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g(u') = 0 (IV)

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Theorem (1). Suppose that assumptions A_1 , A_2 , A_3 , and A_4 hold and in addition that a(t) > 0 and $a'(t) \ge 0$ for $t \ge T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (I), suppose that assumption A_5 also holds and that $\lim_{t\to\infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (II). Suppose that assumptions A_1 , A_2 , A_3 and A_4 hold and in addition that $a'(t) \leq 0$ for $t \geq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (II), suppose that assumption A_5 also holds and $\lim_{t\to\infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (III). Suppose that assumptions A_1 , A_2 , A_3 , and A_4 hold and in addition that $a(t) \ge a_0 > 0$ for $t \ge T$, and $\int_0^\infty |a'(t)| dt < \infty$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (III), suppose that assumption A_5 also holds; then all solutions of (1.1) and their derivatives are bounded.

The method of proof for the above results is based essentially on the well-known lemma of Gronwall [10], which is also known as the Bellman's lemma. In this paper, we use in addition to this fundamental lemma, its generalizations [11], [12], and techniques borrowed from Lyapunov's stability theory.

It might be of interest to note that quite a few results in [4] are incorrect; in particular Theorems 5 and 6. Also, Theorems 3 and 4 are stated incorrectly.

2. Boundedness Theorems I

Theorem 1. Suppose that assumptions A_1 , A_2 , A_3 and A_4 hold and that a(t) > 0 for $t \ge T$ and there exists a non-negative function $\alpha(t)$ such that $-a'(t) \le \alpha(t) a(t)$ with $\int_0^\infty \alpha(s) ds < \infty$. Then all solutions of (1.1) are bounded.

Proof. Write equation (1.1) in its system form $(y_1 = u)$:

$$\begin{cases} \frac{dy_1}{dt} = y_2, \\ \frac{dy_2}{dt} = -a(t) f(y_1) g(y_2). \end{cases}$$
 (2.1)

For system (2.1), we construct the following function:

$$V(t, y_1, y_2) = \int_{0}^{y_1} f(s) \, ds + \frac{1}{a(t)} \int_{0}^{y_2} \frac{s \, ds}{g(s)}$$
 (2.2)

Clearly, under the hypothesis of the theorem, we have V > 0 whenever $y_1^2 + y_2^2 \neq 0$, and by A_4 , $V \to \infty$ as $y_1 \to \infty$. Differentiating with respect to t, we obtain

$$V'(t, y_1, y_2) \le -\frac{a'(t)}{a^2(t)} \int_0^{y_2} \frac{s \, ds}{g(s)} \le \alpha(t) V(t, y_1, y_2),$$

hence,

$$V(t, y_1, y_2) \le V(T, y_1(T), \quad y_2(T)) \left\{ \exp \int_T^t \alpha(s) \, ds \right\} < \infty$$
 (2.3)

for all t; and therefore all solutions of (1.1) are bounded. Furthermore, if assumption A_5 also holds and $a(t) \leq a_1$ for all $t \geq T$, then u'(t) is also bounded for in this case $V \to \infty$ as $y_2 \to \infty$. Thus,

Corollary. In addition to the hypothesis of Theorem 4, suppose that assumption A_5 holds and $a(t) \leq a_1$ for all $t \geq T$; then all solutions and their derivatives are bounded.

Theorem 1 and its corollary generalize a result of Zhang ([4], Theorem 3). By taking $\alpha(s) \equiv 0$, Theorem 1 reduces to Theorem I. All these results are extensions of a theorem of Klokov ([13], Theorem 1). We remark that a slightly weaker version of Klokov's theorem may also be found in Waltman [14].

By the above result, we may conclude for example that all solutions u(t) and their derivatives u'(t) of the following equation:

$$u''(t) + (1 + e^{-t} \sin t) u^{\frac{3}{2}}(t) [2 + \cos u(t)] = 0$$

are bounded. On the other hand, no previously available result seems to yield such a conclusion.

Theorem 2. Suppose that assumptions A_1 , A_2 , A_3 , and A_4 hold and that a(t) > 0, $a(t) \to 0$, and there exists a $\alpha(t) \ge 0$ such that $a'(t) \le \alpha(t)$ and while $\int_{u(t)}^{\infty} \alpha(s) ds < \infty$. Then all solutions of (1.1) satisfy: $a(t) F(u(t)) = a(t) \int_{u(t)}^{\infty} f(s) ds < \infty$ 1) for $t \ge T$, and all its derivatives are bounded.

Proof. Consider the following function:

$$V(t, y_1, y_2) = a(t) \int_{0}^{y_1} f(s) ds + \int_{0}^{y_2} \frac{s ds}{g(s)}$$
 (2.4)

and note that

$$V'(t, y_1, y_2) \leq a'(t) \int_{o}^{y_1} f(s) ds$$

$$\leq \alpha(t) a(t) \int_{o}^{y_1} f(s) ds$$

$$\leq \alpha(t) V(t, y_1, y_2).$$

Hence again, we arrive at (2.3), from which the conclusion and the following corollary follow at once.

Corollary. Under the hypothesis of Theorem 5, if in addition assumption A_5 holds and $a(t) \ge a_0 > 0$ for $t \ge T$; then all solutions of (1.1) are also bounded.

