Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 13 (1967)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NATURAL SETTING FOR THE EXTENSIONS OF A GROUP WITH TRIVIAL CENTRE BY AN ARBITRARY GROUP

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DOI: https://doi.org/10.5169/seals-41540

A NATURAL SETTING FOR THE EXTENSIONS OF A GROUP WITH TRIVIAL CENTRE BY AN ARBITRARY GROUP

by John S. Rose

Let A and H be groups. The intention of the present note is to point out that if A has trivial centre, then any extension of A by H is equivalent (in the sense of extension theory) to one determined in a natural way by a suitable subgroup of Aut $A \times H$. This fact is already implicit in an early paper of R. Baer [1][1], but the aim here is to provide a more explicit formulation; and to deduce that the non-equivalent extensions of a group A with trivial centre by an arbitrary group H stand in one-to-one correspondence with the distinct homomorphisms of H into the group Aut A/Inn A of automorphism classes of A. This latter result is obtained in the treatment of Kurosh [5, p. 148] as a corollary of some cohomological theorems of S. Eilenberg and S. MacLane [3]. The proof offered here is an entirely elementary application of the fact that it is possible to work within Aut $A \times H$.

The notation and terminology used are largely standard. For an arbitrary group A, Aut A denotes the group of all automorphisms of A and Inn A the group of all inner automorphisms of A. We shall denote by $\mathfrak{A}(A)$ the group Aut A/Inn A of automorphism classes of A. If B is a subgroup of A, $C_A(B)$ denotes the centralizer of B in A. Then $C_A(A) = Z(A)$, the centre of A. For an arbitrary element a of A, the inner automorphism of A induced by a is denoted by τ_a; this notation is relative to the group A, which is here presumed to be fixed. The groups $A/Z(A)$ and Inn A may be identified in the natural way by identifying, for each a in A, the elements $aZ(A)$ and τ_a: this identification will be made. An automorphism α of A is called a central automorphism if, for every a in A, $(a\alpha)a^{-1} \in Z(A)$. It is easy to show that the set of all central automorphisms of A forms a subgroup of Aut A which is in fact precisely $C_{\mathrm{Aut}A}(\mathrm{Inn}\, A)$: see Zassenhaus [6, p. 52].

1) See also H. Fitting [4, § 21].

Let A and H be arbitrary groups. An *extension* of A by H is a pair (G, φ) consisting of a group G, containing A as a normal subgroup, and a homomorphism φ of G onto H such that $\operatorname{Ker} \varphi = A$. (Reference to the particular homomorphism φ involved in an extension is often omitted, for instance in KUROSH [5, Chapter XII], but φ is tacitly assumed to be specified in the development of the theory.) Two extensions (G, φ) and (G^*, φ^*) of A by H are said to be *equivalent* if there is an isomorphism Θ of G onto G^* mapping A identically onto itself and such that $\Theta\varphi^* = \varphi$.

Suppose that (G, φ) is an extension of A by H, and that B is a characteristic subgroup of A. Then B is a normal subgroup of G, and (G, φ) induces naturally an extension $(G/B, \bar{\varphi})$ of A/B by H: $\bar{\varphi}$ is defined by

$$(gB)\,\bar{\varphi} = g\varphi\,, \qquad \text{for any } g \text{ in } G;$$

this is well defined since $B \leqq A = \operatorname{Ker} \varphi$. We shall be concerned with the special case in which $B = Z(A)$.

Let $\bar{A} = A/Z(A)$. It is possible, for arbitrary groups A and H, to construct extensions of $\bar{A}$ by H rather transparently by means of suitable subgroups of $\operatorname{Aut} A \times H$ (the external direct product). $\operatorname{Aut} A$ is identified with a subgroup of this direct product in the obvious way by identification of α with $(\alpha, 1)$, for each α in $\operatorname{Aut} A$. Then $\bar{A}$, which is identified with $\operatorname{Inn} A$, is also identified with a subgroup of $\operatorname{Aut} A \times H$. Let π denote the projection homomorphism of $\operatorname{Aut} A \times H$ onto H:

$$(\alpha, h)\,\pi = h, \quad \text{for any } \alpha \text{ in } \operatorname{Aut} A \text{ and } h \text{ in } H.$$

