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- (b) $\int |a(s)| s^\alpha ds < \infty$,
- (c) $0 < g(v) \leq K$ for all v ;

then the derivative of any solution of (1.1) has a limit.

Proof. Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{|u(t)|}{t} \leq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^t s^\alpha KM |a(s)| h\left(\frac{|u(s)|}{s}\right) ds,$$

from which we conclude from a result of Bihari [14] that

$$\frac{|u(t)|}{t} \leq H^{-1}(H(|u(t_0)| + |u'(t_0)|) + KM \int_{t_0}^t |a(s)| s^\alpha ds)$$

which is bounded for t on account of assumption (a). The remaining proof follows verbatim that of Theorem 3.

3. BOUNDEDNESS THEOREMS II

Theorem 5. Suppose that assumptions A_1 , A_2 , A_3 and A_4 hold and in addition that

- (i) $a(t) > 0$, $a'(t) \geq 0$, for $t \geq T$,
- (ii) $\frac{d}{dt}\left(\frac{b}{a}\right) \leq \beta(t)\left(1 + \frac{b}{a}\right)$, with $\int_0^\infty \beta(s) ds < \infty$

and

$$\left(1 + \frac{b}{a}\right) \geq \varepsilon > 0;$$

then every solution of (1.1) with $(a(t) + b(t))$ replacing $a(t)$ is bounded.

Proof. Make the following substitution for the independent variable, $x = \int_0^t \sqrt{a(s)} ds$ which tends to infinity as $t \rightarrow \infty$, and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2} \left(\frac{a}{a^{3/2}} \right) \frac{du}{dx} + \left(1 + \frac{b}{a}\right) f(u) g(u') = 0 \quad (3.1)$$

where “dot” denotes differentiation with respect to t . Now write equation in its system form, letting $y_1 = u$:

$$\begin{cases} \frac{dy_1}{dx} = y_2 \\ \frac{dy_2}{dx} = -\frac{1}{2}\left(\frac{a}{a^{3/2}}\right)y_2 - \left(1 + \frac{b}{a}\right)f(y_1)g(\sqrt{a}y_2). \end{cases} \quad (3.2)$$

Define for (3.2) the following function:

$$V(x, y_1, y_2) = \left(1 + \frac{b}{a}\right) \int f(s) ds + \int \frac{s ds}{g(\sqrt{a}s)},$$

and observe:

$$\begin{aligned} \frac{dV}{dx} &\leq \frac{\beta(t)}{\sqrt{a(t)}} \left(1 + \frac{b}{a}\right) \int f(s) ds - \frac{1}{2} \frac{a}{a^{3/2}} y_2^2 \\ &\leq \frac{\beta(t)}{\sqrt{a(t)}} V. \end{aligned}$$

Hence we have

$$V(x, y_1, y_2) \leq V(x(T), y_1(x(T)), y_2(x(T))) \exp \int_T^t \beta(s) ds$$

which is finite. From (ii) we note that $V \rightarrow \infty$ as $y_1 \rightarrow \infty$ and $V > 0$ if $y_1^2 + y_2^2 \neq 0$. Thus, every solution of (1.1) is bounded.

Corollary. Suppose in addition to the hypothesis of Theorem 5 that assumption A_5 also holds and that $\lim_{t \rightarrow \infty} a(t) = a_1 < \infty$, then every solution of (1.1) and its derivative are bounded.

From the above result we may conclude for example that all solutions of the following equation:

$$u''(t) + (c_1 t^\alpha + c_2 t^\beta) u^\lambda(t) (1 + \exp u'(t) \sin u'(t)) = 0$$

are bounded for all $c_1, c_2 > 0$, $\alpha > \beta \geq 0$, and $\lambda > 0$.

We now consider the following inhomogeneous equation:

$$u''(t) + a(t)f(u)g(u') = h(t, u, u') \quad (3.3)$$

and assume that $|u' h(t, u, u')| \leq \gamma(t) g(u')$ where $\int \gamma(s) ds < \infty$.

Theorem 6. Suppose that assumptions A_1 , A_2 , A_3 and A_4 hold and in addition that $a(t) > 0$ and $a'(t) \geq 0$ for $t \geq T$; then all solutions of (3.3) are bounded.

Proof. Integrate (3.3) in the following manner:

$$\begin{aligned} G(u'(t)) - G(u'(t_0)) + a(t)F(u(t)) - a(t_0)F(u(t_0)) \\ = \int_{t_0}^t a'(s)F(u(s))ds + \int_{t_0}^t \frac{h(t, u, u')u'(s)ds}{g(u')} \end{aligned} \quad (3.4)$$

where $G(v) = \int_0^\gamma \frac{s ds}{g(s)}$ and $F(u) = \int_0^u f(s) ds$. Taking absolute values and

noting that $G(v) \geq 0$ and $F(u) \geq 0$, we obtain

$$a(t)F(u(t)) \leq c_0 + c_1 + \int_{t_0}^t a'(s)F(u(s))ds \quad (3.5)$$

where $c_0 = G(u'(t_0)) + a(t_0)F(u(t_0))$ and $c_1 = \int_{t_0}^\infty \gamma(s) ds$ are non-

negative constants. From (3.5) and A_4 it is now clear that every solution of (3.3) are bounded (cf. [1]).

Corollary. In addition to the hypothesis of Theorem 6, suppose that assumption A_5 also holds and that $\lim_{t \rightarrow \infty} a(t) = k > 0$; then all solutions of (3.3) and their derivatives are bounded.

We note that by setting $h(t, u, u') \equiv 0$, the above result again reduces to Theorem 1 and its corollary. Other comparison theorems may be formulated in a similar way as Theorem 6 by extending the corresponding result for the homogeneous equation. Since the procedure is clear, the statements and proofs of these results will be omitted.

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