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Theorem (I). Suppose that assumptions A_1, A_2, A_3 , and A_4 hold and in addition that $a(t) > 0$ and $a'(t) \geq 0$ for $t \geq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (I), suppose that assumption A_5 also holds and that $\lim_{t \rightarrow \infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (II). Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that $a'(t) \leq 0$ for $t \geq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (II), suppose that assumption A_5 also holds and $\lim_{t \rightarrow \infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (III). Suppose that assumptions A_1, A_2, A_3 , and A_4 hold and in addition that $a(t) \geq a_0 > 0$ for $t \geq T$, and $\int_0^{\infty} |a'(t)| dt < \infty$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (III), suppose that assumption A_5 also holds; then all solutions of (1.1) and their derivatives are bounded.

The method of proof for the above results is based essentially on the well-known lemma of Gronwall [10], which is also known as the Bellman's lemma. In this paper, we use in addition to this fundamental lemma, its generalizations [11], [12], and techniques borrowed from Lyapunov's stability theory.

It might be of interest to note that quite a few results in [4] are incorrect; in particular Theorems 5 and 6. Also, Theorems 3 and 4 are stated incorrectly.

2. BOUNDEDNESS THEOREMS I

Theorem 1. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and that $a(t) > 0$ for $t \geq T$ and there exists a non-negative function $\alpha(t)$ such that $-a'(t) \leq \alpha(t)a(t)$ with $\int_0^{\infty} \alpha(s) ds < \infty$. Then all solutions of (1.1) are bounded.

Proof. Write equation (1.1) in its system form ($y_1 = u$):

$$\begin{cases} \frac{dy_1}{dt} = y_2, \\ \frac{dy_2}{dt} = -a(t)f(y_1)g(y_2). \end{cases} \quad (2.1)$$

For system (2.1), we construct the following function:

$$V(t, y_1, y_2) = \int_0^{y_1} f(s) ds + \frac{1}{a(t)} \int_0^{y_2} \frac{s ds}{g(s)} \quad (2.2)$$

Clearly, under the hypothesis of the theorem, we have $V > 0$ whenever $y_1^2 + y_2^2 \neq 0$, and by A_4 , $V \rightarrow \infty$ as $y_1 \rightarrow \infty$. Differentiating with respect to t , we obtain

$$V'(t, y_1, y_2) \leq -\frac{a'(t)}{a^2(t)} \int_0^{y_2} \frac{s ds}{g(s)} \leq \alpha(t) V(t, y_1, y_2),$$

hence,

$$V(t, y_1, y_2) \leq V(T, y_1(T), y_2(T)) \left\{ \exp \int_T^t \alpha(s) ds \right\} < \infty \quad (2.3)$$

for all t ; and therefore all solutions of (1.1) are bounded. Furthermore, if assumption A_5 also holds and $a(t) \leq a_1$ for all $t \geq T$, then $u'(t)$ is also bounded for in this case $V \rightarrow \infty$ as $y_2 \rightarrow \infty$. Thus,

Corollary. In addition to the hypothesis of Theorem 4, suppose that assumption A_5 holds and $a(t) \leq a_1$ for all $t \geq T$; then all solutions and their derivatives are bounded.

Theorem 1 and its corollary generalize a result of Zhang ([4], Theorem 3). By taking $\alpha(s) \equiv 0$, Theorem 1 reduces to Theorem I. All these results are extensions of a theorem of Klovov ([13], Theorem 1). We remark that a slightly weaker version of Klovov's theorem may also be found in Waltman [14].

By the above result, we may conclude for example that all solutions $u(t)$ and their derivatives $u'(t)$ of the following equation:

$$u''(t) + (1 + e^{-t} \sin t) u^{\frac{3}{2}}(t) [2 + \cos u(t)] = 0$$

are bounded. On the other hand, no previously available result seems to yield such a conclusion.

Theorem 2. Suppose that assumptions A_1, A_2, A_3 , and A_4 hold and that $a(t) > 0$, $a(t) \rightarrow 0$, and there exists a $\alpha(t) \geq 0$ such that $a'(t) \leq \alpha(t) a(t)$ while $\int_0^\infty \alpha(s) ds < \infty$. Then all solutions of (1.1) satisfy: $a(t) F(u(t)) = a(t) \int_{u(t)}^\infty f(s) ds < \infty$ ¹⁾ for $t \geq T$, and all its derivatives are bounded.

Proof. Consider the following function:

$$V(t, y_1, y_2) = a(t) \int_0^{y_1} f(s) ds + \int_0^{y_2} \frac{s ds}{g(s)} \quad (2.4)$$

and note that

$$\begin{aligned} V'(t, y_1, y_2) &\leq a'(t) \int_0^{y_1} f(s) ds \\ &\leq \alpha(t) a(t) \int_0^{y_1} f(s) ds \\ &\leq \alpha(t) V(t, y_1, y_2). \end{aligned}$$

Hence again, we arrive at (2.3), from which the conclusion and the following corollary follow at once.

Corollary. Under the hypothesis of Theorem 5, if in addition assumption A_5 holds and $a(t) \geq a_0 > 0$ for $t \geq T$; then all solutions of (1.1) are also bounded.

By taking $\alpha(t) \equiv 0$, Theorem 2 and its corollary reduce to Theorem (II); similarly by taking $g(u') \equiv 1$, $f(u) = u^{2n-1}$ where n is a positive integer, the above result reduces to Theorem 4 of [4].

