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It should be noted that a morphism may be bijective and bicontinuous and still fail to be an isomorphism. As an example we consider the map $t \rightarrow (t^2, t^3)$ of $X = \mathbf{C}$ into the space Y of all pairs (x, y) satisfying $x^3 - y^2 = 0$. This is a bijective and bicontinuous morphism, but its inverse ψ is no morphism since $\psi^* f_0 \notin \mathcal{O}_{Y,0}$ if $f(t) = t$.

Real analytic sets are not as well behaved as complex ones. To illustrate this we consider "Cartan's umbrella" which is the subset of \mathbf{R}^3 defined by the equation $z(x^2 + y^2) - x^3 = 0$. Its intersection with the plane $z = 1$ has an isolated double point at $(0, 0, 1)$ and so it has a stick (the z -axis) joining the rest of the "umbrella" at the origin. Here the Oka-Cartan theorem fails. Indeed, suppose that the sheaf \mathcal{J} of germs of real-analytic functions vanishing on the umbrella were generated by sections $s_1, \dots, s_n \in \Gamma(U, \mathcal{J})$ over some neighborhood U of the origin. Then, denoting by f_1, \dots, f_n the corresponding real-analytic functions in U , we find (using a complexification and the Nullstellensatz for principal ideals) that every f_j is a multiple of $z(x^2 + y^2) - x^3$ for it can easily be seen that this polynomial defines in the complex domain an irreducible germ at the origin. Hence the germ in \mathcal{J} defined by the coordinate function x at a point $(0, 0, z)$, $z \neq 0$, cannot be a linear combination of S_1, \dots, S_n which is a contradiction.

1.2. Definition of general analytic spaces.

Let U be an open subset of \mathbf{C}^n (or \mathbf{R}^n) and let \mathcal{J} be an arbitrary coherent sheaf of ideals in \mathcal{O}_U , the sheaf on U of germs of holomorphic (or real-analytic) functions. Then $V = \text{supp } \mathcal{O}_U/\mathcal{J}$ is an analytic subset of U . The restriction of $\mathcal{O}_U/\mathcal{J}$ to V will be denoted by \mathcal{O}_V . It is, in general, not a subsheaf of \mathcal{O}_V . The definition of a general analytic space will be based on *local models* (V, \mathcal{O}_V) of the type just constructed. Note that a model (V, \mathcal{O}_V) is of the previously considered reduced type if and only if \mathcal{J} is the sheaf of *all* germs of holomorphic functions vanishing on V . In the general case the set V does not determine the local model; one has to specify the structure sheaf.

Before proceeding to the formal definitions we shall look at a few examples.

Example 1. Let $U = \mathbf{C}$, \mathcal{J} the sheaf of ideals generated by x^2 . Here $V = \{0\}$ and $\mathcal{O}_{V,0} = \mathbf{C}\{x\}/(x^2)$ ($\mathbf{C}\{x\}$ denotes the space of converging power series in the variable x). Thus $\mathcal{O}_{V,0}$ is the space of "dual numbers" representable as $a + b\varepsilon$ where $a, b \in \mathbf{C}$ and $\varepsilon^2 = 0$, ε being the class of x . Evidently $\mathcal{O}_{V,0}$ cannot be a subring of the continuous functions on $\{0\}$. The

only prime ideal of $\mathcal{O}_{V,0}$ is that generated by ε , hence the Krull dimension of $\mathcal{O}_{V,0}$ is 0. (Recall that the Krull dimension of a commutative ring A is the supremum of all numbers k such that there exists a strictly increasing chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_k$$

of prime ideals \mathfrak{p}_j .)