Then any subgroup Q of $\operatorname{Aut} A \times H$ such that $Q \cap \operatorname{Aut} A = \bar{A}$ and $Q\pi = H$ determines an extension (Q, π_0) of $\bar{A}$ by H, where π_0 is simply the restriction of π to Q: for π_0 is a homomorphism of Q onto H, since $Q\pi = H$, and $\operatorname{Ker} \pi_0 = Q \cap \operatorname{Ker} \pi = Q \cap \operatorname{Aut} A = \bar{A}$. For convenience, we introduce a term for such an extension: we shall call it a *sited extension* of $\bar{A}$ by H.

We shall prove the

THEOREM. *Let A and H be arbitrary groups. Suppose that (G, φ) is an extension of A by H, and let $(\bar{G}, \bar{\varphi})$ be the induced extension of $\bar{A}$ by H, where $\bar{G} = G/Z(A)$, $\bar{A} = A/Z(A)$. Then $(\bar{G}, \bar{\varphi})$ is equivalent to a sited extension of $\bar{A}$ by H. Moreover, if the only central automorphisms of A are* *inner automorphism, then sited extensions of* $\bar{A}$ *by* H *corresponding to distinct subgroups of* Aut $A \times H$ *are non-equivalent.*

Proof. For any element g of G, let σ_g denote the restriction to A of the inner automorphism of G induced by g. (Thus $\sigma_a = \tau_a$, for each a in A.) We define a map $\psi \colon G \to \operatorname{Aut} A \times H$ by

$$g\psi = (\sigma_g, g\varphi)\,, \qquad \text{for every } g \text{ in } G.$$

Clearly ψ is a homomorphism, and

$$\begin{aligned} \operatorname{Ker} \psi &= \{ g \in G \mid g^{-1} ag = a \text{ for all } a \text{ in } A \} \cap \operatorname{Ker} \varphi \\ &= C_G(A) \cap A \\ &= Z(A)\,. \end{aligned}$$

Then ψ induces naturally an isomorphism $\bar{\psi}$ of $\bar{G}$ onto a subgroup Q of Aut $A \times H$; and

$$\begin{aligned} Q \cap \operatorname{Aut} A &= \{ (\sigma_g, g\varphi) \mid g \in G, g\varphi = 1 \} \\ &= \{ (\sigma_g, 1) \mid g \in \operatorname{Ker} \varphi \} \\ &= \bar{A}\,, \\ Q\pi &= \{ g\varphi \mid g \in G \} \\ &= \operatorname{Im} \varphi \\ &= H\,. \end{aligned}$$

Hence Q determines a sited extension (Q, π_0) of $\bar{A}$ by H, where π_0 is the restriction of π to Q.

We show that $(\bar{G}, \bar{\varphi})$ is equivalent to (Q, π_0). For this purpose we can use $\bar{\psi}$, which is an isomorphism of $\bar{G}$ onto Q. For any element g of G, let $\bar{g} = g\, Z(A)$. Then

$$\bar{g}\,(\bar{\psi}\pi_0) = (g\psi)\,\pi_0 = (\sigma_g, g\varphi)\,\pi = g\varphi = \bar{g}\bar{\varphi}\,,$$

so that

$$\bar{\psi}\pi_0 = \bar{\varphi}\,.$$

Also, for any a in A,

$$\bar{a}\bar{\psi} = a\psi = (\sigma_a, a\varphi) = (\tau_a, 1) = \bar{a}\,,$$

by identification, so that $\bar{\psi}$ maps $\bar{A}$ identically onto itself. This establishes the equivalence of $(\bar{G}, \bar{\varphi})$ and (Q, π_0).

Now assume that the only central automorphisms of A are inner, that is that $C_{\mathrm{AutA}}(\bar{A}) \leqslant \bar{A}$. Suppose that Q, Q^* are subgroups of $\mathrm{Aut}\, A \times H$ such that $Q \cap \mathrm{Aut}\, A = \bar{A} = Q^* \cap \mathrm{Aut}\, A$ and $Q\pi = H = Q^*\pi$, so that Q, Q^* determine sited extensions (Q, π_0), (Q^*, π_0^*) of $\bar{A}$ by H. Suppose that these extensions are equivalent. Then there is an isomorphism Θ of Q onto Q^* mapping $\bar{A}$ identically onto itself and such that $\Theta\pi_0^* = \pi_0$.

