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# A NOTE ON ABSOLUTE SUMMABILITY FACTORS

by Ronald Lee IRWIN <sup>1)</sup>

In this paper results which appear in [2] are extended, namely Theorem 3. The notation used here and explanation of the problems are given there. The notation  $\varepsilon_v \in (|A|, |B|)_r$  means  $\sum a_v \varepsilon_v \in |B|$  whenever  $\sum a_v \in |A|$ .

The following theorem is a special case of Theorem 3 in [2]. However, strictly speaking somewhat stronger, since it does not require the representation (4) in [2]. This is important later on.

**THEOREM 1.** If  $A$  is absolutely regular, normal,  $A' \geq 0$  and  $a_{vv} > 0 \downarrow$  for  $v \uparrow$  ( $A' \geq 0$  and  $a_{nn} > 0$  imply  $A \geq 0$ , see footnote 1) in [2]), then  $\varepsilon_v = 0 (a_{vv})$  implies  $\varepsilon_v \in (|A|, |I|)_r$ .

*Proof.* It suffices to prove  $\sum_{n=v}^{\infty} |a'_{nv} \varepsilon_n| \leq M$ . Since  $\sum_{n=v}^{\infty} |a'_{nv}| = \frac{2}{a_{vv}} - 1$  (see, Lemma 1 in [1]), and  $|\varepsilon_n| \leq K a_{nn}$ , we have  $\sum_{n=v}^{\infty} |a'_{nv} \varepsilon_n| \leq 2K$  using  $a_{nn} \downarrow$ .

Proofs of the following lemmas can be found in [1]. The notation  $|B| \subseteq |A|$  means  $\sum a_v \in |A|$  whenever  $\sum a_v \in |B|$ .

*Lemma 1.* If  $A$  and  $B$  are absolutely regular and normal, and if  $|B| \subseteq |A|$ , then given a bounded sequence  $c = \{c_v\}$  there exists a bounded sequence  $c' = \{c'_v\}$  such that  $\varepsilon_v(A, c) = \varepsilon_v(B, c')$ .

*Lemma 2.* Let  $A = BP$ ,  $P$  a weighted mean. Then,

$$[\varepsilon_{v+1}(A, c) - \varepsilon_v(A, c)] \frac{P_{v-1}}{p_v} = \varepsilon_v(A, c) - \varepsilon_v(B, c).$$

We now extend Theorem 1 to matrices of the form  $AP^k$  where  $k$  is a positive integer and  $P$  is a weighted mean. If  $P$  is a weighted mean, then

$$p_{nv} = \frac{P_n}{P_n P_{n-1}} P_{v-1} (p_{00} = 1).$$

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THEOREM 2. Suppose  $P$  is a weighted mean with  $p_v > 0$ ,  $a_{vv} > 0 \downarrow (v \uparrow)$ , and  $p_{v+1} = 0(p_v)$ . If  $\varepsilon_v = 0(a_{vv})$  implies  $\varepsilon_v \in (|A|, |I|)_r$ , then  $\delta_v = 0\left(\frac{a_{vv} p_v}{P_v}\right)$  implies  $\delta_v \in (|AP|, |I|)_r$ .

*Proof.* It suffices to prove  $\sum |a_v a_{vv} \frac{p_v}{P_v}| < \infty$  whenever  $\sum a_v \in |AP|$ .

Using the identity

$$a_v = \frac{1}{P_{v-1}} \sum_{k=1}^v P_{k-1} a_k - \frac{1}{P_{v-1}} \sum_{k=1}^{v-1} P_{k-1} a_k \quad (v \geq 1)$$

we conclude

$$\sum_{v=1}^{\infty} |a_v \frac{a_{vv} p_v}{P_v}| \leq \sum_{v=1}^{\infty} |a_{vv} P_v(a)| + \sum_{v=1}^{\infty} \left| \frac{a_{v+1, v+1}}{a_{vv}} \cdot \frac{p_{v+1}}{p_v} \cdot a_{vv} P_v(a) \right|$$

( $P_v(a)$  denotes the  $P$ -transform of  $\sum a_v$ .) Since,  $\sum P_v(a) \in |A|$  we have  $\sum |a_{vv} P_v(a)| < \infty$  by hypothesis. The same holds true for the second sum, since all of the additional factors are bounded. This completes the proof. To conclude the results of Theorem 2 with  $AP^k$  ( $k$  an integer,  $k > 1$ ) in place of  $AP$  we need only show  $a_{v+1, v+1} = 0(a_{vv})$  implies  $(AP^k)_{v+1, v+1} = 0((AP^k)_{vv})$ . This is done inductively using  $p_{v+1} = 0(p_v)$ . Thus we have by induction.

THEOREM 3. Suppose  $P$  is a weighted mean which satisfies the hypotheses of Theorem 2. If  $\varepsilon_v = 0(a_{vv})$  implies  $\varepsilon_v \in (|A|, |I|)_r$ , then  $\delta_v = 0\left(a_{vv} \left(\frac{p_v}{P_v}\right)^k\right)$  implies  $\delta_v \in (|AP^k|, |I|)_r$ .

We now state and prove the main result of the paper.

