, \mathcal{O}_X) \to (U, \mathcal{O}_U)$ which satisfy $g \circ \psi = 0$, where $g = (g_0, g^1) : (U, \mathcal{O}_U) \to (G, \mathcal{O}_G)$ is the morphism of analytic spaces defined by g_0.

The proof will be left as an exercise to the reader.

On the other hand, the morphisms $(X, \mathcal{O}_X) \to (U, \mathcal{O}_U)$ are obviously these morphisms $(X, \mathcal{O}_X) \to \mathbf{C}^n$ such that $\varphi_0(X) \subset U$; this fact, combined with propositions 1.2.4. and 1.2.5. gives the description of the morphisms: $(X, \mathcal{O}_X) \to$ (special model).

We end this section with the definition of analytic subspace. First we state

Definition. 1.2.6. An *analytic coherent sheaf* on an analytic space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$- modules such that every $x \in X$ has an open neighborhood U over which there exists an exact sequence

$$\mathcal{O}_X^q \mid U \to \mathcal{O}_X^p \mid U \to \mathcal{F} \mid U \to 0 .$$

Definition. 1.2.7. A *closed analytic subspace* of an analytic space $(X, \mathcal{O}_X)$ is a ringed space $(Y, \mathcal{O}_Y)$ where $Y = \operatorname{supp}(\mathcal{O}_X/\mathcal{I})$ and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I} \mid Y$

for some coherent sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_X$. An *open analytic subspace* of $(X, \mathcal{O}_X)$ is just a restriction $(U, \mathcal{O}_X \mid U)$, U open in X. An *analytic subspace* of an analytic space $(X, \mathcal{O}_X)$ is a closed analytic subspace $(Y, \mathcal{O}_Y)$ of the open analytic subspace $(\complement\, \bar{Y} \cup Y, \mathcal{O}_{\complement \bar{Y} \cup Y})$ of $(X, \mathcal{O}_X)$, provided $\complement\, \bar{Y} \cup Y$ is indeed open in X, i.e. Y is locally closed in X.

Examples. The " single point " $(0, \mathbf{C})$ is an analytic subspace of the " double point " $(0, \mathbf{C}\{x\}/(x^2))$, but not conversely. The double point is, however, a closed analytic subspace of, e.g., $(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. A " point " of an analytic space will always mean a single point embedded in $(X, \mathcal{O}_X)$ by means of a map $(0, \mathbf{C}) \to (X, \mathcal{O}_X)$.

1.3. Operations on analytic spaces.

In this section we shall write X for the analytic space $(X, \mathcal{O}_X)$.

a) *Product.* By a general definition in the theory of categories, a product of two analytic spaces X, X' is a triple (Z, π, π') where Z is an analytic space and $\pi : Z \to X$, $\pi' : Z \to X'$ are two morphisms with the following property:

Given any analytic space Y and any pair $f : Y \to X, f' : Y \to X'$ of morphisms there exists a unique morphism $g : Y \to Z$ such that $f = \pi \circ g$, $f' = \pi' \circ g$.

For example, the product of $\mathbf{C}^p$ and $\mathbf{C}^q$ is $\mathbf{C}^{p+q}$, according to proposition 1.2.4.

We shall see that a product of analytic spaces always exists. The uniqueness of g clearly implies the uniqueness of the product (Z, π, π') up to isomorphism; we denote one such Z by $X \times X'$.

To prove that the product always exists, let us suppose first that X and X' are special models, i.e. X is defined by a triple (U, f, F) where U is open in $\mathbf{C}^n$, F is a finite-dimensional complex linear space, and $f : U \to F$ is an analytic map; similarly for X'. We claim that the special model Z defined by $(U \times U', f \times f', F \times F')$ is a product. Indeed, from the description of the morphisms into a special model provided by Proposition 1.2.5. it follows that we have natural maps $\pi : Z \to X$, $\pi' : Z \to X'$ induced by the projections $U \times U' \to U$, $U \times U' \to U'$. Also, if f: $Y \to X$ and f': $Y \to X'$ are given, $g : Y \to Z$ is determined by

$$\begin{tikzcd}[row sep=small] & X \arrow[r] & U \arrow[dr] & \\ Y \arrow[ur, "f"] \arrow[dr, "f'"'] & & & U \times U' . \\ & X' \arrow[r] & U' \arrow[ur] & \end{tikzcd}$$