For each h in H, we choose α_h in $\mathrm{Aut}\, A$ such that $(\alpha_h, h) \in Q$: this is possible since $Q\pi = H$. (In general h does not determine a unique such element α_h, but we make a choice of one element for each h.) Let

$$(\alpha_h, h)\,\Theta = (\alpha_h^*, h^*), \qquad \text{with } \alpha_h^* \text{ in Aut } A \text{ and } h^* \text{ in } H.$$

Since $\bar{A}$ is a normal subgroup of Aut A,

$$\alpha_h^{-1}\bar{a}\ \alpha_h \in \bar{A} \qquad \text{for any } \bar{a} \text{ in } \bar{A}\,,$$

and therefore

$$(\alpha_h^{-1}\bar{a}\,\alpha_h)\ \Theta = \alpha_h^{-1}\,\bar{a}\,\alpha_h\,. \tag{1}$$

Now (by identification)

$$\alpha_h^{-1}\bar{a}\,\alpha_h = (\alpha_h, h)^{-1}\bar{a}\,(\alpha_h, h)\,. \tag{2}$$

Since (α_h, h) and $\bar{a}$ both belong to Q, (1) and (2) give

$$\begin{aligned}\alpha_h^{-1}\bar{a}\,\alpha_h &= \big((\alpha_h, h)\,\Theta\big)^{-1}(\bar{a}\Theta)\big((\alpha_h, h)\,\Theta\big)\\ &= (\alpha_h^*, h^*)^{-1}\,\bar{a}\,(\alpha_h^*, h^*)\,,\end{aligned}$$

that is

$$\alpha_h^{-1}\bar{a}\,\alpha_h = (\alpha_h^*)^{-1}\bar{a}\ \alpha_h^*\,. \tag{3}$$

Hence $\alpha_h^*\,\alpha_h^{-1} \in C_{\mathrm{Aut}\,A}(\bar{A}) \leqslant \bar{A}$, by hypothesis. Thus for each h in H, there is an element η_h in $\bar{A}$ such that

$$\alpha_h^* = \eta_h\,\alpha_h\,. \tag{4}$$

Also

$$h^* = (\alpha_h, h)\,\Theta\pi_0^* = (\alpha_h, h)\,\pi_0 = h\,,$$

so that

$$(\alpha_h, h)\,\Theta = (\alpha_h^*, h)\,,$$

that is

$$(\alpha_h, h)\,\Theta = \eta_h(\alpha_h, h)\,. \tag{5}$$

Now we consider an arbitrary element of Q, say (α, h) with α in Aut A and h in H. Since also $(\alpha_h, h) \in Q$ and $Q \cap \text{Aut } A = \bar{A}$, there is an element $\bar{a}$ in $\bar{A}$ such that

$$(\alpha, h) = \bar{a}\,(\alpha_h, h)\,.$$

Then

$$\begin{aligned}(\alpha, h)\,\Theta &= (\bar{a}\Theta)\big((\alpha_h, h)\,\Theta\big)\\ &= \bar{a}\,\eta_h\,(\alpha_h, h)\,, \qquad \text{by (5).}\end{aligned}$$

Since $\bar{A} \leqq Q$, this shows that $(\alpha, h)\,\Theta \in Q$. Hence $Q^* = Q\Theta \leqq Q$. Similarly $Q \leqq Q^*$. Therefore $Q = Q^*$. This complete the proof.

We observe now that the distinct subgroups of Aut $A \times H$ determining sited extensions of $\bar{A}$ by H stand in one-to-one correspondence with the distinct homomorphisms of H into $\mathfrak{A}\,(A)$. To see this, suppose first that Q is a subgroup of Aut $A \times H$ determining a sited extension of $\bar{A}$ by H, that is such that $Q \cap \text{Aut } A = \bar{A}$ and $Q\pi = H$. Then Q determines a homomorphism $\lambda_Q \colon H \to \mathfrak{A}\,(A)$ as follows:

$$\text{for any } h \text{ in } H,\ h\lambda_Q = \alpha\bar{A} \text{ if and only if } (\alpha, h) \in Q\,,$$

where $\alpha \in \text{Aut } A$.

