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BOUNDEDNESS THEOREMS FOR
SOLUTIONS OF $u''(t) + a(t)f(u)g(u') = 0$ (IV)

by James S. W. WONG

1. INTRODUCTION

In our previous work [1-3], we have presented rather fragmentary results concerning the boundedness of solutions to certain second order non-linear differential equations of the following form:

$$u''(t) + a(t)f(u)g(u') = 0 \quad (1.1)$$

where $a(t)$, $f(u)$ and $g(u')$ satisfy certain assumptions to be described below. The purpose of the present paper is to further extend these results and establish comparison theorems. Some of our results presented here may be considered as generalizations to the results of Zhang [4], where a special case of equation (1.1):

$$u''(t) + a(t)f(u) = 0 \quad (1.2)$$

was treated ¹⁾.

Throughout the discussion of this paper, we will need the following assumptions:

(A₁) $g(u')$ is a positive continuous function of u' ,

(A₂) $f(u)$ is a continuous function of u satisfying $uf(u) > 0$, if $u \neq 0$,

(A₃) $a(t)$ is continuous in t ,

$$(A_4) \lim_{|u| \rightarrow \infty} \int_0^u f(s) ds = \infty,$$

$$(A_5) \lim_{|v| \rightarrow \infty} \int_0^v \frac{g(s)}{s} ds = \infty.$$

We also list in the following a brief résumé of our previous results on boundedness.

¹⁾ For other boundedness result concerning (1.2), see [5]-[7], [13], [14].

Theorem (I). Suppose that assumptions $A_1, A_2, A_3,$ and A_4 hold and in addition that $a(t) > 0$ and $a'(t) \geq 0$ for $t \geq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (I), suppose that assumption A_5 also holds and that $\lim_{t \rightarrow \infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (II). Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that $a'(t) \leq 0$ for $t \geq T$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (II), suppose that assumption A_5 also holds and $\lim_{t \rightarrow \infty} a(t) = k > 0$; then all solutions of (1.1) and their derivatives are bounded.

Theorem (III). Suppose that assumptions $A_1, A_2, A_3,$ and A_4 hold and in addition that $a(t) \geq a_0 > 0$ for $t \geq T$, and $\int_0^{\infty} |a'(t)| dt < \infty$. Then all solutions of (1.1) are bounded.

Corollary. In addition to the hypothesis of Theorem (III), suppose that assumption A_5 also holds; then all solutions of (1.1) and their derivatives are bounded.

The method of proof for the above results is based essentially on the well-known lemma of Gronwall [10], which is also known as the Bellman's lemma. In this paper, we use in addition to this fundamental lemma, its generalizations [11], [12], and techniques borrowed from Lyapunov's stability theory.

It might be of interest to note that quite a few results in [4] are incorrect; in particular Theorems 5 and 6. Also, Theorems 3 and 4 are stated incorrectly.

2. BOUNDEDNESS THEOREMS I

Theorem 1. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and that $a(t) > 0$ for $t \geq T$ and there exists a non-negative function $\alpha(t)$ such that $-a'(t) \leq \alpha(t)a(t)$ with $\int_0^{\infty} \alpha(s) ds < \infty$. Then all solutions of (1.1) are bounded.

Proof. Write equation (1.1) in its system form ($y_1 = u$):

$$\begin{cases} \frac{dy_1}{dt} = y_2, \\ \frac{dy_2}{dt} = -a(t)f(y_1)g(y_2). \end{cases} \quad (2.1)$$

For system (2.1), we construct the following function:

$$V(t, y_1, y_2) = \int_0^{y_1} f(s) ds + \frac{1}{a(t)} \int_0^{y_2} \frac{s ds}{g(s)} \quad (2.2)$$