We now present two results on the boundedness of solutions by linear functions, i.e. $|u(t)| \leq c|t|$ for some positive constant c , and for $t \geq T$; and the existence of limit of $u'(t)$ as $t \rightarrow \infty$.

Theorem 3. Suppose that assumptions A_1, A_2 and A_3 hold and that

- (i) $|f(u)| \leq M|u|^\alpha$, where $M, \alpha > 0$,
- (ii) $\int_0^\infty |a(s)| s^\alpha ds < \infty$,
- (iii) $0 < g(v) \leq K$ for all v ;

then the derivative $u'(t)$ of any solution $u(t)$ of (1.1) has a limit if the initial conditions satisfy: for $\alpha > 1$,

$$\left\{ KM(\alpha - 1) \int_{t_0}^\infty s^\alpha |a(s)| ds \right\}^{\frac{1}{1-\alpha}} \geq \left\{ |u(t_0)| + |u'(t_0)| \right\} \quad (2.5)$$

¹⁾ $a(t) F(u(t)) < \infty$ means that a solution $u(t)$ of (1.1) is either bounded or unbounded, but in case of unboundedness must satisfy $a(t) F(u(t)) < \infty$. (Note that $a(t) \rightarrow 0$, as $t \rightarrow \infty$ and assumption A_4 .)

Proof. Consider equation (1.1) in its equivalent integral equation form:

$$u(t) = u(t_0) + u'(t_0)t - \int_{t_0}^t (t-s) a(s) f(u(s)) g(u'(s)) ds.$$

From the hypothesis of the theorem, we have for $t \geq t_0 \geq 1$ the following estimate:

$$\frac{|u(t)|}{t} \leq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^t s^\alpha KM |a(s)| \left(\frac{|u(s)|}{s} \right)^\alpha ds \quad (2.6)$$

By a variation of Gronwall's lemma (see e.g. [15], [16]), we obtain for $t \geq t_0 \geq 1$,

$$\frac{|u(t)|}{t} \leq \{ (|u(t_0)| + |u'(t_0)|)^{1-\alpha} + KM(1-\alpha) \int_{t_0}^t s^\alpha |a(s)| ds \}^{\frac{1}{1-\alpha}} \quad (2.7)$$

which is finite on account of (2.5) and (i). Now from

$$u'(t) = u'(t_0) - \int_{t_0}^t a(s) f(u(s)) g(u'(s)) ds$$

and that

$$\begin{aligned} \left| \int_t^{t_0} a(s) f(u(s)) g(u'(s)) ds \right| &\leq MK \int_t^{t_0} |a(s) u^\alpha(s)| ds \\ &\leq MKC^\alpha \int_{t_0}^t |a(s)| s^\alpha ds \end{aligned}$$

where C denotes the bound given in (2.7); we conclude that the limit $\lim_{t \rightarrow \infty} u'(t) = L$ exists.

We remark that the method of the above proof has also been applied by the author [17] to prove a generalization of a recent result of Waltman [18].

Theorem 4. Suppose that assumptions A_1 , A_2 and A_3 hold and in addition that

$$(a) \quad |f(u)| \leq M h(|u|),$$

where $M > 0$ and $h(r)$ is a non-decreasing continuous function such that $h(\lambda r) \leq \lambda^\alpha h(r)$, where λ is positive and α is a positive constant; and

$$H(x) = \int_{\infty}^x \frac{dr}{h(r)} \rightarrow \infty \text{ as } x \rightarrow \infty,$$

$$(b) \quad \int |a(s)| s^\alpha ds < \infty,$$

$$(c) \quad 0 < g(v) \leq K \quad \text{for all } v;$$

then the derivative of any solution of (1.1) has a limit.

Proof. Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{|u(t)|}{t} \leq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^t s^\alpha KM |a(s)| h\left(\frac{|u(s)|}{s}\right) ds,$$

from which we conclude from a result of Bihari [14] that

$$\frac{|u(t)|}{t} \leq H^{-1} \left(H(|u(t_0)| + |u'(t_0)|) + KM \int_{t_0}^t |a(s)| s^\alpha ds \right)$$

which is bounded for t on account of assumption (a). The remaining proof follows verbatim that of Theorem 3.

3. BOUNDEDNESS THEOREMS II

Theorem 5. Suppose that assumptions A_1 , A_2 , A_3 and A_4 hold and in addition that

$$(i) \quad a(t) > 0, \quad a'(t) \geq 0, \quad \text{for } t \geq T,$$

$$(ii) \quad \frac{d}{dt} \left(\frac{b}{a} \right) \leq \beta(t) \left(1 + \frac{b}{a} \right), \quad \text{with } \int_0^\infty \beta(s) ds < \infty$$

and

$$\left(1 + \frac{b}{a} \right) \geq \varepsilon > 0;$$

then every solution of (1.1) with $(a(t) + b(t))$ replacing $a(t)$ is bounded.

Proof. Make the following substitution for the independent variable, $x = \int_0^t \sqrt{a(s)} ds$ which tends to infinity as $t \rightarrow \infty$, and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2} \left(\frac{a}{a^{3/2}} \right) \frac{du}{dx} + \left(1 + \frac{b}{a} \right) f(u) g(u') = 0 \quad (3.1)$$