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1.1 Reduced analytic spaces.

To prepare for the general definition we shall first introduce reduced analytic spaces and their local models. Let U be an open set in $\mathbf{C}^n$ and V an analytic subset of U. The sheaf $\mathscr{I}$ on U of all germs of holomorphic functions vanishing on V is coherent by the Oka-Cartan theorem (for a proof, see e.g. Narasimhan [9, Theorem 5, p. 77]). The support of $\mathcal{O}_U/\mathscr{I}$ is V, and we shall denote by $\mathcal{O}_V$ the restriction of $\mathcal{O}_V/\mathscr{I}$ to V ($\mathcal{O}_U$ denotes the sheaf on U of germs of holomorphic functions). The *local models for reduced analytic spaces* shall be the pairs $(V, \mathcal{O}_V)$. Obviously we may consider $\mathcal{O}_V$ as a subsheaf of $\mathscr{C}_V$, the sheaf on V of germs of continuous functions.

Definition 1.1.1. A *reduced analytic space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space (not necessarily separated) and $\mathcal{O}_X$ is a sheaf of sub-$\mathbf{C}$-algebras of $\mathscr{C}_X$ which is locally isomorphic to a local model.

To be explicit, the last property means that every point $x \in X$ has a neighborhood U such that for some local model $(V, \mathcal{O}_V)$ there is a homeomorphism $\varphi : U \to V$ with the property that for $y \in U, f \in \mathscr{C}_{U,y}$ belongs to $\mathcal{O}_{U,y}$ if and only if $f = g \circ \varphi$ for some germ $g \in \mathcal{O}_{V,\varphi(y)}$.

As a common abuse of language we shall sometimes write X instead of $(X, \mathcal{O}_X)$.

Reduced analytic spaces need not be separated. Consider for example the disjoint union of two copies of $\mathbf{C}$, with all points except the origins identified. This topological space is in a natural way a reduced analytic space, indeed a complex manifold.

Reduced analytic spaces were introduced by Cartan-Serre (under the name of " analytic spaces ").

Definition 1.1.2. A *morphism*, or *holomorphic map* of one reduced analytic space $(X, \mathcal{O}_X)$ into another, $(Y, \mathcal{O}_Y)$, is a continuous map $\varphi : X \to Y$ such that $\varphi^* (\mathcal{O}_{Y,\varphi(x)}) \subset \mathcal{O}_X$ for all $x \in X$.

This definition, of course, gives us also the notion of isomorphism of reduced analytic spaces, which we have already used in a special case in Definition 1.1.1.

Example 1. If X, Y are complex manifolds, the morphisms of $(X,\mathcal{O}_X)$ into $(Y, \mathcal{O}_Y)$ are the holomorphic maps $X \to Y$ in the usual sense.

Example 2. The morphisms of $(X, \mathcal{O}_X)$ into $\mathbf{C}$, regarded as a reduced analytic space $(\mathbf{C}, \mathcal{O}_\mathbf{C})$, can be identified with the sections $\Gamma (X, \mathcal{O}_X)$.

Example 3. The morphisms of $(X,\mathcal{O}_X)$ into $\mathbf{C}^n$ can be identified with n-tuples of sections of $\mathcal{O}_X$, or, again, with sections of $\mathcal{O}_X^n$.

It should be noted that a morphism may be bijective and bicontinuous and still fail to be an isomorphism. As an example we consider the map $t \to (t^2, t^3)$ of $X = \mathbf{C}$ into the space Y of all pairs (x,y) satisfying $x^3 - y^2 = 0$. This is a bijective and bicontinuous morphism, but its inverse ψ is no morphism since $\psi^* f_0 \notin \mathcal{O}_{Y,0}$ if $f(t) = t$.

Real analytic sets are not as well behaved as complex ones. To illustrate this we consider " Cartan's umbrella " which is the subset of $\mathbf{R}^3$ defined by the equation $z(x^2 + y^2) - x^3 = 0$. Its intersection with the plane $z = 1$ has an isolated double point at (0, 0, 1) and so it has a stick (the z-axis) joining the rest of the " umbrella " at the origin. Here the Oka-Cartan theorem fails. Indeed, suppose that the sheaf $\mathcal{J}$ of germs of real-analytic functions vanishing on the umbrella were generated by sections $s_1, \ldots, s_n \in \Gamma(U, \mathcal{J})$ over some neighborhood U of the origin. Then, denoting by $f_1, \ldots, f_n$ the corresponding real-analytic functions in U, we find (using a complexification and the Nullstellensatz for principal ideals) that every f_j is a multiple of $z(x^2 + y^2) - x^3$ for it can easily be seen that this polynomial defines in the complex domain an irreducible germ at the origin. Hence the germ in $\mathcal{J}$ defined by the coordinate function x at a point $(0, 0, z)$, $z \neq 0$, cannot be a linear combination of $S_1, \ldots, S_n$ which is a contradiction.

1.2. Definition of general analytic spaces.

Let U be an open subset of $\mathbf{C}^n$ (or $\mathbf{R}^n$) and let $\mathcal{J}$ be an arbitrary coherent sheaf of ideals in $\mathcal{O}_U$, the sheaf on U of germs of holomorphic (or real-analytic) functions. Then $V = \operatorname{supp} \mathcal{O}_U/\mathcal{J}$ is an analytic subset of U. The restriction of $\mathcal{O}_U/\mathcal{J}$ to V will be denoted by $\mathcal{O}_V$. It is, in general, not a subsheaf of $\mathcal{C}_V$. The definition of a general analytic space will be based on *local models* $(V, \mathcal{O}_V)$ of the type just constructed. Note that a model $(V, \mathcal{O}_V)$ is of the previously considered reduced type if and only if $\mathcal{J}$ is the sheaf of *all* germs of holomorphic functions vanishing on V. In the general case the set V does not determine the local model; one has to specify the structure sheaf.

Before proceeding to the formal definitions we shall look at a few examples.

Example 1. Let $U = \mathbf{C}$, $\mathcal{J}$ the sheaf of ideals generated by x^2. Here $V = \{0\}$ and $\mathcal{O}_{V,0} = \mathbf{C}\{x\}/(x^2)$ ($\mathbf{C}\{x\}$ denotes the space of converging power series in the variable x). Thus $\mathcal{O}_{V,0}$ is the space of " dual numbers " representable as $a + b\,\varepsilon$ where $a, b \in \mathbf{C}$ and $\varepsilon^2 = 0$, ε being the class of x. Evidently $\mathcal{O}_{V,0}$ cannot be a subring of the continuous functions on $\{0\}$. The