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| <b>Zeitschrift:</b> | L'Enseignement Mathématique   |
| <b>Herausgeber:</b> | Commission Internationale de l'Enseignement Mathématique                              |
| <b>Band:</b>        | 13 (1967)   |
| <b>Heft:</b>        | 1: L'ENSEIGNEMENT MATHÉMATIQUE  |
| <br>                |   |
| <b>Artikel:</b>     | BOUNDEDNESS THEOREMS FOR SOLUTIONS OF $u''(t) + a(t)f(u) = 0$ (IV)                    |
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| <b>Kapitel:</b>     | 2. BOUNDEDNESS THEOREMS I   |
| <b>DOI:</b>         | <a href="https://doi.org/10.5169/seals-41539">https://doi.org/10.5169/seals-41539</a> |

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*Theorem (I).* Suppose that assumptions  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_4$  hold and in addition that  $a(t) > 0$  and  $a'(t) \geq 0$  for  $t \geq T$ . Then all solutions of (1.1) are bounded.

*Corollary.* In addition to the hypothesis of Theorem (I), suppose that assumption  $A_5$  also holds and that  $\lim_{t \rightarrow \infty} a(t) = k > 0$ ; then all solutions of (1.1) and their derivatives are bounded.

*Theorem (II).* Suppose that assumptions  $A_1$ ,  $A_2$ ,  $A_3$  and  $A_4$  hold and in addition that  $a'(t) \leq 0$  for  $t \geq T$ . Then all solutions of (1.1) are bounded.

*Corollary.* In addition to the hypothesis of Theorem (II), suppose that assumption  $A_5$  also holds and  $\lim_{t \rightarrow \infty} a(t) = k > 0$ ; then all solutions of (1.1) and their derivatives are bounded.

*Theorem (III).* Suppose that assumptions  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_4$  hold and in addition that  $a(t) \geq a_0 > 0$  for  $t \geq T$ , and  $\int_t^\infty |a'(s)| ds < \infty$ . Then all solutions of (1.1) are bounded.

*Corollary.* In addition to the hypothesis of Theorem (III), suppose that assumption  $A_5$  also holds; then all solutions of (1.1) and their derivatives are bounded.

The method of proof for the above results is based essentially on the well-known lemma of Gronwall [10], which is also known as the Bellman's lemma. In this paper, we use in addition to this fundamental lemma, its generalizations [11], [12], and techniques borrowed from Lyapunov's stability theory.

It might be of interest to note that quite a few results in [4] are incorrect; in particular Theorems 5 and 6. Also, Theorems 3 and 4 are stated incorrectly.

## 2. BOUNDEDNESS THEOREMS I

*Theorem 1.* Suppose that assumptions  $A_1$ ,  $A_2$ ,  $A_3$  and  $A_4$  hold and that  $a(t) > 0$  for  $t \geq T$  and there exists a non-negative function  $\alpha(t)$  such that  $-a'(t) \leq \alpha(t)a(t)$  with  $\int_t^\infty \alpha(s) ds < \infty$ . Then all solutions of (1.1) are bounded.

*Proof.* Write equation (1.1) in its system form ( $y_1 = u$ ):

$$\begin{cases} \frac{dy_1}{dt} = y_2, \\ \frac{dy_2}{dt} = -a(t)f(y_1)g(y_2). \end{cases} \quad (2.1)$$

For system (2.1), we construct the following function:

$$V(t, y_1, y_2) = \int_0^{y_1} f(s) ds + \frac{1}{a(t)} \int_0^{y_2} \frac{s}{g(s)} ds \quad (2.2)$$

Clearly, under the hypothesis of the theorem, we have  $V > 0$  whenever  $y_1^2 + y_2^2 \neq 0$ , and by  $A_4$ ,  $V \rightarrow \infty$  as  $y_1 \rightarrow \infty$ . Differentiating with respect to  $t$ , we obtain

$$V'(t, y_1, y_2) \leq -\frac{a'(t)}{a^2(t)} \int_0^{y_2} \frac{s}{g(s)} ds \leq \alpha(t) V(t, y_1, y_2),$$

hence,

$$V(t, y_1, y_2) \leq V(T, y_1(T), y_2(T)) \left\{ \exp \int_T^t \alpha(s) ds \right\} < \infty \quad (2.3)$$

for all  $t$ ; and therefore all solutions of (1.1) are bounded. Furthermore, if assumption  $A_5$  also holds and  $\alpha(t) \leq a_1$  for all  $t \geq T$ , then  $u'(t)$  is also bounded for in this case  $V \rightarrow \infty$  as  $y_2 \rightarrow \infty$ . Thus,

*Corollary.* In addition to the hypothesis of Theorem 4, suppose that assumption  $A_5$  holds and  $\alpha(t) \leq a_1$  for all  $t \geq T$ ; then all solutions and their derivatives are bounded.

