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Artikel: BOUNDEDNESS THEOREMS FOR SOLUTIONS OF u'' (t) + a(t)f{u) g(u') =0 (IV)

Autor: Wong, James S. W.

Kapitel: 2. BOUNDEDNESS THEOREMS I

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Theorem (I). Suppose that assumptions A_1, A_2, A_3, and A_4 hold and in addition that $a(t) > 0$ and $a'(t) \geqq 0$ for $t \geqq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (I), suppose that assumption A_5 also holds and that $\lim_{t \to \infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (II). Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that $a'(t) \leqq 0$ for $t \geqq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (II), suppose that assumption A_5 also holds and $\lim_{t \to \infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (III). Suppose that assumptions A_1, A_2, A_3, and A_4 hold and in addition that $a(t) \geqq a_0 > 0$ for $t \geqq T$, and $\int^{\infty} |a'(t)| \, dt < \infty$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (III), suppose that assumption A_5 also holds; then all solutions of (1.1) and their derivatives are bounded.

The method of proof for the above results is based essentially on the well-known lemma of Gronwall [10], which is also known as the Bellman's lemma. In this paper, we use in addition to this fundamental lemma, its generalizations [11], [12], and techniques borrowed from Lyapunov's stability theory.

It might be of interest to note that quite a few results in [4] are incorrect; in particular Theorems 5 and 6. Also, Theorems 3 and 4 are stated incorrectly.

2. Boundedness Theorems I

Theorem 1. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and that $a(t) > 0$ for $t \geqq T$ and there exists a non-negative function $\alpha(t)$ such that $-a'(t) \leqq \alpha(t) a(t)$ with $\int^{\infty} \alpha(s) \, ds < \infty$. Then all solutions of (1.1) are bounded.

Proof. Write equation (1.1) in its system form ($y_1 = u$):

$$\begin{cases} \dfrac{dy_1}{dt} = y_2, \\ \\ \dfrac{dy_2}{dt} = -a(t) f(y_1) g(y_2). \end{cases} \tag{2.1}$$

For system (2.1), we construct the following function:

$$V(t, y_1, y_2) = \int_0^{y_1} f(s)\, ds + \frac{1}{a(t)} \int_0^{y_2} \frac{s\, ds}{g(s)} \tag{2.2}$$

Clearly, under the hypothesis of the theorem, we have $V > 0$ whenever $y_1^2 + y_2^2 \neq 0$, and by A_4, $V \to \infty$ as $y_1 \to \infty$. Differentiating with respect to t, we obtain

$$V'(t, y_1, y_2) \leqq -\frac{a'(t)}{a^2(t)} \int_0^{y_2} \frac{s\, ds}{g(s)} \leqq \alpha(t)\, V(t, y_1, y_2),$$

hence,

$$V(t, y_1, y_2) \leqq V(T, y_1(T), \quad y_2(T)) \left\{ \exp \int_T^t \alpha(s)\, ds \right\} < \infty \tag{2.3}$$

for all t; and therefore all solutions of (1.1) are bounded. Furthermore, if assumption A_5 also holds and $a(t) \leqq a_1$ for all $t \geqq T$, then $u'(t)$ is also bounded for in this case $V \to \infty$ as $y_2 \to \infty$. Thus,

Corollary. In addition to the hypothesis of Theorem 4, suppose that assumption A_5 holds and $a(t) \leqq a_1$ for all $t \geqq T$; then all solutions and their derivatives are bounded.

Theorem 1 and its corollary generalize a result of Zhang ([4], Theorem 3). By taking $\alpha(s) \equiv 0$, Theorem 1 reduces to Theorem I. All these results are extensions of a theorem of Klokov ([13], Theorem 1). We remark that a slightly weaker version of Klokov's theorem may also be found in Waltman [14].

By the above result, we may conclude for example that all solutions $u(t)$ and their derivatives $u'(t)$ of the following equation:

$$u''(t) + (1 + e^{-t} \sin t)\, u^{\frac{3}{2}}(t) \left[2 + \cos u(t)\right] = 0$$

are bounded. On the other hand, no previously available result seems to yield such a conclusion.

Theorem 2. Suppose that assumptions A_1, A_2, A_3, and A_4 hold and that $a(t) > 0$, $a(t) \to 0$, and there exists a $\alpha(t) \geqq 0$ such that $a'(t) \leqq \alpha(t)\,a(t)$ while $\int^{\infty} \alpha(s)\,ds < \infty$. Then all solutions of (1.1) satisfy: $a(t)\,F(u(t)) = a(t)\int^{u(t)} f(s)\,ds < \infty$ [1]) for $t \geqq T$, and all its derivatives are bounded.

Proof. Consider the following function:

$$V(t, y_1, y_2) = a(t)\int_0^{y_1} f(s)\,ds + \int_0^{y_2} \frac{s\,ds}{g(s)} \tag{2.4}$$

and note that

$$\begin{aligned} V'(t, y_1, y_2) &\leqq a'(t)\int_0^{y_1} f(s)\,ds \\ &\leqq \alpha(t)\,a(t)\int_0^{y_1} f(s)\,ds \\ &\leqq \alpha(t)\,V(t, y_1, y_2)\,. \end{aligned}$$

Hence again, we arrive at (2.3), from which the conclusion and the following corollary follow at once.

Corollary. Under the hypothesis of Theorem 5, if in addition assumption A_5 holds and $a(t) \geqq a_0 > 0$ for $t \geqq T$; then all solutions of (1.1) are also bounded.

