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A NATURAL SETTING FOR THE EXTENSIONS OF A GROUP WITH TRIVIAL CENTRE BY AN ARBITRARY GROUP

by John S. Rose

Let A and H be groups. The intention of the present note is to point out that if A has trivial centre, then any extension of A by H is equivalent (in the sense of extension theory) to one determined in a natural way by a suitable subgroup of Aut $A \times H$. This fact is already implicit in an early paper of R. BAER [1]¹, but the aim here is to provide a more explicit formulation; and to deduce that the non-equivalent extensions of a group A with trivial centre by an arbitrary group H stand in one-to-one correspondence with the distinct homomorphisms of H into the group Aut A/Inn A of automorphism classes of A. This latter result is obtained in the treatment of Kurosh [5, p. 148] as a corollary of some cohomological theorems of S. Eilenberg and S. Maclane [3]. The proof offered here is an entirely elementary application of the fact that it is possible to work within Aut $A \times H$.

The notation and terminology used are largely standard. For an arbitrary group A, Aut A denotes the group of all automorphisms of A and Inn A the group of all inner automorphisms of A. We shall denote by $\mathfrak{A}(A)$ the group Aut $A/\mathrm{Inn}\ A$ of automorphism classes of A. If B is a subgroup of A, $C_A(B)$ denotes the centralizer of B in A. Then $C_A(A) = Z(A)$, the centre of A. For an arbitrary element a of A, the inner automorphism of A induced by a is denoted by τ_a ; this notation is relative to the group A, which is here presumed to be fixed. The groups A/Z(A) and Inn A may be identified in the natural way by identifying, for each a in A, the elements aZ(A) and τ_a : this identification will be made. An automorphism α of A is called a central automorphism if, for every a in A, $(a\alpha) a^{-1} \in Z(A)$. It is easy to show that the set of all central automorphisms of A forms a subgroup of Aut A which is in fact precisely C_{AutA} (Inn A): see Zassenhaus [6, p. 52].

¹⁾ See also H. FITTING [4, § 21].

Let A and H be arbitrary groups. An extension of A by H is a pair (G, φ) consisting of a group G, containing A as a normal subgroup, and a homomorphism φ of G onto H such that $\operatorname{Ker} \varphi = A$. (Reference to the particular homomorphism φ involved in an extension is often omitted, for instance in Kurosh [5, Chapter XII], but φ is tacitly assumed to be specified in the development of the theory.) Two extensions (G, φ) and (G^*, φ^*) of A by H are said to be equivalent if there is an isomorphism Θ of G onto G^* mapping A identically onto itself and such that $\Theta \varphi^* = \varphi$.

Suppose that (G, φ) is an extension of A by H, and that B is a characteristic subgroup of A. Then B is a normal subgroup of G, and (G, φ) induces naturally an extension $(G/B, \overline{\varphi})$ of A/B by $H: \overline{\varphi}$ is defined by

$$(gB) \overline{\varphi} = g\varphi$$
, for any g in G;

this is well defined since $B \le A = \text{Ker } \varphi$. We shall be concerned with the special case in which B = Z(A).

Let $\overline{A} = A/Z(A)$. It is possible, for arbitrary groups A and H, to construct extensions of \overline{A} by H rather transparently by means of suitable subgroups of Aut $A \times H$ (the external direct product). Aut A is identified with a subgroup of this direct product in the obvious way by identification of α with $(\alpha, 1)$, for each α in Aut A. Then \overline{A} , which is identified with Inn A, is also identified with a subgroup of Aut $A \times H$. Let π denote the projection homomorphism of Aut $A \times H$ onto H:

$$(\alpha, h) \pi = h$$
, for any α in Aut A and h in H.

Then any subgroup Q of Aut $A \times H$ such that $Q \cap \operatorname{Aut} A = \overline{A}$ and $Q\pi = H$ determines an extension (Q, π_0) of \overline{A} by H, where π_0 is simply the restriction of π to Q: for π_0 is a homomorphism of Q onto H, since $Q\pi = H$, and $\operatorname{Ker} \pi_0 = Q \cap \operatorname{Ker} \pi = Q \cap \operatorname{Aut} A = \overline{A}$. For convenience, we introduce a term for such an extension: we shall call it a *sited extension* of \overline{A} by H.

We shall prove the

THEOREM. Let A and H be arbitrary groups. Suppose that (G, φ) is an extension of A by H, and let $(\overline{G}, \overline{\varphi})$ be the induced extension of \overline{A} by H, where $\overline{G} = G/Z(A)$, $\overline{A} = A/Z(A)$. Then $(\overline{G}, \overline{\varphi})$ is equivalent to a sited extension of \overline{A} by H. Moreover, if the only central automorphisms of A are

inner automorphism, then sited extensions of \overline{A} by H corresponding to distinct subgroups of $\operatorname{Aut} A \times H$ are non-equivalent.

