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surface, one might expect that the direction of the normal to each face of the polyhedron should not differ very much from the directions of the normals to the part of the surface which is subtended by the particular face. One sees that the polyhedra constructed by Schwarz do not have this property; that in fact, as the norms of the polyhedra converge to zero, the angle between the normal to each face and the normals to the surface subtended by the particular face approaches $\pi/2$. It is this pleating effect that produces a set of polyhedral areas that is unbounded.

The present paper is an attempt to take into consideration the angular or directional deviation of the faces of the polyhedra.

We shall here confine ourselves to non-paramatric surfaces. Such a surface is the locus in E^3 of an equation $z = f(x, y)$ where the domain is the closure of a bounded, open, and connected set E in E^2, and f is continupus on $\bar{E}$.

THE BASIS

We shall make use of the following properties of E^3.

1) Let U and V be any two vectors in E^3 such that $|\cos(U, V)| < k$, where $0 < k < 1$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if U_1 and V_1 are any two vectors such that $|\sin(U_1, U)| < \delta$ and $|\sin(V_1, V)| < \delta$ then $|\sin(U \times V, U_1 \times V_1)| < \epsilon$.

Let E be an open, bounded, and connected set on the xy plane. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. Then

2) The directional derivative of f is uniformly continuous on $\bar{E}$, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that if (x_1, y_1) and (x_2, y_2) are in $\bar{E}$ and $0 < \rho((x_1, y_1), (x_2, y_2)) < \delta$ then $|D_{x_1,y_1;x_2,y_2} f(x_1, y_1) - D_{x_1,y_1;x_2,y_2} f(x_2, y_2)| < \epsilon$. Here $\rho((x_1, y_1), (x_2, y_2))$ is the distance between (x_1, y_1) and (x_2, y_2). $D_{x_1,y_1;x_2,y_2}(x_1, y_1)$ is the directional derivative of f at (x_1, y_1) in the direction of the vector from (x_1, y_1) to (x_2, y_2).

The directional derivative is uniformly Lipschitzian over $\bar{E}$.

3) There exist positive numbers k and δ, $k < 1$ such that if P, P_1, and P_2 are any three distinct points of $\bar{E}$ such that

a) $\rho(P, P_1) < \delta$

b) $\rho(P, P_2) < \delta$ and

c) $\cos(\overrightarrow{PP_1}, \overrightarrow{PP_2}) = 0$, then

$|\cos(\overrightarrow{QQ_1}, \overrightarrow{QQ_2})| < k$, where $Q = f(P)$, $Q_1 = f(P_1)$ and $Q_2 = f(P_2)$.

4) Let P_1 and P_2 be any two distinct points on $\bar{E}$, $Q_1 = f(P_1)$, and $Q_2 = f(P_2)$. Let $P_1 P_2$ denote the closed interval determined by P_1 and P_2 and $Q_1 Q_2$ the closed interval determined by Q_1 and Q_2. Let the curve $C = f(P_1 P_2)$. Then there exists a point R on C such that the tangent line to C at R is parallel to $Q_1 Q_2$.

5) With the notation as in 4), let the *deviation* $D(P_1 P_2)$ denote the L.U.B. of the acute angles φ between the surface chord $Q_1 Q_2$ and any tangent line to C. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < \rho(P_1, P_2) < \delta$ then $D(P_1 P_2) < \epsilon$.

6) For every $\epsilon > 0$ there exists $\delta > 0$ such that if P_1 and P_2 are any two distinct points of $\bar{E}$ such that $\rho(P_1, P_2) < \delta$ then $\psi < \epsilon$, where ψ is the acute angle between the surface normals at $f(P_1)$ and at $f(P_2)$.

We need to give some preliminary definitions.

Definitions

We shall call a surface $S = f(\bar{E})$ simple when the boundary of $\bar{E}$ is a simple closed polygon. We shall first be concerned only with simple surfaces.

A polyhedron Π is said to be inscribed on S when all the vertices of Π are in S and the orthogonal projection, Proj Π, on the xy plane is $\bar{E}$. By the norm of a polyhedron we shall mean the greatest of the diameters of the faces (triangles) of Π.

Let Π be inscribed on S and let A be a face of Π. By the deviation $D(A)$ of A we shall mean the L.U.B. of the acute angles between the normal to A and the surface normal at a point of the surface *subtended* by A. By the deviation norm of Π we shall mean the greatest of the deviations of its faces.

We shall consider sequences of polyhedra which are inscribed on S. A sequence $\{\Pi_1, \Pi_2, \ldots\}$ of such polyhedra is said to be a *proper* sequence of polyhedra inscribed on S when the corresponding sequence of norms $\{N_1, N_2, \ldots\}$ converges to zero and the corresponding sequence $\{\phi_1, \phi_2, \ldots\}$ of deviation norms also converges to zero.

We now give our basic definition of surface area:

Let E be a bounded set on the xy plane whose boundary is a simple closed polygon. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. If to every proper sequence of polyhedra inscribed on $S = f(\bar{E})$ the corresponding sequence of polyhedral areas $\{A_1, A_2, \ldots\}$ converges, then