Example 2. Let V be the subspace of \mathbb{C}^4 defined by the requirement that $M(x) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ be nilpotent. It can easily be seen that V can be defined by

$$(1) \quad \det M(x) = \operatorname{tr} M(x) = 0$$

and as well by

$$(2) \quad M(x)^2 = 0.$$

Let \mathcal{I} and \mathcal{I}' denote the sheaves of ideals defined by (1) and (2), respectively. Explicitly this means that \mathcal{I} is generated by $x_1 + x_4$, $x_1 x_4 - x_2 x_3$ and \mathcal{I}' by $x_1^2 + x_2 x_3$, $x_2(x_1 + x_4)$, $x_3(x_1 + x_4)$, $x_2 x_3 + x_4^2$. It can be seen easily that $\mathcal{I}' \subset \mathcal{I}$ but this inclusion is strict since the generators of \mathcal{I}' are all of the second degree. Thus the two ideals provide two different structure sheaves on the same set V .

Example 3. Let us note here some less pleasant properties of real local models. Take, for example, $U = \mathbb{R}^2$, and let \mathcal{I} be the sheaf of ideals generated by $x^2 + y^2$. Then $V = \{0\}$ and $\mathcal{O}_{V,0} = \mathbb{R}\{x, y\}/(x^2 + y^2)$. Here $\{0\}$ and (x, y) are prime ideals so the Krull dimension of $\mathcal{O}_{V,0}$ is at least 1 (in fact it is 1) and therefore not equal to the geometric dimension of V as in the complex example above.

To give the definition of a general analytic space we first introduce that of a ringed space:

Definition 1.2.1. A **C**-ringed space is a pair (X, \mathcal{O}_X) where X is a topological space and \mathcal{O}_X is a sheaf of local **C**-algebras. (This means that $\mathcal{O}_{X,x}$ are local algebras for $x \in X$ arbitrary; all algebras are assumed to be commutative and with units; furthermore $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is assumed to be isomorphic to **C** where \mathfrak{m}_x is the maximal ideal of $\mathcal{O}_{X,x}$.)

Definition 1.2.2. A *morphism*

$$\varphi : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$$

of one **C**-ringed space into another is a pair $\varphi = (\varphi_0, \varphi^1)$ where $\varphi_0 : X \rightarrow Y$

is a continuous map, and $\varphi^1 : \varphi_0^* (\mathcal{O}_Y) \rightarrow \mathcal{O}_X$ is a morphism of sheaves of \mathbf{C} -algebras (morphisms of algebras are always assumed to be unitary).

\mathbf{R} -ringed spaces and their morphisms are of course defined similarly.

Let $f \in \Gamma(U, \mathcal{O}_X)$ be a section of a \mathbf{C} -ringed space (X, \mathcal{O}_X) over an open set $U \subset X$. We may then define the *value* $f(x)$ of f at a point $x \in U$ as $f_x \in \mathcal{O}_{X,x}$ taken modulo \mathfrak{m}_x . Since $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbf{C}$, $f(x)$ is a complex number.

Example 4. The values $f(x)$ of f do not determine f completely. In the example

$$(\{0\}, \mathbf{C}\{x\}/(x^2))$$

we considered earlier, the sections are given by dual numbers $a + b\varepsilon$, and since $\mathfrak{m}_0 = (\varepsilon)$, we get $f(0) = a$. Hence one has to consider also “higher order terms” to determine f .

If $\varphi : A \rightarrow B$ is a unitary homomorphism of local \mathbf{C} -algebras it follows that $\varphi(\mathfrak{m}(A)) \subset \mathfrak{m}(B)$, $\mathfrak{m}(A)$ denoting the maximal ideal of A ; in other words, the homomorphism is local. To see this, let us note that $\varphi^{-1}(\mathfrak{m}(B))$ is an ideal of A and that φ induces an injective (in fact bijective) map of $A/\varphi^{-1}(\mathfrak{m}(B))$ into $B/\mathfrak{m}(B) \cong \mathbf{C}$, hence $\varphi^{-1}(\mathfrak{m}(B))$ is either all of A or a maximal ideal in A , but the first possibility is ruled out by the condition $\varphi(1) = 1$. It therefore follows that $\varphi^{-1}(\mathfrak{m}(B)) = \mathfrak{m}(A)$, hence $\mathfrak{m}(B) \supset \varphi(\mathfrak{m}(A))$. A consequence of this is that a morphism $(\varphi_0, \varphi^1) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of ringed spaces preserves the values of the sections, in symbols

$$(*) \quad \varphi^1(f)(x) = f(\varphi_0(x)),$$

if $x \in X$ and f is a section of \mathcal{O}_Y over some open set containing $\varphi_0(x)$. Thus φ^1 and φ_0 are related, but our example “the double point” shows that φ^1 is not in general determined by φ_0 :