Proof. For any element g of G, let σ_g denote the restriction to A of the inner automorphism of G induced by g. (Thus $\sigma_a = \tau_a$, for each a in A.) We define a map $\psi \colon G \to \operatorname{Aut} A \times H$ by

$$g\psi = (\sigma_g, g\varphi)$$
, for every g in G .

Clearly ψ is a homomorphism, and

Ker
$$\psi = \{ g \in G \mid g^{-1} \ ag = a \text{ for all } a \text{ in } A \} \cap \text{Ker } \varphi$$

= $C_G(A) \cap A$
= $Z(A)$.

Then ψ induces naturally an isomorphism $\overline{\psi}$ of \overline{G} onto a subgroup Q of Aut $A \times H$; and

$$Q \cap \operatorname{Aut} A = \{ (\sigma_g, g\varphi) \mid g \in G, g\varphi = 1 \}$$

$$= \{ (\sigma_g, 1) \mid g \in \operatorname{Ker} \varphi \}$$

$$= \overline{A},$$

$$Q\pi = \{ g\varphi \mid g \in G \}$$

$$= \operatorname{Im} \varphi$$

$$= H.$$

Hence Q determines a sited extension (Q, π_0) of \overline{A} by H, where π_0 is the restriction of π to Q.

We show that $(\overline{G}, \overline{\varphi})$ is equivalent to (Q, π_0) . For this purpose we can use $\overline{\psi}$, which is an isomorphism of \overline{G} onto Q. For any element g of G, let $\overline{g} = g Z(A)$. Then

$$\overline{g}(\overline{\psi}\pi_0) = (g\psi)\,\pi_0 = (\sigma_g,g\varphi)\,\pi = g\varphi = \overline{g}\overline{\varphi},$$

so that

$$\overline{\psi}\pi_0=\overline{\varphi}$$
 .

Also, for any a in A,

$$\bar{a}\bar{\psi}=a\psi=(\sigma_a,a\varphi)=(\tau_a,1)=\bar{a}$$
,

by identification, so that $\overline{\psi}$ maps \overline{A} identically onto itself. This establishes the equivalence of $(\overline{G}, \overline{\varphi})$ and (Q, π_0) .

Now assume that the only central automorphisms of A are inner, that is that $C_{\text{AutA}}(\overline{A}) \leqslant \overline{A}$. Suppose that Q, Q^* are subgroups of Aut $A \times H$ such that $Q \cap \text{Aut } A = \overline{A} = Q^* \cap \text{Aut } A$ and $Q\pi = H = Q^* \pi$, so that Q, Q^* determine sited extensions (Q, π_0) , (Q^*, π_0^*) of \overline{A} by H. Suppose that these extensions are equivalent. Then there is an isomorphism Θ of Q onto Q^* mapping \overline{A} identically onto itself and such that $\Theta\pi_0^* = \pi_0$.

For each h in H, we choose α_h in Aut A such that $(\alpha_h, h) \in Q$: this is possible since $Q\pi = H$. (In general h does not determine a unique such element α_h , but we make a choice of one element for each h.) Let

$$(\alpha_h, h) \Theta = (\alpha_h^*, h^*),$$
 with α_h^* in Aut A and h^* in H.

Since \overline{A} is a normal subgroup of Aut A,

$$\alpha_h^{-1}\bar{a} \ \alpha_h \in \bar{A}$$
 for any \bar{a} in \bar{A} ,

and therefore

$$(\alpha_h^{-1}\bar{a} \alpha_h) \Theta = \alpha_h^{-1}\bar{a} \alpha_h. \tag{1}$$

Now (by identification)

$$\alpha_h^{-1}\bar{a}\ \alpha_h = (\alpha_h, h)^{-1}\bar{a}\ (\alpha_h, h)\ . \tag{2}$$

Since (α_h, h) and \bar{a} both belong to Q, (1) and (2) give

$$\alpha_h^{-1}\bar{a} \alpha_h = ((\alpha_h, h) \Theta)^{-1} (\bar{a} \Theta) ((\alpha_h, h) \Theta)$$
$$= (\alpha_h^*, h^*)^{-1} \bar{a} (\alpha_h^*, h^*),$$

that is

$$\alpha_h^{-1}\bar{a} \ \alpha_h = (\alpha_h^*)^{-1}\bar{a} \ \alpha_h^* \ . \tag{3}$$

Hence $\alpha_h^* \alpha_h^{-1} \in C_{\text{Aut } A}(\overline{A}) \leqslant \overline{A}$, by hypothesis. Thus for each h in H, there is an element η_h in \overline{A} such that

$$\alpha_h^* = \eta_h \, \alpha_h \,. \tag{4}$$

Also

$$h^* = (\alpha_h, h) \Theta \pi_0^* = (\alpha_h, h) \pi_0 = h,$$

so that

$$(\alpha_h, h) \Theta = (\alpha_h^*, h),$$

that is

$$(\alpha_h, h) \Theta = \eta_h(\alpha_h, h). \tag{5}$$

Now we consider an arbitrary element of Q, say (α, h) with α in Aut A and h in H. Since also $(\alpha_h, h) \in Q$ and $Q \cap \text{Aut } A = \overline{A}$, there is an element \overline{a} in \overline{A} such that

$$(\alpha, h) = \bar{a}(\alpha_h, h).$$

Then

$$(\alpha, h) \Theta = (\bar{a}\Theta) ((\alpha_h, h) \Theta)$$

= $\bar{a} \eta_h (\alpha_h, h)$, by (5).