Clearly, under the hypothesis of the theorem, we have $V > 0$ whenever $y_1^2 + y_2^2 \neq 0$, and by A_4 , $V \rightarrow \infty$ as $y_1 \rightarrow \infty$. Differentiating with respect to t , we obtain

$$V'(t, y_1, y_2) \leq -\frac{a'(t)}{a^2(t)} \int_0^{y_2} \frac{s ds}{g(s)} \leq \alpha(t) V(t, y_1, y_2),$$

hence,

$$V(t, y_1, y_2) \leq V(T, y_1(T), y_2(T)) \left\{ \exp \int_T^t \alpha(s) ds \right\} < \infty \quad (2.3)$$

for all t ; and therefore all solutions of (1.1) are bounded. Furthermore, if assumption A_5 also holds and $a(t) \leq a_1$ for all $t \geq T$, then $u'(t)$ is also bounded for in this case $V \rightarrow \infty$ as $y_2 \rightarrow \infty$. Thus,

Corollary. In addition to the hypothesis of Theorem 4, suppose that assumption A_5 holds and $a(t) \leq a_1$ for all $t \geq T$; then all solutions and their derivatives are bounded.

Theorem 1 and its corollary generalize a result of Zhang ([4], Theorem 3). By taking $\alpha(s) \equiv 0$, Theorem 1 reduces to Theorem I. All these results are extensions of a theorem of Klovov ([13], Theorem 1). We remark that a slightly weaker version of Klovov's theorem may also be found in Waltman [14].

By the above result, we may conclude for example that all solutions $u(t)$ and their derivatives $u'(t)$ of the following equation:

$$u''(t) + (1 + e^{-t} \sin t) u^{\frac{3}{2}}(t) [2 + \cos u(t)] = 0$$

are bounded. On the other hand, no previously available result seems to yield such a conclusion.

Theorem 2. Suppose that assumptions $A_1, A_2, A_3,$ and A_4 hold and that $a(t) > 0, a(t) \rightarrow 0,$ and there exists a $\alpha(t) \geq 0$ such that $a'(t) \leq \alpha(t) a(t)$ while $\int_0^{\infty} \alpha(s) ds < \infty.$ Then all solutions of (1.1) satisfy: $a(t) F(u(t)) = a(t) \int_0^{u(t)} f(s) ds < \infty$ ¹⁾ for $t \geq T,$ and all its derivatives are bounded.

Proof. Consider the following function:

$$V(t, y_1, y_2) = a(t) \int_0^{y_1} f(s) ds + \int_0^{y_2} \frac{s ds}{g(s)} \quad (2.4)$$

and note that

$$\begin{aligned} V'(t, y_1, y_2) &\leq a'(t) \int_0^{y_1} f(s) ds \\ &\leq \alpha(t) a(t) \int_0^{y_1} f(s) ds \\ &\leq \alpha(t) V(t, y_1, y_2). \end{aligned}$$

Hence again, we arrive at (2.3), from which the conclusion and the following corollary follow at once.

Corollary. Under the hypothesis of Theorem 5, if in addition assumption A_5 holds and $a(t) \geq a_0 > 0$ for $t \geq T;$ then all solutions of (1.1) are also bounded.

By taking $\alpha(t) \equiv 0,$ Theorem 2 and its corollary reduce to Theorem (II); similarly by taking $g(u') \equiv 1, f(u) = u^{2n-1}$ where n is a positive integer, the above result reduces to Theorem 4 of [4].

We now present two results on the boundedness of solutions by linear functions, i.e. $|u(t)| \leq c|t|$ for some positive constant $c,$ and for $t \geq T;$ and the existence of limit of $u'(t)$ as $t \rightarrow \infty.$

Theorem 3. Suppose that assumptions A_1, A_2 and A_3 hold and that

- (i) $|f(u)| \leq M|u|^\alpha,$ where $M, \alpha > 0,$
- (ii) $\int_0^{\infty} |a(s)| s^\alpha ds < \infty,$
- (iii) $0 < g(v) \leq K$ for all $v;$

then the derivative $u'(t)$ of any solution $u(t)$ of (1.1) has a limit if the initial conditions satisfy: for $\alpha > 1,$

$$\left\{ KM(\alpha - 1) \int_{t_0}^{\infty} s^\alpha |a(s)| ds \right\}^{\frac{1}{1-\alpha}} \geq \left\{ |u(t_0)| + |u'(t_0)| \right\} \quad (2.5)$$

¹⁾ $a(t) F(u(t)) < \infty$ means that a solution $u(t)$ of (1.1) is either bounded or unbounded, but in case of unboundedness must satisfy $a(t) F(u(t)) < \infty.$ (Note that $a(t) \rightarrow 0,$ as $t \rightarrow \infty$ and assumption $A_4.$)

