

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ANALYTIC SPACES
Autor: Malgrange, Bernard
Kapitel: 1.1 Reduced analytic spaces.
DOI: <https://doi.org/10.5169/seals-42341>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

1.1 Reduced analytic spaces.

To prepare for the general definition we shall first introduce reduced analytic spaces and their local models. Let U be an open set in \mathbf{C}^n and V an analytic subset of U . The sheaf \mathcal{I} on U of all germs of holomorphic functions vanishing on V is coherent by the Oka-Cartan theorem (for a proof, see e.g. Narasimhan [9, Theorem 5, p. 77]). The support of $\mathcal{O}_U/\mathcal{I}$ is V , and we shall denote by \mathcal{O}_V the restriction of $\mathcal{O}_U/\mathcal{I}$ to V (\mathcal{O}_U denotes the sheaf on U of germs of holomorphic functions). The *local models for reduced analytic spaces* shall be the pairs (V, \mathcal{O}_V) . Obviously we may consider \mathcal{O}_V as a subsheaf of \mathcal{C}_V , the sheaf on V of germs of continuous functions.

Definition 1.1.1. A *reduced analytic space* is a pair (X, \mathcal{O}_X) where X is a topological space (not necessarily separated) and \mathcal{O}_X is a sheaf of sub- \mathbf{C} -algebras of \mathcal{C}_X which is locally isomorphic to a local model.

To be explicit, the last property means that every point $x \in X$ has a neighborhood U such that for some local model (V, \mathcal{O}_V) there is a homeomorphism $\varphi : U \rightarrow V$ with the property that for $y \in U$, $f \in \mathcal{C}_{U,y}$ belongs to $\mathcal{O}_{U,y}$ if and only if $f = g \circ \varphi$ for some germ $g \in \mathcal{O}_{V,\varphi(y)}$.

As a common abuse of language we shall sometimes write X instead of (X, \mathcal{O}_X) .

Reduced analytic spaces need not be separated. Consider for example the disjoint union of two copies of \mathbf{C} , with all points except the origins identified. This topological space is in a natural way a reduced analytic space, indeed a complex manifold.

Reduced analytic spaces were introduced by Cartan-Serre (under the name of “analytic spaces”).

Definition 1.1.2. A *morphism*, or *holomorphic map* of one reduced analytic space (X, \mathcal{O}_X) into another, (Y, \mathcal{O}_Y) , is a continuous map $\varphi : X \rightarrow Y$ such that $\varphi^*(\mathcal{O}_{Y,\varphi(x)}) \subset \mathcal{O}_X$ for all $x \in X$.

This definition, of course, gives us also the notion of isomorphism of reduced analytic spaces, which we have already used in a special case in Definition 1.1.1.

Example 1. If X, Y are complex manifolds, the morphisms of (X, \mathcal{O}_X) into (Y, \mathcal{O}_Y) are the holomorphic maps $X \rightarrow Y$ in the usual sense.

Example 2. The morphisms of (X, \mathcal{O}_X) into \mathbf{C} , regarded as a reduced analytic space $(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$, can be identified with the sections $\Gamma(X, \mathcal{O}_X)$.

Example 3. The morphisms of (X, \mathcal{O}_X) into \mathbf{C}^n can be identified with n -tuples of sections of \mathcal{O}_X , or, again, with sections of \mathcal{O}_X^n .

It should be noted that a morphism may be bijective and bicontinuous and still fail to be an isomorphism. As an example we consider the map $t \rightarrow (t^2, t^3)$ of $X = \mathbf{C}$ into the space Y of all pairs (x, y) satisfying $x^3 - y^2 = 0$. This is a bijective and bicontinuous morphism, but its inverse ψ is no morphism since $\psi^* f_0 \notin \mathcal{O}_{Y,0}$ if $f(t) = t$.

Real analytic sets are not as well behaved as complex ones. To illustrate this we consider “Cartan’s umbrella” which is the subset of \mathbf{R}^3 defined by the equation $z(x^2 + y^2) - x^3 = 0$. Its intersection with the plane $z = 1$ has an isolated double point at $(0, 0, 1)$ and so it has a stick (the z -axis) joining the rest of the “umbrella” at the origin. Here the Oka-Cartan theorem fails. Indeed, suppose that the sheaf \mathcal{S} of germs of real-analytic functions vanishing on the umbrella were generated by sections $s_1, \dots, s_n \in \Gamma(U, \mathcal{S})$ over some neighborhood U of the origin. Then, denoting by f_1, \dots, f_n the corresponding real-analytic functions in U , we find (using a complexification and the Nullstellensatz for principal ideals) that every f_j is a multiple of $z(x^2 + y^2) - x^3$ for it can easily be seen that this polynomial defines in the complex domain an irreducible germ at the origin. Hence the germ in \mathcal{S} defined by the coordinate function x at a point $(0, 0, z)$, $z \neq 0$, cannot be a linear combination of S_1, \dots, S_n which is a contradiction.

1.2. Definition of general analytic spaces.

Let U be an open subset of \mathbf{C}^n (or \mathbf{R}^n) and let \mathcal{S} be an arbitrary coherent sheaf of ideals in \mathcal{O}_U , the sheaf on U of germs of holomorphic (or real-analytic) functions. Then $V = \text{supp } \mathcal{O}_U/\mathcal{S}$ is an analytic subset of U . The restriction of $\mathcal{O}_U/\mathcal{S}$ to V will be denoted by \mathcal{O}_V . It is, in general, not a subsheaf of \mathcal{C}_V . The definition of a general analytic space will be based on *local models* (V, \mathcal{O}_V) of the type just constructed. Note that a model (V, \mathcal{O}_V) is of the previously considered reduced type if and only if \mathcal{S} is the sheaf of *all* germs of holomorphic functions vanishing on V . In the general case the set V does not determine the local model; one has to specify the structure sheaf.

Before proceeding to the formal definitions we shall look at a few examples.

Example 1. Let $U = \mathbf{C}$, \mathcal{S} the sheaf of ideals generated by x^2 . Here $V = \{0\}$ and $\mathcal{O}_{V,0} = \mathbf{C}\{x\}/(x^2)$ ($\mathbf{C}\{x\}$ denotes the space of converging power series in the variable x). Thus $\mathcal{O}_{V,0}$ is the space of “dual numbers” representable as $a + b\varepsilon$ where $a, b \in \mathbf{C}$ and $\varepsilon^2 = 0$, ε being the class of x . Evidently $\mathcal{O}_{V,0}$ cannot be a subring of the continuous functions on $\{0\}$. The