Example 5. Let X be the \mathbf{C} -ringed space $(\{0\}, \mathbf{C}\{x\}/(x^2))$, and let $Y = \mathbf{C}^n$ regarded as a \mathbf{C} -ringed space (with the sheaf $\mathcal{O}_{\mathbf{C}^n}$ of germs of holomorphic functions). Let (φ_0, φ^1) be a morphism of X into Y with $\varphi(0) = 0$, say. Then φ^1 is a homomorphism.

$$\varphi^1 : \mathbf{C}\{y_1, \dots, y_n\} \rightarrow \mathbf{C}\{x\}/(x^2).$$

Let us express $\varphi^1(f)$ as $a(f) + \varepsilon b(f)$ (see the example¹). Since the maximal ideal of $\mathbf{C}\{x\}/(x^2)$ is (ε) , the value of $\varphi^1(f)$ is $a(f)$. From (*) it follows that

$$a(f) = \varphi^1(f)(0) = f(0) = \varphi_0^*(f).$$

Thus φ_0 determines the “zero order term” of $\varphi^1(f)(0)$. As to the proper-

ties of $b(f)$, it follows from the multiplication rule $\varepsilon^2 = 0$ that

$$b(fg) = f(0)b(g) + g(0)b(f),$$

hence that b is a tangent vector, or derivation, at $O \in \mathbb{C}^n$.

It is clear what the restriction of a ringed space (X, \mathcal{O}_X) to an open subset U of X should mean: it is the ringed space $(U, \mathcal{O}_X|_U)$. The following definition therefore makes sense.

Definition 1.2.3. (Grothendieck [4]). A \mathbb{C} -analytic space is a \mathbb{C} -ringed space (X, \mathcal{O}_X) where every point $x \in X$ has an open neighborhood U such that the restriction of (X, \mathcal{O}_X) to U is isomorphic (in the sense of \mathbb{C} -ringed spaces) to a model (defined at the beginning of Section 1.2.). A morphism of analytic spaces is a morphism in the sense of ringed spaces.

We shall determine the morphisms of (X, \mathcal{O}_X) into (Y, \mathcal{O}_Y) in two important special cases, viz. when (X, \mathcal{O}_X) is arbitrary and (Y, \mathcal{O}_Y) is either \mathbb{C}^n or defined by the vanishing of finitely many analytic functions in an open set in \mathbb{C}^n .

Proposition 1.2.4. The morphisms of a \mathbb{C} -analytic space (X, \mathcal{O}_X) into \mathbb{C}^n can be identified in a natural way with $\Gamma(X, \mathcal{O}_X)^n$ (or $\Gamma(X, \mathcal{O}_X^n)$).

Proof. Given a morphism $\varphi = (\varphi_0, \varphi^1)$ of (X, \mathcal{O}_X) into \mathbb{C}^n we shall construct an n -tuple $T\varphi = (f_1, \dots, f_n)$ of sections of \mathcal{O}_X .

To define T we proceed as follows. Let $x \in X$. Recall that φ^1 maps $\mathcal{O}_{\mathbb{C}^n, \varphi_0(x)}$ into $\mathcal{O}_{X, x}$. Define $(f_j)_x \in \mathcal{O}_{X, x}$ as the image under φ^1 of the germ at $\varphi_0(x)$ of the coordinate function y_j in \mathbb{C}^n . Somewhat less precisely, $f_j = \varphi^1(y_j)$. This defines $f_j \in \Gamma(X, \mathcal{O}_X)$ and hence T .