Proof. Consider equation (1.1) in its equivalent integral equation form:

$$u(t) = u(t_0) + u'(t_0)t - \int_{t_0}^t (t-s) a(s) f(u(s)) g(u'(s)) ds.$$

From the hypothesis of the theorem, we have for $t \geq t_0 \geq 1$ the following estimate:

$$\frac{|u(t)|}{t} \leq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^t s^\alpha KM |a(s)| \left(\frac{|u(s)|}{s}\right)^\alpha ds \quad (2.6)$$

By a variation of Gronwall's lemma (see e.g. [15], [16]), we obtain for $t \geq t_0 \geq 1$,

$$\frac{|u(t)|}{t} \leq \{(|u(t_0)| + |u'(t_0)|)^{1-\alpha} + KM(1-\alpha) \int_{t_0}^t s^\alpha |a(s)| ds\}^{\frac{1}{1-\alpha}} \quad (2.7)$$

which is finite on account of (2.5) and (i). Now from

$$u'(t) = u'(t_0) - \int_{t_0}^t a(s) f(u(s)) g(u'(s)) ds$$

and that

$$\begin{aligned} \left| \int_t^{t_0} a(s) f(u(s)) g(u'(s)) ds \right| &\leq MK \int_t^{t_0} |a(s) u^\alpha(s)| ds \\ &\leq MKC^\alpha \int_{t_0}^t |a(s)| s^\alpha ds \end{aligned}$$

where C denotes the bound given in (2.7); we conclude that the limit $\lim_{t \rightarrow \infty} u'(t) = L$ exists.

We remark that the method of the above proof has also been applied by the author [17] to prove a generalization of a recent result of Waltman [18].

Theorem 4. Suppose that assumptions A_1 , A_2 and A_3 hold and in addition that

$$(a) \quad |f(u)| \leq Mh(|u|),$$

where $M > 0$ and $h(r)$ is a non-decreasing continuous function such that $h(\lambda r) \leq \lambda^\alpha h(r)$, where λ is positive and α is a positive constant; and

$$H(x) = \int_{\infty}^x \frac{dr}{h(r)} \rightarrow \infty \text{ as } x \rightarrow \infty,$$

$$(b) \int |a(s)| s^\alpha ds < \infty,$$

$$(c) 0 < g(v) \leq K \text{ for all } v;$$

then the derivative of any solution of (1.1) has a limit.

Proof. Proceeding as in the above proof, we obtain instead of (3.2) the following estimate:

$$\frac{|u(t)|}{t} \leq (|u(t_0)| + |u'(t_0)|) + \int_{t_0}^t s^\alpha KM |a(s)| h\left(\frac{|u(s)|}{s}\right) ds,$$

from which we conclude from a result of Bihari [14] that

$$\frac{|u(t)|}{t} \leq H^{-1} (H(|u(t_0)| + |u'(t_0)|) + KM \int_{t_0}^t |a(s)| s^\alpha ds)$$

which is bounded for t on account of assumption (a). The remaining proof follows verbatim that of Theorem 3.

3. BOUNDEDNESS THEOREMS II

Theorem 5. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that

$$(i) a(t) > 0, \quad a'(t) \geq 0, \quad \text{for } t \geq T,$$

$$(ii) \frac{d}{dt} \left(\frac{b}{a} \right) \leq \beta(t) \left(1 + \frac{b}{a} \right), \quad \text{with } \int_0^\infty \beta(s) ds < \infty$$

and

$$\left(1 + \frac{b}{a} \right) \geq \varepsilon > 0;$$

then every solution of (1.1) with $(a(t) + b(t))$ replacing $a(t)$ is bounded.