THEOREM 4. Suppose  $A$  and  $B$  are normal, and absolutely regular. Let  $P$  be a weighted mean with  $p_v > 0$  and  $|AP| \cong |A|$ . If

$$(1) \quad \delta_v = 0(a_{vv}) \text{ implies } \delta_v \in (|A|, |I|)_r$$

and

$$(2) \quad \varepsilon_v(A, c) = 0\left(\frac{a_{vv}}{b_{vv}}\right) \text{ implies } \varepsilon_v \in (|A|, |B|)_r,$$

then

$$\varepsilon_v(AP, c) = 0\left(\frac{a_{vv} p_v}{b_{vv} P_v}\right) \text{ implies } \varepsilon_v(AP, c) \in (|AP|, |B|)_r$$

whenever

$$b_{nv} \uparrow \text{ for } v \uparrow (v \leq n),$$

$$a_{v+1, v+1} = 0(a_{vv}), \quad b_{vv} = 0(b_{v+1, v+1}), \quad \text{and } p_{v+1} = 0(p_v).$$

*Proof.* If  $\sum a_v \in |AP|$ , then  $\sum P_v(a) \in |A|$ . Since  $p_v > 0$ , we have

$$a_v = \sum_{\mu=0}^v p'_{v\mu} \alpha_\mu \text{ where } \sum \alpha_\mu \in |A|. \text{ Using this we have}$$

$$\beta_n = \sum_{v=0}^n b_{nv} \varepsilon_v(AP, c) a_v = \sum_{\mu=0}^n \alpha_\mu \sum_{v=\mu}^n b_{nv} \varepsilon_v(AP, c) p'_{v\mu}.$$

The matrix  $P' = (p'_{v\mu})$  is given by

$$p'_{vv} = \frac{P_v}{p_v}, \quad p'_{v, v-1} = -\frac{P_{v-2}}{p_{v-1}}, \quad p_{v\mu} = 0 \text{ otherwise.}$$

Introducing the inverse we have

$$\beta_n = \sum_{\mu=0}^n \alpha_\mu b_{n\mu} \varepsilon_\mu(AP, c) + \sum_{\mu=0}^n \alpha_\mu \frac{P_{\mu-1}}{p_\mu} (b_{n\mu} \varepsilon_\mu(AP, c) - b_{n\mu+1} \varepsilon_{\mu+1}(AP, c)).$$

Write

$$b_{n\mu} \varepsilon_\mu(AP, c) - b_{n, \mu+1} \varepsilon_{\mu+1}(AP, c) = b_{n\mu} (\varepsilon_\mu(AP, c) - \varepsilon_{\mu+1}(AP, c)) + \varepsilon_{\mu+1}(AP, c) (b_{n\mu} - b_{n, \mu+1}).$$

From this it follows by Lemmas 1 and 2 that

$$(3) \quad \sum_{v=0}^n b_{nv} \varepsilon_v(AP, c) a_v = \sum_{\mu=0}^n b_{n\mu} \varepsilon_\mu(A, c') \alpha_\mu + \sum_{\mu=0}^n \alpha_\mu \frac{P_{\mu-1}}{p_\mu} \varepsilon_{\mu+1}(AP, c) (b_{n\mu} - b_{n, \mu+1}),$$

where  $\varepsilon_v(A, c') = 0\left(\frac{a_{vv}}{b_{vv}}\right)$  by Lemma 2. Thus by (2) the first term is absolutely

convergent. Write the second term of (3) in the form  $\sum_{\mu=0}^n a_{\mu\mu} \alpha_\mu A_{n\mu}$ , where

$$A_{n\mu} = \frac{1}{a_{\mu\mu}} \frac{P_{\mu-1}}{p_\mu} \times \varepsilon_{\mu+1}(AP, c) (b_{n\mu} - b_{n, \mu+1}).$$

$$\sum_{n=\mu}^{\infty} |A_{n\mu}| = 0 \left( \frac{b_{\mu\mu}}{a_{\mu\mu}} \frac{P_{\mu-1}}{p_\mu} \frac{a_{\mu+1, \mu+1}}{b_{\mu+1, \mu+1}} \frac{p_{\mu+1}}{p_\mu} \right)$$

By hypothesis (1)  $\sum_{\mu=0}^{\infty} |a_{\mu\mu} \alpha_{\mu}| < \infty$ . Hence the second term of (3) is absolutely convergent if  $\sum_{n=\mu}^{\infty} |A_{n\mu}| \leq M$ . Since  $b_{n\mu} \uparrow$  for  $\mu \uparrow$  we have

which is 0 (1).

**THEOREM 5.** Let  $A, B$  be normal and absolutely regular with  $A \succ 0$ ,  $B \succ 0$ , and  $A' \leq 0$ . Furthermore, assume

$$(4) \quad a_{nv} \uparrow, \quad v \uparrow (v \leq n)$$

$$(5) \quad b_{nv} \uparrow, \quad v \uparrow (v \leq n)$$

$$(6) \quad \frac{a_{nv}}{b_{nv}} \downarrow, \quad v \uparrow (v \leq n)$$

$$(7) \quad \frac{b_{nv}}{a_{nv}} (a_{kn} - a_{kv}) \downarrow, \quad v \uparrow \text{ for all } n \leq k$$

$$(8) \quad a_{v+1, v+1} = 0(a_{vv})$$

$$(9) \quad b_{vv} = 0(b_{v+1, v+1})$$

$$(10) \quad p_{v+1} = 0(p_v).$$

When these conditions are satisfied

$$\varepsilon_v(AP^k, c) = 0 \left( \frac{a_{vv}}{b_{vv}} \left( \frac{p_v}{P_v} \right)^k \right)$$

implies

$$\varepsilon_v(AP^k, c) \in (|AP^k|, |B|)_r.$$

*Proof.* With  $k = 0$  these conditions imply  $\varepsilon_v(A, c) \in (|A|, |B|)_r$  (see Theorem 3 in [2]). Hence, the theorem follows by induction from Theorem 3 and 4.

Theorem 5 extends Theorem 3 in [2] to include all Cesaro methods  $A = (C, \alpha)$ ,  $B = (C, \beta)$  where  $\alpha \geq 0$ , and  $0 \leq \beta \leq 1$ .

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