T is injective. For $T\varphi = T\psi$ means that

$$\mathcal{O}_{\mathbb{C}^n, \varphi_0(x)} \xrightarrow{\varphi^1} \mathcal{O}_{X, x}$$

and

$$\mathcal{O}_{\mathbb{C}^n, \psi_0(x)} \xrightarrow{\psi^1} \mathcal{O}_{X, x}$$

agree on the germs of the coordinate functions. Since in particular the *values* of the sections are preserved, i.e. φ^1 and ψ^1 are the identities modulo the respective maximal ideals, the *values* of the coordinates at $\varphi_0(x)$ and $\psi_0(x)$ must agree, hence $\varphi_0 = \psi_0$. Furthermore, since φ^1 and ψ^1 are homomorphisms, they agree on all polynomials. But the polynomials form a dense set in $\mathcal{O}_{\mathbb{C}^n, \varphi_0(x)}$ and $\mathcal{O}_{X, x}$ is separated (for the Krull topology) in virtue of the Krull theorem (see Appendix). Finally φ^1 and ψ^1 are continuous maps since $\varphi^1(\mathfrak{m}(\mathcal{O}_{\mathbb{C}^n, \varphi_0(x)})) \subset \mathfrak{m}(\mathcal{O}_{X, x})$. Now if two continuous maps

from a topological space to a separated topological space coincide on a dense subset, then they are equal. Hence T is injective.

T is surjective. For if $(f_1, \dots, f_n) \in \Gamma(X, \mathcal{O}_X)^n$ is given we first define $\varphi_0 : X \rightarrow \mathbb{C}^n$ by $\varphi_0(x) = (f_1(x), \dots, f_n(x))$ (recall that $f(x)$ is the equivalence class of f_x modulo $\mathfrak{m}(\mathcal{O}_{X,x})$). Then we may define

$$\mathcal{O}_{\mathbb{C}^n, \varphi_0(x)} \xrightarrow{\varphi^1} \mathcal{O}_{X,x}$$

first on the constants by the requirement that $\varphi^1(1) = 1$; then on the germs of the coordinates by putting $\varphi^1(y_j) = f_j$; next on the polynomials by the multiplicative property of homomorphisms and finally, by uniform continuity, in all of $\mathcal{O}_{\mathbb{C}^n, \varphi_0(x)}$. (Note that we have again used the fact that $\mathcal{O}_{X,x}$ is separated in the last step).

Before the next proposition we introduce the notion of special model. A *special model* (V, \mathcal{O}_V) is a model (see the beginning of this section) where the ideal \mathcal{I} is generated by the components of a vector-valued analytic function $f : U \rightarrow F$ where U is open in \mathbb{C}^n and F is a finite-dimensional complex linear space. Here V is the set of zeros of f and \mathcal{O}_V is the restriction of $\mathcal{O}_U/\mathcal{I}$ to its own support.

Proposition 1.2.5. Let (X, \mathcal{O}_X) be an arbitrary analytic space and (Y, \mathcal{O}_Y) a special model defined by the vanishing of a vector-valued analytic function $g_0 : U \rightarrow G$. Then there is a bijection between the morphisms $\varphi : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ and those morphisms $\psi : (X, \mathcal{O}_X) \rightarrow (U, \mathcal{O}_U)$ which satisfy $g \circ \psi = 0$, where $g = (g_0, g^1) : (U, \mathcal{O}_U) \rightarrow (G, \mathcal{O}_G)$ is the morphism of analytic spaces defined by g_0 .

The proof will be left as an exercise to the reader.

On the other hand, the morphisms $(X, \mathcal{O}_X) \rightarrow (U, \mathcal{O}_U)$ are obviously these morphisms $(X, \mathcal{O}_X) \rightarrow \mathbb{C}^n$ such that $\varphi_0(X) \subset U$; this fact, combined with propositions 1.2.4. and 1.2.5. gives the description of the morphisms: $(X, \mathcal{O}_X) \rightarrow$ (special model).

