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- 4) Let P_1 and P_2 be any two distinct points on \overline{E} , $Q_1 = f(P_1)$, and $Q_2 = f(P_2)$. Let $P_1 P_2$ denote the closed interval determined by P_1 and P_2 and $Q_1 Q_2$ the closed interval determined by Q_1 and Q_2 . Let the curve $C = f(P_1 P_2)$. Then there exists a point R on C such that the tangent line to C at R is parallel to $Q_1 Q_2$.
- 5) With the notation as in 4), let the deviation $D(P_1 P_2)$ denote the L.U.B. of the acute angles φ between the surface chord $Q_1 Q_2$ and any tangent line to C. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < \rho(P_1, P_2) < \delta$ then $D(P_1 P_2) < \epsilon$.
- 6) For every $\in > 0$ there exists $\delta > 0$ such that if P_1 and P_2 are any two distinct points of \overline{E} such that $\rho(P_1, P_2) < \delta$ then $\psi < \in$, where ψ is the acute angle between the surface normals at $f(P_1)$ and at $f(P_2)$.

We need to give some preliminary definitions.

DEFINITIONS

We shall call a surface $S = f(\overline{E})$ simple when the boundary of \overline{E} is a simple closed polygon. We shall first be concerned only with simple surfaces.

A polyhedron Π is said to be inscribed on S when all the vertices of Π are in S and the orthogonal projection, Proj Π , on the xy plane is \overline{E} . By the norm of a polyhedron we shall mean the greatest of the diameters of the faces (triangles) of Π .

Let Π be inscribed on S and let A be a face of Π . By the deviation D(A) of A we shall mean the L.U.B. of the acute angles between the normal to A and the surface normal at a point of the surface *subtended* by A. By the deviation norm of Π we shall mean the greatest of the deviations of its faces.

We shall consider sequences of polyhedra which are inscribed on S. A sequence $\{\Pi_1, \Pi_2, ...\}$ of such polyhedra is said to be a *proper* sequence of polyhedra inscribed on S when the corresponding sequence of norms $\{N_1, N_2, ...\}$ converges to zero and the corresponding sequence $\{\phi_1, \phi_2, ...\}$ of deviation norms also converges to zero.

We now give our basic definition of surface area:

Let E be a bounded set on the xy plane whose boundary is a simple closed polygon. Let f(x, y) be defined and continuously differentiable on \overline{E} . If to every proper sequence of polyhedra inscribed on $S = f(\overline{E})$ the corresponding sequence of polyhedral areas $\{A_1, A_2, ...\}$ converges, then

then we say that S is quadrable and that the necessarily unique limit of $\{A_1, A_2, ...\}$ is the area of the surface S.

THEOREM 1.

Let E be a bounded set on the xy plane whose boundary is a simple closed polygon. Let f(x, y) be defined and continuously differentiable on \overline{E} . Then there exist a proper sequence $\{\Pi_1, \Pi_2, ...\}$ of polyhedra inscribed on S.

Proof:

For every positive number r there exists a decomposition of \overline{E} as the union of closed right triangles whose diameters are all less than r. The vertices of these right triangles determine a finite set of points in S whose projection is precisely the set of these vertices. This set of points in S determines a triangular polyhedron which is inscribed on S. We shall show that by making the norm of the decomposition of \overline{E} sufficiently small we can make the acute angle between the normal to each polyhedral face and the surface normal at any point of the portion of S which is subtended by the particular face to be arbitrarily small. Let $\varepsilon > 0$ be given.

By property 3) there exist positive real numbers k < 1 and δ_1 such that if PP_1P_2 is a right triangle on \overline{E} (P being the right angled vertex) with diameter $< \delta_1$, then $|\cos{(\overrightarrow{QQ_1}, \overrightarrow{QQ_2})}| < k$. Let the decomposition of \overline{E} by right triangles be of norm less than δ_1 .

By property 1) there exists a positive real number θ such that if $|\sin(\overrightarrow{QQ_1},\overrightarrow{QQ_1}')| < \theta$ and $|\sin(\overrightarrow{QQ_2},\overrightarrow{QQ_2}')| < \theta$, then the acute angle between $\overrightarrow{QQ_1} \times \overrightarrow{QQ_2}$ and $\overrightarrow{QQ_1}' \times \overrightarrow{QQ_2}'$ is less than $\varepsilon/3$.

By properties 4) and 5) there exists a positive real number δ_2 such that if PP_1P_2 is a right triangle on \overline{E} with diameter less than δ_2 , then the angle between the chord $\overrightarrow{QQ_1}$ and the tangent line at Q to the curve on S subtented by $\overrightarrow{QQ_1}$ is less than θ . Similarly, the angle between the chord $\overrightarrow{QQ_2}$ and the tangent line at Q to the curve on S subtended by $\overrightarrow{QQ_2}$ is less than θ . It follows that the angle between the normal to the polyhedral face QQ_1 Q_2 and the surface normal at Q is less than $\varepsilon/3$.