By taking $\alpha(t) \equiv 0$, Theorem 2 and its corollary reduce to Theorem (II); similarly by taking $g(u') \equiv 1$, $f(u) = u^{2n-1}$ where n is a positive integer, the above result reduces to Theorem 4 of [4].

We now present two results on the boundedness of solutions by linear functions, i.e. $|u(t)| \le c |t|$ for some positive constant c, and for $t \ge T$; and the existence of limit of u'(t) as $t \to \infty$.

Theorem 3. Suppose that assumptions A_1 , A_2 and A_3 hold and that

- (i) $|f(u)| \le M |u|^{\alpha}$, where M, $\alpha > 0$,
- (ii) $\int_{0}^{\infty} |a(s)| s^{\alpha} ds < \infty,$
- (iii) $0 < g(v) \le K$ for all v;

then the derivative u'(t) of any solution u(t) of (1.1) has a limit if the initial conditions satisfy: for $\alpha > 1$,

$$\left\{ KM(\alpha - 1) \int_{t_0}^{\infty} s^{\alpha} |a(s)| ds \right\}^{\frac{1}{1 - a}} \ge \left\{ |u(t_0)| + |u'(t_0)| \right\}$$
 (2.5)

¹⁾ $a(t) F(u(t)) < \infty$ means that a solution u(t) of (1.1) is either bounded or unbounded, but in case of unboundedness must satisfy $a(t) F(u(t)) < \infty$. (Note that $a(t) \to O$, as $t \to \infty$ and assumption A₄.)

Proof. Consider equation (1.1) in its equivalent integral equation form:

$$u(t) = u(t_0) + u'(t_0)t - \int_{t_0}^{t} (t-s) a(s) f(u(s)) g(u'(s)) ds.$$

From the hypothesis of the theorem, we have for $t \ge t_0 \ge 1$ the following estimate:

$$\frac{|u(t)|}{t} \le (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^t s^\alpha KM |a(s)| \left(\frac{|u(s)|}{s}\right)^\alpha ds \quad (2.6)$$

By a variation of Gronwall's lemma (see e.g. [15], [16]), we obtain for $t \ge t_0 \ge 1$,

$$\frac{|u(t)|}{t} \le \left\{ \left(|u(t_0)| + |u'(t_0)| \right)^{1-\alpha} + KM(1-\alpha) \int_{t_0}^t s^\alpha |a(s)| ds \right\}^{\frac{1}{1-\alpha}} \tag{2.7}$$

which is finite on account of (2.5) and (i). Now from

$$u'(t) = u'(t_0) - \int_{t_0}^t a(s) f(u(s)) g(u'(s)) ds$$

and that

$$\left| \int_{t}^{t_{0}} a(s) f(u(s)) g(u'(s)) ds \right| \leq MK \int_{t}^{t_{0}} |a(s) u^{\alpha}(s)| ds$$
$$\leq MKC^{\alpha} \int_{t_{0}}^{t} |a(s)| s^{\alpha} ds$$

where C denotes the bound given in (2.7); we conclude that the limit $\lim_{t\to\infty} u'(t) = L$ exists.

We remark that the method of the above proof has also been applied by the author [17] to prove a generalization of a recent result of Waltman [18].

Theorem 4. Suppose that assumptions A_1 , A_2 and A_3 hold and in addition that

(a)
$$|f(u)| \leqslant M h(|u|)$$
,

where M > O and h(r) is a non-decreasing continous function such that $h(\lambda r) \leq \lambda^{\alpha} h(r)$, where λ is positive and α is a positive constant; and

$$H(x) = \int_{-\infty}^{x} \frac{dr}{h(r)} \to \infty \text{ as } x \to \infty,$$

(b)
$$\int |a(s)| s^{\alpha} ds < \infty$$
,

(c)
$$0 < g(v) \le K$$
 for all v ;

then the derivative of any solution of (1.1) has a limit.

Proof. Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{|u(t)|}{t} \leq \left(|u(t_0)| + |u'(t_0)|\right) + \int_{t_0}^t s^{\alpha} KM |a(s)| h\left(\frac{|u(s)|}{s}\right) ds$$

from which we conclude from a result of Bihari [14] that

$$\frac{|u(t)|}{t} \le H^{-1} \left(H(|u(t_0)| + |u'(t_0)|) + KM \int_{t_0}^{t} |a(s)| s^{\alpha} ds \right)$$

which is bounded for t on account of assumption (a). The remaining proof follows verbatim that of Theorem 3.

3. Boundedness Theorems II

Theorem 5. Suppose that assumptions A_1 , A_2 , A_3 and A_4 hold and in addition that

(i)
$$a(t) > 0$$
, $a'(t) \ge 0$, for $t \ge T$,

(ii)
$$\frac{d}{dt}\left(\frac{b}{a}\right) \le \beta(t)\left(1 + \frac{b}{a}\right)$$
, with $\int_{0}^{\infty} \beta(s) ds < \infty$

and

$$\left(1 + \frac{b}{a}\right) \ge \varepsilon > 0;$$

then every solution of (1.1) with (a(t) + b(t)) replacing a(t) is bounded.

Proof. Make the following substitution for the independent variable, $x = \int_{0}^{t} \sqrt{a(s)} ds$ which tends to infinity as $t \to \infty$, and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2} \left(\frac{a}{a^{3/2}} \right) \frac{du}{dx} + \left(1 + \frac{b}{a} \right) f(u) g(u') = 0$$
 (3.1)