We end this section with the definition of analytic subspace. First we state

Definition. 1.2.6. An *analytic coherent sheaf* on an analytic space (X, \mathcal{O}_X) is a sheaf \mathcal{F} of \mathcal{O}_X -modules such that every $x \in X$ has an open neighborhood U over which there exists an exact sequence

$$\mathcal{O}_X^q|_U \rightarrow \mathcal{O}_X^p|_U \rightarrow \mathcal{F}|_U \rightarrow 0.$$

Definition. 1.2.7. A *closed analytic subspace* of an analytic space (X, \mathcal{O}_X) is a ringed space (Y, \mathcal{O}_Y) where $Y = \text{supp}(\mathcal{O}_X/\mathcal{I})$ and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}|_Y$

for some coherent sheaf \mathcal{I} of ideals of \mathcal{O}_X . An *open analytic subspace* of (X, \mathcal{O}_X) is just a restriction $(U, \mathcal{O}_X|_U)$, U open in X . An *analytic subspace* of an analytic space (X, \mathcal{O}_X) is a closed analytic subspace (Y, \mathcal{O}_Y) of the open analytic subspace $(\mathbb{C} \bar{Y} \cup Y, \mathcal{O}_{\mathbb{C} \bar{Y} \cup Y})$ of (X, \mathcal{O}_X) , provided $\mathbb{C} \bar{Y} \cup Y$ is indeed open in X , i.e. Y is locally closed in X .

Examples. The “single point” $(0, \mathbb{C})$ is an analytic subspace of the “double point” $(0, \mathbb{C} \{x\}/(x^2))$, but not conversely. The double point is, however, a closed analytic subspace of, e.g., $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$. A “point” of an analytic space will always mean a single point embedded in (X, \mathcal{O}_X) by means of a map $(0, \mathbb{C}) \rightarrow (X, \mathcal{O}_X)$.

1.3. Operations on analytic spaces.

In this section we shall write X for the analytic space (X, \mathcal{O}_X) .

a) *Product.* By a general definition in the theory of categories, a product of two analytic spaces X, X' is a triple (Z, π, π') where Z is an analytic space and $\pi : Z \rightarrow X, \pi' : Z \rightarrow X'$ are two morphisms with the following property:

Given any analytic space Y and any pair $f : Y \rightarrow X, f' : Y \rightarrow X'$ of morphisms there exists a unique morphism $g : Y \rightarrow Z$ such that $f = \pi \circ g, f' = \pi' \circ g$.

For example, the product of \mathbb{C}^p and \mathbb{C}^q is \mathbb{C}^{p+q} , according to proposition 1.2.4.

We shall see that a product of analytic spaces always exists. The uniqueness of g clearly implies the uniqueness of the product (Z, π, π') up to isomorphism; we denote one such Z by $X \times X'$.

To prove that the product always exists, let us suppose first that X and X' are special models, i.e. X is defined by a triple (U, f, F) where U is open in \mathbb{C}^n, F is a finite-dimensional complex linear space, and $f : U \rightarrow F$ is an analytic map; similarly for X' . We claim that the special model Z defined by $(U \times U', f \times f', F \times F')$ is a product. Indeed, from the description of the morphisms into a special model provided by Proposition 1.2.5. it follows that we have natural maps $\pi : Z \rightarrow X, \pi' : Z \rightarrow X'$ induced by the projections $U \times U' \rightarrow U, U \times U' \rightarrow U'$. Also, if $f : Y \rightarrow X$ and $f' : Y \rightarrow X'$ are given, $g : Y \rightarrow Z$ is determined by

$$\begin{array}{ccccc} Y & \xrightarrow{f} & X & \rightarrow & U \\ & & & \searrow & \\ & & & & U \times U' \\ & \searrow & X' & \rightarrow & U' \\ & & & \nearrow & \end{array}$$