By property 6) there exists a positive real number δ_3 such that if the diameter of the triangle PP_1P_2 is less than δ_3 , then the angle between the surface normals at any two points of the portion of S which is subtended by the polyhedral face QQ_1Q_2 is less than $\varepsilon/3$.

Let δ be the least of δ_1 , δ_2 , and δ_3 . If D is any decomposition of \overline{E} int closed right triangles of norm less than δ , then if QQ_1 Q_2 is any of the

polyhedral faces, the L.U.B. of the angles between the normal to $QQ_1 Q_2$ and the surface normals at any point of the portion of the surface subtended by $QQ_1 Q_2$ is less than ε .

Thus corresponding to a sequence $\{\varepsilon_1, \varepsilon_2, ...\}$ converging to zero, there exists a sequence of polyhedra with corresponding sequence of norms converging to zero and also with corresponding sequence of deviation norms converging to zero.

THEOREM 2.

Let E be an open set on the xy plane whose boundary is a simple closed polygon. Let f(x, y) be defined and continuously differentiable on \overline{E} . Then for every proper sequence of polyhedra inscribed on S the corresponding sequence $\{A_1, A_2, ...\}$ of polyhedral areas converges and moreover it converges to the double integral

$$\int_{\overline{E}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} d(x, y).$$

Proof:

For each n, the projection of the faces of Π_n constitute a decomposition D_n of \overline{E} as the union of a finite set of closed triangles. Let the triangle $\Delta_{mn} = QQ_1 \ Q_2$ be the face of Π_n and let $\Delta'_{mn} = \operatorname{Proj} \ QQ_1 \ Q_2 = \operatorname{PP}_1 \ P_2$. Let β_{mn} be the acute angle between the normals to Δ_{mn} and to Δ'_{mn} . Let A_{mn} and Δ'_{mn} denote the areas of Δ_{mn} and Δ'_{mn} , respectively. Then $A_{mn} = A'_{mn}$ sec β_{mn} and the area A_n of Π_n is $\Sigma A'_{mn}$ sec β_{mn} .

Let P_{mn} be any point in Δ'_{mn} and let Q_{mn} be the point of S whose projection is P_{mn} . Let θ_{mn} denote the acute angle between the surface normal at Q_{mn} and the z-axis.

Let $\{\Pi_1, \Pi_2, \Pi_3, ...\}$ be any proper sequence of polyhedra inscribed on S. We shall associate to $\{\Pi_1, \Pi_2, \Pi_3, ...\}$ certain related sequences.

$$\Pi_{1}, \Pi_{2}, \Pi_{3}, \dots$$
 $\phi_{1}, \phi_{2}, \phi_{3}, \dots$
 $\Sigma_{1}, \Sigma_{2}, \Sigma_{3}, \dots$
 $\Sigma_{1}^{'}, \Sigma_{2}^{'}, \Sigma_{3}^{'}, \dots$

The sequence $\{\phi_1, \phi_2, \phi_3, ...\}$ is the corresponding sequence of deviation norms. The sequence $\{\Sigma_1, \Sigma_2, \Sigma_3, ...\}$ is the corresponding sequence

of polyhedral areas. $\Sigma_n = \sum_{m} A'_{mn} \sec \beta_{mn}$. In the fourth sequence $\Sigma'_n = \sum_{m} A'_{mn} \sec \theta_{mn}$. Here $\sec \theta_{mn}$ is the value of $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ at some point of A'_{mn} . Thus the sequence $\left\{\sum'_1, \sum'_2, \sum'_3, ...\right\}$ is a sequence of Riemann sums of the function $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ on \overline{E} with corresponding sequence of norms converging to zero. Since $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ is continuous on \overline{E} , this converges to the double integral $\oint = \int_{\overline{E}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, d(x, y)$.

We will now consider the sequence $\{\Sigma_1, \Sigma_2, \Sigma_3, ...\}$.