Theorem 1 and its corollary generalize a result of Zhang ([4], Theorem 3). By taking  $\alpha(s) \equiv 0$ , Theorem 1 reduces to Theorem I. All these results are extensions of a theorem of Klokov ([13], Theorem 1). We remark that a slightly weaker version of Klokov's theorem may also be found in Waltman [14].

By the above result, we may conclude for example that all solutions  $u(t)$  and their derivatives  $u'(t)$  of the following equation:

$$u''(t) + (1 + e^{-t} \sin t) u^{\frac{3}{2}}(t) [2 + \cos u(t)] = 0$$

are bounded. On the other hand, no previously available result seems to yield such a conclusion.

**Theorem 2.** Suppose that assumptions  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_4$  hold and that  $a(t) > 0$ ,  $a(t) \rightarrow 0$ , and there exists a  $\alpha(t) \geq 0$  such that  $a'(t) \leq \alpha(t) a(t)$  while  $\int_0^\infty \alpha(s) ds < \infty$ . Then all solutions of (1.1) satisfy:  $a(t) F(u(t)) = a(t) \int_0^{u(t)} f(s) ds < \infty$ <sup>1)</sup> for  $t \geq T$ , and all its derivatives are bounded.

**Proof.** Consider the following function:

$$V(t, y_1, y_2) = a(t) \int_0^{y_1} f(s) ds + \int_0^{y_2} \frac{s}{g(s)} ds \quad (2.4)$$

and note that

$$\begin{aligned} V'(t, y_1, y_2) &\leq a'(t) \int_0^{y_1} f(s) ds \\ &\leq \alpha(t) a(t) \int_0^{y_1} f(s) ds \\ &\leq \alpha(t) V(t, y_1, y_2). \end{aligned}$$

Hence again, we arrive at (2.3), from which the conclusion and the following corollary follow at once.

**Corollary.** Under the hypothesis of Theorem 5, if in addition assumption  $A_5$  holds and  $a(t) \geq a_0 > 0$  for  $t \geq T$ ; then all solutions of (1.1) are also bounded.

By taking  $\alpha(t) \equiv 0$ , Theorem 2 and its corollary reduce to Theorem (II); similarly by taking  $g(u) \equiv 1$ ,  $f(u) = u^{2n-1}$  where  $n$  is a positive integer, the above result reduces to Theorem 4 of [4].

We now present two results on the boundedness of solutions by linear functions, i.e.  $|u(t)| \leq c|t|$  for some positive constant  $c$ , and for  $t \geq T$ ; and the existence of limit of  $u'(t)$  as  $t \rightarrow \infty$ .

**Theorem 3.** Suppose that assumptions  $A_1$ ,  $A_2$  and  $A_3$  hold and that

- (i)  $|f(u)| \leq M|u|^\alpha$ , where  $M, \alpha > 0$ ,
- (ii)  $\int_0^\infty |a(s)| s^\alpha ds < \infty$ ,
- (iii)  $0 < g(v) \leq K$  for all  $v$ ;

then the derivative  $u'(t)$  of any solution  $u(t)$  of (1.1) has a limit if the initial conditions satisfy: for  $\alpha > 1$ ,

$$\left\{ KM(\alpha-1) \int_{t_0}^\infty s^\alpha |a(s)| ds \right\}^{\frac{1}{1-\alpha}} \geq \left\{ |u(t_0)| + |u'(t_0)| \right\} \quad (2.5)$$

<sup>1)</sup>  $a(t) F(u(t)) < \infty$  means that a solution  $u(t)$  of (1.1) is either bounded or unbounded, but in case of unboundedness must satisfy  $a(t) F(u(t)) < \infty$ . (Note that  $a(t) \rightarrow 0$ , as  $t \rightarrow \infty$  and assumption  $A_4$ .)