Since $Q \cap \text{Aut } A = \bar{A}$, λ_Q is well defined by this rule, and is defined on the whole of H since $Q\pi = H$. Conversely, suppose that λ is a homomorphism of H into $\mathfrak{A}\,(A)$. Then λ determines a subgroup Q of Aut $A \times H$, defined as

$$Q = \{\,(\alpha, h)\;|\;\alpha \in \text{Aut } A,\ h \in H \text{ and } h\lambda = \alpha\bar{A}\,\},$$

and it is clear that then $Q \cap \text{Aut } A = \bar{A}$ and $Q\pi = H$, so that Q determines a sited extension of $\bar{A}$ by H. Furthermore, $\lambda_Q = \lambda$. Finally, distinct homomorphisms of H into $\mathfrak{A}\,(A)$ evidently determine distinct subgroups of Aut $A \times H$, and so the correspondence between homomorphisms and subgroups is one-to-one.

If A is a group with trivial centre, then A is naturally identified with $\bar{A}$ and the Theorem shows that any extension of A by H is equivalent to a sited extension of A by H. Moreover, the only central automorphism of A is the identity automorphism, so that we obtain

Corollary 1. *Let A be a group with trivial centre and H an arbitrary group. Then every extension of A by H is equivalent to a sited extension* *of A by H. The non-equivalent extensions of A by H stand in one-to-one correspondence with the distinct homomorphisms of H into $\mathfrak{A}(A)$.*

If the only homomorphism of H into $\mathfrak{A}(A)$ is the trivial homomorphism, then the only sited extension of $\bar{A}$ by H is $(\bar{A} \times H, \pi)$, where π denotes the projection map of $\bar{A} \times H$ onto H. Thus in particular we have

Corollary 2. *Let A be a group with trivial centre and H a group. Then (up to equivalence) the only extension of A by H is $(A \times H, \pi)$, where π denotes the projection map of $A \times H$ onto H, in any of the following cases:*

(i) *$\mathfrak{A}(A)$ is trivial.*

(ii) *$\mathfrak{A}(A)$ is soluble and* H *is perfect.*

(iii) *$\mathfrak{A}(A)$ is a ϖ-group and H is a ϖ' group, where ϖ is a set of prime numbers and ϖ' the set of all prime numbers not belonging to ϖ.*

(iv) *H is simple and cannot be embedded in $\mathfrak{A}(A)$.*

Here (i) is the well known case of a *complete* group A.

According to a celebrated conjecture of O. Schreier, $\mathfrak{A}(E)$ ought to be soluble for any finite non-abelian simple group E. Schreier's Conjecture is valid for every known finite non-abelian simple group. Thus (ii) applies if A is any known finite non-abelian simple group.

Another result can be derived from (ii) and a Lemma due to H. Fitting [4, Satz 12], which may be expressed as follows.

Lemma. *Let E be a finite non-abelian simple group. Then, if* n *is a positive integer and D is the direct product of n copies of E,* Aut *D is isomorphic to the wreath product of* Aut *E by the symmetric group of degree n, formed according to the natural representation.*

A group is called *completely reducible* if it can be decomposed as a direct product of a finite number of simple groups (Kurosh [5, p. 203]). An easy inductive proof, using (ii) and the Lemma, establishes

Corollary 3. *Let G be a non-trivial finite group. Associated with G there is a set of non-isomorphic simple groups $E_1, \ldots, E_k$ and a set of positive integers $n_1, \ldots, n_k$ such that every composition series of G has precisely n_i composition factors isomorphic to E_i $(i = 1, \ldots, k)$ and no others. If every E_i is non-abelian and satisfies* Schreier's *Conjecture, and if every $n_i < 5$, then G is completely reducible.*

This is a particular case of a recent result of R. Bercov [2].

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(Received August 18, 1967.)