Since $\overline{A} \leq Q$, this shows that $(\alpha, h) \Theta \in Q$. Hence $Q^* = Q\Theta \leq Q$. Similarly $Q \leq Q^*$. Therefore $Q = Q^*$. This complete the proof.

We observe now that the distinct subgroups of Aut $A \times H$ determining sited extensions of \overline{A} by H stand in one-to-one correspondence with the distinct homomorphisms of H into $\mathfrak{A}(A)$. To see this, suppose first that Q is a subgroup of Aut $A \times H$ determining a sited extension of \overline{A} by H, that is such that $Q \cap \operatorname{Aut} A = \overline{A}$ and $Q\pi = H$. Then Q determines a homomorphism $\lambda_Q \colon H \to \mathfrak{A}(A)$ as follows:

for any h in H,
$$h\lambda_Q = \alpha \overline{A}$$
 if and only if $(\alpha, h) \in Q$,

where $\alpha \in Aut A$.

Since $Q \cap \operatorname{Aut} A = \overline{A}$, λ_Q is well defined by this rule, and is defined on the whole of H since $Q\pi = H$. Conversely, suppose that λ is a homomorphism of H into $\mathfrak{A}(A)$. Then λ determines a subgroup Q of $\operatorname{Aut} A \times H$, defined as

$$Q = \{ (\alpha, h) \mid \alpha \in \text{Aut } A, h \in H \text{ and } h\lambda = \alpha \overline{A} \},$$

and it is clear that then $Q \cap \operatorname{Aut} A = \overline{A}$ and $Q\pi = H$, so that Q determines a sited extension of \overline{A} by H. Furthermore, $\lambda_Q = \lambda$. Finally, distinct homomorphisms of H into $\mathfrak{A}(A)$ evidently determine distinct subgroups of $\operatorname{Aut} A \times H$, and so the correspondence between homomorphisms and subgroups is one-to-one.

If A is a group with trivial centre, then A is naturally identified with \overline{A} and the Theorem shows that any extension of A by H is equivalent to a sited extension of A by H. Moreover, the only central automorphism of A is the identity automorphism, so that we obtain

COROLLARY 1. Let A be a group with trivial centre and H an arbitrary group. Then every extension of A by H is equivalent to a sited extension

of A by H. The non-equivalent extensions of A by H stand in one-to-one correspondence with the distinct homomorphisms of H into $\mathfrak{A}(A)$.

If the only homomorphism of H into $\mathfrak{A}(A)$ is the trivial homomorphism, then the only sited extension of \overline{A} by H is $(\overline{A} \times H, \pi)$, where π denotes the projection map of $\overline{A} \times H$ onto H. Thus in particular we have

COROLLARY 2. Let A be a group with trivial centre and H a group. Then (up to equivalence) the only extension of A by H is $(A \times H, \pi)$, where π denotes the projection map of $A \times H$ onto H, in any of the following cases:

- (i) $\mathfrak{A}(A)$ is trivial.
- (ii) $\mathfrak{A}(A)$ is soluble and H is perfect.
- (iii) $\mathfrak{A}(A)$ is a ϖ -group and H is a ϖ' group, where ϖ is a set of prime numbers and ϖ' the set of all prime numbers not belonging to ϖ .
- (iv) H is simple and cannot be embedded in $\mathfrak{A}(A)$.

Here (i) is the well known case of a complete group A.

According to a celebrated conjecture of O. Schreier, $\mathfrak{A}(E)$ ought to be soluble for any finite non-abelian simple group E. Schreier's Conjecture is valid for every known finite non-abelian simple group. Thus (ii) applies if A is any known finite non-abelian simple group.

Another result can be derived from (ii) and a Lemma due to H. FITTING [4, Satz 12], which may be expressed as follows.

LEMMA. Let E be a finite non-abelian simple group. Then, if n is a positive integer and D is the direct product of n copies of E, Aut D is isomorphic to the wreath product of Aut E by the symmetric group of degree n, formed according to the natural representation.

A group is called *completely reducible* if it can be decomposed as a direct product of a finite number of simple groups (Kurosh [5, p. 203]). An easy inductive proof, using (ii) and the Lemma, establishes

COROLLARY 3. Let G be a non-trivial finite group. Associated with G there is a set of non-isomorphic simple groups $E_1, ..., E_k$ and a set of positive integers $n_1, ..., n_k$ such that every composition series of G has precisely n_i composition factors isomorphic to E_i (i = 1, ..., k) and no others. If every E_i is non-abelian and satisfies Schreier's Conjecture, and if every $n_i < 5$, then G is completely reducible.

This is a particular case of a recent result of R. Bercov [2].

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