*Proof.* Consider equation (1.1) in its equivalent integral equation form:

$$u(t) = u(t_0) + u'(t_0)t - \int_{t_0}^t (t-s)a(s)f(u(s))g(u'(s))ds.$$

From the hypothesis of the theorem, we have for  $t \geq t_0 \geq 1$  the following estimate:

$$\frac{|u(t)|}{t} \leq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^t s^\alpha KM |a(s)| \left(\frac{|u(s)|}{s}\right)^\alpha ds \quad (2.6)$$

By a variation of Gronwall's lemma (see e.g. [15], [16]), we obtain for  $t \geq t_0 \geq 1$ ,

$$\frac{|u(t)|}{t} \leq \left\{ (|u(t_0)| + |u'(t_0)|)^{1-\alpha} + KM(1-\alpha) \int_{t_0}^t s^\alpha |a(s)| ds \right\}^{\frac{1}{1-\alpha}} \quad (2.7)$$

which is finite on account of (2.5) and (i). Now from

$$u'(t) = u'(t_0) - \int_{t_0}^t a(s)f(u(s))g(u'(s))ds$$

and that

$$\begin{aligned} \left| \int_t^{t_0} a(s)f(u(s))g(u'(s))ds \right| &\leq MK \int_t^{t_0} |a(s)| u^\alpha(s) ds \\ &\leq MKC^\alpha \int_{t_0}^t |a(s)| s^\alpha ds \end{aligned}$$

where  $C$  denotes the bound given in (2.7); we conclude that the limit  $\lim_{t \rightarrow \infty} u'(t) = L$  exists.

We remark that the method of the above proof has also been applied by the author [17] to prove a generalization of a recent result of Waltman [18].

*Theorem 4.* Suppose that assumptions  $A_1$ ,  $A_2$  and  $A_3$  hold and in addition that

$$(a) \quad |f(u)| \leq M h(|u|),$$

where  $M > 0$  and  $h(r)$  is a non-decreasing continuous function such that  $h(\lambda r) \leq \lambda^\alpha h(r)$ , where  $\lambda$  is positive and  $\alpha$  is a positive constant; and

$$H(x) = \int_{-\infty}^x \frac{dr}{h(r)} \rightarrow \infty \text{ as } x \rightarrow \infty,$$

- (b)  $\int |a(s)| s^\alpha ds < \infty$ ,
- (c)  $0 < g(v) \leq K$  for all  $v$ ;

then the derivative of any solution of (1.1) has a limit.

*Proof.* Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{|u(t)|}{t} \leq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^t s^\alpha KM |a(s)| h\left(\frac{|u(s)|}{s}\right) ds,$$

from which we conclude from a result of Bihari [14] that

$$\frac{|u(t)|}{t} \leq H^{-1} (H(|u(t_0)| + |u'(t_0)|) + KM \int_{t_0}^t |a(s)| s^\alpha ds)$$

which is bounded for  $t$  on account of assumption (a). The remaining proof follows verbatim that of Theorem 3.

### 3. BOUNDEDNESS THEOREMS II

*Theorem 5.* Suppose that assumptions  $A_1$ ,  $A_2$ ,  $A_3$  and  $A_4$  hold and in addition that

- (i)  $a(t) > 0$ ,  $a'(t) \geq 0$ , for  $t \geq T$ ,
- (ii)  $\frac{d}{dt}\left(\frac{b}{a}\right) \leq \beta(t)\left(1 + \frac{b}{a}\right)$ , with  $\int_0^\infty \beta(s) ds < \infty$

and

$$\left(1 + \frac{b}{a}\right) \geq \varepsilon > 0;$$

then every solution of (1.1) with  $(a(t) + b(t))$  replacing  $a(t)$  is bounded.

*Proof.* Make the following substitution for the independent variable,  $x = \int_0^t \sqrt{a(s)} ds$  which tends to infinity as  $t \rightarrow \infty$ , and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2} \left( \frac{a}{a^{3/2}} \right) \frac{du}{dx} + \left(1 + \frac{b}{a}\right) f(u) g(u') = 0 \quad (3.1)$$