Let θ denote the acute angle between the surface normal at a point of S and the z-axis. Sec $\theta = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$ is bounded on \overline{E} . Thus there exists an acute angle $\theta^* > 0$ such that $\theta < \theta^*$ for all points of \overline{E} (i.e. for all points of S). Since sec θ is uniformly continuous on the closed interval $[0, \theta^*]$, for every $\eta > 0$ there exists $\tau > 0$ such that if $0 < \theta_1 < \theta^*, 0 < \theta_2 < \theta^*$, and $\left| \theta_1 - \theta_2 \right| < \tau$, then $\left| \sec \theta_1 - \sec \theta_2 \right| < \eta$. We now compare the corresponding sequences

$$\{ \Sigma_{1}, \Sigma_{2}, \Sigma_{3}, \dots \}$$

 $\{ \Sigma_{1}^{'}, \Sigma_{2}^{'}, \Sigma_{3}^{'}, \dots \}.$

Let $\varepsilon > 0$ be given. Take $\frac{\varepsilon}{2A}$, where $A = \text{area of } \overline{E}$. There exists $\tau > 0$ such that if $|\theta_1 - \theta_2| < \tau$, then $|\sec \theta_1 - \sec \theta_2| < \frac{\varepsilon}{2A}$. Since $\{\phi_1, \phi_2, \phi_3, ...\}$ converges to zero, there exists a positive integer N_1 such that if $n > N_1$ then $\phi_n < \tau$. Thus if $n > N_1$, then

$$|\Sigma_{n} - \Sigma_{n}^{'}| = |\Sigma A_{mn}^{'} (\sec \beta_{mn} - \sec \theta_{mn})| < \frac{\varepsilon}{2A} \sum_{m} A_{mn}^{'} = \frac{\varepsilon}{2}.$$

Since $\{\Sigma_n^{'}, \Sigma_2^{'}, \Sigma_3^{'}, ...\}$ converges to \mathfrak{p} , there exists a positive integer N_2 . such that if $n > N_2$, then $|\Sigma_n^{'} - \mathfrak{p}| < \frac{\varepsilon}{2}$. Let N be the larger of N_1 and N_2 . If n > N then

$$|\Sigma_{n} - \mathfrak{p}| = |\Sigma_{n} - \Sigma_{n}^{'} + \Sigma_{n}^{'} - \mathfrak{p}| \leqslant |\Sigma_{n} - \Sigma_{n}^{'}| + |\Sigma_{n}^{'} - \mathfrak{p}| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{\Sigma_1, \Sigma_2, \Sigma_3, ...\}$ converges to \oplus .

Thus far we have defined the concept of area only for surfaces which are not only continuously differentiable but are also simple. We now remove this latter restriction.

Let E be any quadrable (i.e. Jordan measurable) open set on the xy plane having for boundary a simple closed curve. Let f be defined and continuously differentiable on \overline{E} . Let P be any subset of \overline{E} whose boundary is a simple closed polygon. The surface $S_p = f(P)$ is quadrable. Denote its area by A_p . Consider now the set of all such areas A_p . Since sec θ is bounded on \overline{E} , for every polygonal subset P of \overline{E} , $A_p \leq AM$, where A is the area of \overline{E} and M is an upper bound of $|\sec \theta|$ on \overline{E} . We now define the area of $S = f(\overline{E})$ as the L.U.B. of the set [all A_p].

THEOREM 3.

Let E be a quadrable open set on the xy plane having for boundary a simple closed curve. Let f be defined and continuously differentiable on \overline{E} . Then the area of $S = f(\overline{E})$ is given by

$$\oint = \int_{\overline{E}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} d(x, y).$$

Proof:

Let B denote the L.U.B. of the set [all A_p]. For each P, $A_p \leqslant \oplus$ and hence $B \leqslant \oplus$. Suppose now that $\oplus -B = 2\varepsilon > 0$.

Let $\{D_1, D_2, D_3, ...\}$ be any sequence of triangular "decompositions" of \overline{E} with corresponding sequence of norms converging to zero. Here we permit the triangles to abut beyond the boundary of \overline{E} . On each D_n form

a Riemann sum of
$$F(x, y) = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$
 in the following manner:

If a triangle does not abut beyond the boundary of \overline{E} , then take for the point P any point of the triangle. However, if a triangle does abut beyond the boundary of \overline{E} , let its contribution to the Riemann sum be zero. Now every sequence $\{S_1, S_2, S_3, ...\}$ of such Riemann sums converges and moreover, it converges to \mathfrak{P} . Since $\{S_1, S_2, S_3, ...\}$ converges to \mathfrak{P} , there

exists a positive integer N such that if
$$n > N$$
 then $\left| \int_{n} - S_{n} \right| < \frac{\varepsilon}{2}$.

On D_n , the set of the triangles which do not abut beyond the boundaries of \overline{E} constitutes a polygonal subset of \overline{E} . Call it P_n . There exists a triangular

decomposition D_n' of P_n such that if S_n' is a Riemann sum of f(x, y) on D_n' , then $|A_{p_n} - S_n'| < \frac{\varepsilon}{4}$ and $|\text{th} - S_n'| < \frac{\varepsilon}{2}$. It follows that $A_{p_n} > B$. This contradiction shows that B = th.

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