Proof. Make the following substitution for the independent variable, $x = \int_0^t \sqrt{a(s)} ds$ which tends to infinity as $t \rightarrow \infty$, and obtain instead of (1.1) its transformed equation:

$$\frac{d^2 u}{dx^2} + \frac{1}{2} \left(\frac{a}{a^{3/2}} \right) \frac{du}{dx} + \left(1 + \frac{b}{a} \right) f(u) g(u') = 0 \quad (3.1)$$

where “dot” denotes differentiation with respect to t . Now write equation in its system form, letting $y_1 = u$:

$$\begin{cases} \frac{d y_1}{d x} = y_2 \\ \frac{d y_2}{d x} = -\frac{1}{2} \left(\frac{a}{a^{3/2}} \right) y_2 - \left(1 + \frac{b}{a} \right) f(y_1) g(\sqrt{a} y_2). \end{cases} \quad (3.2)$$

Define for (3.2) the following function:

$$V(x, y_1, y_2) = \left(1 + \frac{b}{a} \right) \int^{y_1} f(s) ds + \int^{y_2} \frac{s ds}{g(\sqrt{a} s)},$$

and observe:

$$\begin{aligned} \frac{dV}{dx} &\leq \frac{\beta(t)}{\sqrt{a(t)}} \left(1 + \frac{b}{a} \right) \int^{y_1} f(s) ds - \frac{1}{2} \frac{a}{a^{3/2}} y_2^2 \\ &\leq \frac{\beta(t)}{\sqrt{a(t)}} V. \end{aligned}$$

Hence we have

$$V(x, y_1, y_2) \leq V(x(T), y_1(x(T)), y_2(x(T))) \exp \int_T^t \beta(s) ds$$

which is finite. From (ii) we note that $V \rightarrow \infty$ as $y_1 \rightarrow \infty$ and $V > 0$ if $y_1^2 + y_2^2 \neq 0$. Thus, every solution of (1.1) is bounded.

Corollary. Suppose in addition to the hypothesis of Theorem 5 that assumption A_5 also holds and that $\lim_{t \rightarrow \infty} a(t) = a_1 < \infty$, then every solution of (1.1) and its derivative are bounded.

From the above result we may conclude for example that all solutions of the following equation:

$$u''(t) + (c_1 t^\alpha + c_2 t^\beta) u^\lambda(t) (1 + \exp u'(t) \sin u'(t)) = 0$$

are bounded for all $c_1, c_2 > 0$, $\alpha > \beta \geq 0$, and $\lambda > 0$.

We now consider the following inhomogeneous equation:

$$u''(t) + a(t) f(u) g(u') = h(t, u, u') \quad (3.3)$$

and assume that $|u' h(t, u, u')| \leq \gamma(t) g(u')$ where $\int_0^\infty \gamma(s) ds < \infty$.

Theorem 6. Suppose that assumptions A_1, A_2, A_3 and A_4 hold and in addition that $a(t) > 0$ and $a'(t) \geq 0$ for $t \geq T$; then all solutions of (3.3) are bounded.

Proof. Integrate (3.3) in the following manner:

$$G(u'(t)) - G(u'(t_0)) + a(t)F(u(t)) - a(t_0)F(u(t_0)) \\ = \int_{t_0}^t a'(s)F(u(s))ds + \int_{t_0}^t \frac{h(t, u, u')u'(s)ds}{g(u')} \quad (3.4)$$

where $G(v) = \int_0^v \frac{s ds}{g(s)}$ and $F(u) = \int_0^u f(s) ds$. Taking absolute values and noting that $G(v) \geq 0$ and $F(u) \geq 0$, we obtain

$$a(t)F(u(t)) \leq c_0 + c_1 + \int_{t_0}^t a'(s)F(u(s))ds \quad (3.5)$$

where $c_0 = G(u'(t_0)) + a(t_0)F(u(t_0))$ and $c_1 = \int_{t_0}^{\infty} \gamma(s) ds$ are non-negative constants. From (3.5) and A_4 it is now clear that every solution of (3.3) are bounded (cf. [1]).

Corollary. In addition to the hypothesis of Theorem 6, suppose that assumption A_5 also holds and that $\lim_{t \rightarrow \infty} a(t) = k > 0$; then all solutions of (3.3) and their derivatives are bounded.

We note that by setting $h(t, u, u') \equiv 0$, the above result again reduces to Theorem 1 and its corollary. Other comparison theorems may be formulated in a similar way as Theorem 6 by extending the corresponding result for the homogeneous equation. Since the procedure is clear, the statements and proofs of these results will be omitted.

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