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1.1 Reduced analytic spaces.

To prepare for the general definition we shall first introduce reduced analytic spaces and their local models. Let U be an open set in \mathbb{C}^n and V an analytic subset of U . The sheaf \mathcal{I} on U of all germs of holomorphic functions vanishing on V is coherent by the Oka-Cartan theorem (for a proof, see e.g. Narasimhan [9, Theorem 5, p. 77]). The support of $\mathcal{O}_U/\mathcal{I}$ is V , and we shall denote by \mathcal{O}_V the restriction of $\mathcal{O}_U/\mathcal{I}$ to V (\mathcal{O}_U denotes the sheaf on U of germs of holomorphic functions). The *local models for reduced analytic spaces* shall be the pairs (V, \mathcal{O}_V) . Obviously we may consider \mathcal{O}_V as a subsheaf of \mathcal{C}_V , the sheaf on V of germs of continuous functions.

Definition 1.1.1. A *reduced analytic space* is a pair (X, \mathcal{O}_X) where X is a topological space (not necessarily separated) and \mathcal{O}_X is a sheaf of sub- \mathbb{C} -algebras of \mathcal{C}_X which is locally isomorphic to a local model.

To be explicit, the last property means that every point $x \in X$ has a neighborhood U such that for some local model (V, \mathcal{O}_V) there is a homeomorphism $\varphi : U \rightarrow V$ with the property that for $y \in U, f \in \mathcal{C}_{U,y}$ belongs to $\mathcal{O}_{U,y}$ if and only if $f = g \circ \varphi$ for some germ $g \in \mathcal{O}_{V,\varphi(y)}$.

As a common abuse of language we shall sometimes write X instead of (X, \mathcal{O}_X) .

Reduced analytic spaces need not be separated. Consider for example the disjoint union of two copies of \mathbb{C} , with all points except the origins identified. This topological space is in a natural way a reduced analytic space, indeed a complex manifold.

Reduced analytic spaces were introduced by Cartan-Serre (under the name of “analytic spaces”).

Definition 1.1.2. A *morphism*, or *holomorphic map* of one reduced analytic space (X, \mathcal{O}_X) into another, (Y, \mathcal{O}_Y) , is a continuous map $\varphi : X \rightarrow Y$ such that $\varphi^*(\mathcal{O}_{Y,\varphi(x)}) \subset \mathcal{O}_X$ for all $x \in X$.

This definition, of course, gives us also the notion of isomorphism of reduced analytic spaces, which we have already used in a special case in Definition 1.1.1.

Example 1. If X, Y are complex manifolds, the morphisms of (X, \mathcal{O}_X) into (Y, \mathcal{O}_Y) are the holomorphic maps $X \rightarrow Y$ in the usual sense.

Example 2. The morphisms of (X, \mathcal{O}_X) into \mathbb{C} , regarded as a reduced analytic space $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$, can be identified with the sections $\Gamma(X, \mathcal{O}_X)$.

Example 3. The morphisms of (X, \mathcal{O}_X) into \mathbb{C}^n can be identified with n -tuples of sections of \mathcal{O}_X , or, again, with sections of \mathcal{O}_X^n .

It should be noted that a morphism may be bijective and bicontinuous and still fail to be an isomorphism. As an example we consider the map $t \rightarrow (t^2, t^3)$ of $X = \mathbf{C}$ into the space Y of all pairs (x, y) satisfying $x^3 - y^2 = 0$. This is a bijective and bicontinuous morphism, but its inverse ψ is no morphism since $\psi^* f_0 \notin \mathcal{O}_{Y,0}$ if $f(t) = t$.

Real analytic sets are not as well behaved as complex ones. To illustrate this we consider "Cartan's umbrella" which is the subset of \mathbf{R}^3 defined by the equation $z(x^2 + y^2) - x^3 = 0$. Its intersection with the plane $z = 1$ has an isolated double point at $(0, 0, 1)$ and so it has a stick (the z -axis) joining the rest of the "umbrella" at the origin. Here the Oka-Cartan theorem fails. Indeed, suppose that the sheaf \mathcal{J} of germs of real-analytic functions vanishing on the umbrella were generated by sections $s_1, \dots, s_n \in \Gamma(U, \mathcal{J})$ over some neighborhood U of the origin. Then, denoting by f_1, \dots, f_n the corresponding real-analytic functions in U , we find (using a complexification and the Nullstellensatz for principal ideals) that every f_j is a multiple of $z(x^2 + y^2) - x^3$ for it can easily be seen that this polynomial defines in the complex domain an irreducible germ at the origin. Hence the germ in \mathcal{J} defined by the coordinate function x at a point $(0, 0, z)$, $z \neq 0$, cannot be a linear combination of S_1, \dots, S_n which is a contradiction.

1.2. Definition of general analytic spaces.

Let U be an open subset of \mathbf{C}^n (or \mathbf{R}^n) and let \mathcal{J} be an arbitrary coherent sheaf of ideals in \mathcal{O}_U , the sheaf on U of germs of holomorphic (or real-analytic) functions. Then $V = \text{supp } \mathcal{O}_U/\mathcal{J}$ is an analytic subset of U . The restriction of $\mathcal{O}_U/\mathcal{J}$ to V will be denoted by \mathcal{O}_V . It is, in general, not a subsheaf of \mathcal{O}_V . The definition of a general analytic space will be based on *local models* (V, \mathcal{O}_V) of the type just constructed. Note that a model (V, \mathcal{O}_V) is of the previously considered reduced type if and only if \mathcal{J} is the sheaf of *all* germs of holomorphic functions vanishing on V . In the general case the set V does not determine the local model; one has to specify the structure sheaf.

Before proceeding to the formal definitions we shall look at a few examples.

Example 1. Let $U = \mathbf{C}$, \mathcal{J} the sheaf of ideals generated by x^2 . Here $V = \{0\}$ and $\mathcal{O}_{V,0} = \mathbf{C}\{x\}/(x^2)$ ($\mathbf{C}\{x\}$ denotes the space of converging power series in the variable x). Thus $\mathcal{O}_{V,0}$ is the space of "dual numbers" representable as $a + b\varepsilon$ where $a, b \in \mathbf{C}$ and $\varepsilon^2 = 0$, ε being the class of x . Evidently $\mathcal{O}_{V,0}$ cannot be a subring of the continuous functions on $\{0\}$. The