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A NOTE ON ABSOLUTE SUMMABILITY FACTORS

by Ronald Lee IRWIN ¹⁾

In this paper results which appear in [2] are extended, namely Theorem 3. The notation used here and explanation of the problems are given there. The notation $\varepsilon_v \in (|A|, |B|)_r$ means $\sum a_v \varepsilon_v \in |B|$ whenever $\sum a_v \in |A|$.

The following theorem is a special case of Theorem 3 in [2]. However, strictly speaking somewhat stronger, since it does not require the representation (4) in [2]. This is important later on.

THEOREM 1. If A is absolutely regular, normal, $A' \geq 0$ and $a_{vv} > 0 \downarrow$ for $v \uparrow$ ($A' \geq 0$ and $a_{nn} > 0$ imply $A \geq 0$, see footnote 1) in [2]), then $\varepsilon_v = 0 (a_{vv})$ implies $\varepsilon_v \in (|A|, |I|)_r$.

Proof. It suffices to prove $\sum_{n=v}^{\infty} |a'_{nv} \varepsilon_n| \leq M$. Since $\sum_{n=v}^{\infty} |a'_{nv}| = \frac{2}{a_{vv}} - 1$ (see, Lemma 1 in [1]), and $|\varepsilon_n| \leq K a_{nn}$, we have $\sum_{n=v}^{\infty} |a'_{nv} \varepsilon_n| \leq 2K$ using $a_{nn} \downarrow$.

Proofs of the following lemmas can be found in [1]. The notation $|B| \subseteq |A|$ means $\sum a_v \in |A|$ whenever $\sum a_v \in |B|$.

Lemma 1. If A and B are absolutely regular and normal, and if $|B| \subseteq |A|$, then given a bounded sequence $c = \{c_v\}$ there exists a bounded sequence $c' = \{c'_v\}$ such that $\varepsilon_v(A, c) = \varepsilon_v(B, c')$.

Lemma 2. Let $A = BP$, P a weighted mean. Then,

$$[\varepsilon_{v+1}(A, c) - \varepsilon_v(A, c)] \frac{P_{v-1}}{p_v} = \varepsilon_v(A, c) - \varepsilon_v(B, c).$$

We now extend Theorem 1 to matrices of the form AP^k where k is a positive integer and P is a weighted mean. If P is a weighted mean, then

$$p_{nv} = \frac{P_n}{P_n P_{n-1}} P_{v-1} (p_{00} = 1).$$

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THEOREM 2. Suppose P is a weighted mean with $p_v > 0$, $a_{vv} > 0 \downarrow (v \uparrow)$, and $p_{v+1} = 0(p_v)$. If $\varepsilon_v = 0(a_{vv})$ implies $\varepsilon_v \in (|A|, |I|)_r$, then $\delta_v = 0\left(\frac{a_{vv} p_v}{P_v}\right)$ implies $\delta_v \in (|AP|, |I|)_r$.

Proof. It suffices to prove $\sum |a_v a_{vv} \frac{p_v}{P_v}| < \infty$ whenever $\sum a_v \in |AP|$.

Using the identity

$$a_v = \frac{1}{P_{v-1}} \sum_{k=1}^v P_{k-1} a_k - \frac{1}{P_{v-1}} \sum_{k=1}^{v-1} P_{k-1} a_k \quad (v \geq 1)$$

we conclude

$$\sum_{v=1}^{\infty} |a_v \frac{a_{vv} p_v}{P_v}| \leq \sum_{v=1}^{\infty} |a_{vv} P_v(a)| + \sum_{v=1}^{\infty} \left| \frac{a_{v+1, v+1}}{a_{vv}} \cdot \frac{p_{v+1}}{p_v} \cdot a_{vv} P_v(a) \right|$$

($P_v(a)$ denotes the P -transform of $\sum a_v$.) Since, $\sum P_v(a) \in |A|$ we have $\sum |a_{vv} P_v(a)| < \infty$ by hypothesis. The same holds true for the second sum, since all of the additional factors are bounded. This completes the proof. To conclude the results of Theorem 2 with AP^k (k an integer, $k > 1$) in place of AP we need only show $a_{v+1, v+1} = 0(a_{vv})$ implies $(AP^k)_{v+1, v+1} = 0((AP^k)_{vv})$. This is done inductively using $p_{v+1} = 0(p_v)$. Thus we have by induction.

THEOREM 3. Suppose P is a weighted mean which satisfies the hypotheses of Theorem 2. If $\varepsilon_v = 0(a_{vv})$ implies $\varepsilon_v \in (|A|, |I|)_r$, then $\delta_v = 0\left(a_{vv} \left(\frac{p_v}{P_v}\right)^k\right)$ implies $\delta_v \in (|AP^k|, |I|)_r$.

We now state and prove the main result of the paper.