By taking $\alpha(t) \equiv 0$, Theorem 2 and its corollary reduce to Theorem (II); similarly by taking $g(u') \equiv 1$, $f(u) = u^{2n-1}$ where n is a positive integer, the above result reduces to Theorem 4 of [4].

We now present two results on the boundedness of solutions by linear functions, i.e. $|u(t)| \leqq c\,|t|$ for some positive constant c, and for $t \geqq T$; and the existence of limit of $u'(t)$ as $t \to \infty$.

Theorem 3. Suppose that assumptions A_1, A_2 and A_3 hold and that

(i) $|f(u)| \leqq M\,|u|^{\alpha}$, where M, $\alpha > 0$,

(ii) $\int^{\infty} |a(s)|\,s^{\alpha}\,ds < \infty$,

(iii) $0 < g(v) \leqq K$ for all v;

then the derivative $u'(t)$ of any solution $u(t)$ of (1.1) has a limit if the initial conditions satisfy: for $\alpha > 1$,

$$\left\{KM(\alpha-1)\int_{t_0}^{\infty} s^{\alpha}\,|a(s)|\,ds\right\}^{\frac{1}{1-a}} \geqq \left\{|u(t_0)| + |u'(t_0)|\right\} \tag{2.5}$$

1) $a(t)\,F(u(t)) < \infty$ means that a solution $u(t)$ of (1.1) is either bounded or unbounded, but in case of unboundedness must satisfy $a(t)F(u(t)) < \infty$. (Note that $a(t) \to O$, as $t \to \infty$ and assumption A_4.)

Proof. Consider equation (1.1) in its equivalent integral equation form:

$$u(t) = u(t_0) + u'(t_0)t - \int_{t_0}^{t} (t-s)\,a(s)f(u(s))g(u'(s))ds\,.$$

From the hypothesis of the theorem, we have for $t \geqq t_0 \geqq 1$ the following estimate:

$$\frac{|u(t)|}{t} \leqq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^{t} s^\alpha KM\,|a(s)|\left(\frac{|u(s)|}{s}\right)^\alpha ds \qquad (2.6)$$

By a variation of Gronwall's lemma (see e.g. [15], [16]), we obtain for $t \geqq t_0 \geqq 1$,

$$\frac{|u(t)|}{t} \leqq \left\{(|u(t_0)| + |u'(t_0)|)^{1-\alpha} + KM(1-\alpha)\int_{t_0}^{t} s^\alpha\,|a(s)|\,ds\right\}^{\frac{1}{1-a}} \qquad (2.7)$$

which is finite on account of (2.5) and (i). Now from

$$u'(t) = u'(t_0) - \int_{t_0}^{t} a(s)f(u(s))g(u'(s))ds$$

and that

$$\left|\int_{t}^{t_0} a(s)f(u(s))g(u'(s))ds\right| \leqq MK\int_{t}^{t_0} |a(s)\,u^\alpha(s)|\,ds$$

$$\leqq MKC^\alpha \int_{t_0}^{t} |a(s)|\,s^\alpha\,ds$$

where C denotes the bound given in (2.7); we conclude that the limit $\lim_{t\to\infty} u'(t) = \mathrm{L}$ exists.

We remark that the method of the above proof has also been applied by the author [17] to prove a generalization of a recent result of Waltman [18].

Theorem 4. Suppose that assumptions A_1, A_2 and A_3 hold and in addition that

(a) $|f(u)| \leqslant M\,h(|u|)$,

where $M > O$ aud $h(r)$ is a non-decreasing continous function such that $h(\lambda r) \leqq \lambda^\alpha h(r)$, where λ is positive and α is a positive constant; and

$$H(x) = \int_{\infty}^{x} \frac{dr}{h(r)} \to \infty \text{ as } x \to \infty,$$

(b) $\int |a(s)| s^{\alpha} ds < \infty$,

(c) $0 < g(v) \leqq K$ for all v;

then the derivative of any solution of (1.1) has a limit.

Proof. Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{|u(t)|}{t} \leqq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^{t} s^{\alpha} KM |a(s)| h\left(\frac{|u(s)|}{s}\right) ds,$$

from which we conclude from a result of Bihari [14] that

$$\frac{|u(t)|}{t} \leqq H^{-1}\left(H(|u(t_0)| + |u'(t_0)|) + KM \int_{t_0}^{t} |a(s)| s^{\alpha} ds\right)$$

which is bounded for t on account of assumption (a). The remaining proof follows verbatim that of Theorem 3.

3. Boundedness Theorems II

Theorem 5. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that

(i) $a(t) > 0, \quad a'(t) \geqq 0, \quad$ for $\quad t \geqq T$,

(ii) $\frac{d}{dt}\left(\frac{b}{a}\right) \leqq \beta(t)\left(1 + \frac{b}{a}\right), \quad$ with $\quad \int^{\infty} \beta(s) ds < \infty$

and

$$\left(1 + \frac{b}{a}\right) \geqq \varepsilon > 0;$$

then every solution of (1.1) with $(a(t) + b(t))$ replacing $a(t)$ is bounded.

Proof. Make the following substitution for the independent variable, $x = \int^{t} \sqrt{a(s)}\, ds$ which tends to infinity as $t \to \infty$, and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2}\left(\frac{a}{a^{3/2}}\right)\frac{du}{dx} + \left(1 + \frac{b}{a}\right) f(u) g(u') = 0 \qquad (3.1)$$