THEOREM 4. Suppose A and B are normal, and absolutely regular. Let P be a weighted mean with $p_v > 0$ and $|AP| \cong |A|$. If

$$(1) \quad \delta_v = 0(a_{vv}) \text{ implies } \delta_v \in (|A|, |I|)_r$$

and

$$(2) \quad \varepsilon_v(A, c) = 0\left(\frac{a_{vv}}{b_{vv}}\right) \text{ implies } \varepsilon_v \in (|A|, |B|)_r,$$

then

$$\varepsilon_v(AP, c) = 0\left(\frac{a_{vv} p_v}{b_{vv} P_v}\right) \text{ implies } \varepsilon_v(AP, c) \in (|AP|, |B|)_r$$

whenever

$$b_{nv} \uparrow \text{ for } v \uparrow (v \leq n),$$

$$a_{v+1, v+1} = 0(a_{vv}), \quad b_{vv} = 0(b_{v+1, v+1}), \quad \text{and } p_{v+1} = 0(p_v).$$

Proof. If $\sum a_v \in |AP|$, then $\sum P_v(a) \in |A|$. Since $p_v > 0$, we have

$$a_v = \sum_{\mu=0}^v p'_{v\mu} \alpha_\mu \text{ where } \sum \alpha_\mu \in |A|. \text{ Using this we have}$$

$$\beta_n = \sum_{v=0}^n b_{nv} \varepsilon_v(AP, c) a_v = \sum_{\mu=0}^n \alpha_\mu \sum_{v=\mu}^n b_{nv} \varepsilon_v(AP, c) p'_{v\mu}.$$

The matrix $P' = (p'_{v\mu})$ is given by

$$p'_{vv} = \frac{P_v}{p_v}, \quad p'_{v, v-1} = -\frac{P_{v-2}}{p_{v-1}}, \quad p_{v\mu} = 0 \text{ otherwise.}$$

Introducing the inverse we have

$$\beta_n = \sum_{\mu=0}^n \alpha_\mu b_{n\mu} \varepsilon_\mu(AP, c) + \sum_{\mu=0}^n \alpha_\mu \frac{P_{\mu-1}}{p_\mu} (b_{n\mu} \varepsilon_\mu(AP, c) - b_{n\mu+1} \varepsilon_{\mu+1}(AP, c)).$$

Write

$$b_{n\mu} \varepsilon_\mu(AP, c) - b_{n, \mu+1} \varepsilon_{\mu+1}(AP, c) = b_{n\mu} (\varepsilon_\mu(AP, c) - \varepsilon_{\mu+1}(AP, c)) + \varepsilon_{\mu+1}(AP, c) (b_{n\mu} - b_{n\mu+1}).$$

From this it follows by Lemmas 1 and 2 that

$$(3) \quad \sum_{v=0}^n b_{nv} \varepsilon_v(AP, c) a_v = \sum_{\mu=0}^n b_{n\mu} \varepsilon_\mu(A, c') \alpha_\mu + \sum_{\mu=0}^n \alpha_\mu \frac{P_{\mu-1}}{p_\mu} \varepsilon_{\mu+1}(AP, c) (b_{n\mu} - b_{n\mu+1}),$$

where $\varepsilon_v(A, c') = 0\left(\frac{a_{vv}}{b_{vv}}\right)$ by Lemma 2. Thus by (2) the first term is absolutely

convergent. Write the second term of (3) in the form $\sum_{\mu=0}^n a_{\mu\mu} \alpha_\mu A_{n\mu}$, where

$$A_{n\mu} = \frac{1}{a_{\mu\mu}} \frac{P_{\mu-1}}{p_\mu} \times \varepsilon_{\mu+1}(AP, c) (b_{n\mu} - b_{n\mu+1}).$$

$$\sum_{n=\mu}^{\infty} |A_{n\mu}| = 0 \left(\frac{b_{\mu\mu}}{a_{\mu\mu}} \frac{P_{\mu-1}}{p_\mu} \frac{a_{\mu+1, \mu+1}}{b_{\mu+1, \mu+1}} \frac{p_{\mu+1}}{p_\mu} \right)$$

By hypothesis (1) $\sum_{\mu=0}^{\infty} |a_{\mu\mu} \alpha_{\mu}| < \infty$. Hence the second term of (3) is absolutely convergent if $\sum_{n=\mu}^{\infty} |A_{n\mu}| \leq M$. Since $b_{n\mu} \uparrow$ for $\mu \uparrow$ we have

which is 0 (1).

THEOREM 5. Let A, B be normal and absolutely regular with $A \succ 0$, $B \succ 0$, and $A' \leq 0$. Furthermore, assume

$$(4) \quad a_{nv} \uparrow, \quad v \uparrow (v \leq n)$$

$$(5) \quad b_{nv} \uparrow, \quad v \uparrow (v \leq n)$$

$$(6) \quad \frac{a_{nv}}{b_{nv}} \downarrow, \quad v \uparrow (v \leq n)$$

$$(7) \quad \frac{b_{nv}}{a_{nv}} (a_{kn} - a_{kv}) \downarrow, \quad v \uparrow \text{ for all } n \leq k$$

$$(8) \quad a_{v+1, v+1} = 0(a_{vv})$$

$$(9) \quad b_{vv} = 0(b_{v+1, v+1})$$

$$(10) \quad p_{v+1} = 0(p_v).$$

When these conditions are satisfied

$$\varepsilon_v(AP^k, c) = 0 \left(\frac{a_{vv}}{b_{vv}} \left(\frac{p_v}{P_v} \right)^k \right)$$

implies

$$\varepsilon_v(AP^k, c) \in (|AP^k|, |B|)_r.$$

Proof. With $k = 0$ these conditions imply $\varepsilon_v(A, c) \in (|A|, |B|)_r$ (see Theorem 3 in [2]). Hence, the theorem follows by induction from Theorem 3 and 4.

Theorem 5 extends Theorem 3 in [2] to include all Cesaro methods $A = (C, \alpha)$, $B = (C, \beta)$ where $\alpha \geq 0$, and $0 \leq \beta \leq 1$.

BIBLIOGRAPHY

- [1] IRWIN, R. L., Absolute Summability Factors I, *Tohoku Math. Journal*, 1., 247-254 (1966).
- [2] — and A. PEYERIMHOFF, On Absolute Summability Factors, to appear in *L'Enseignement Mathématique*.

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