Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 13 (1967)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: DIRECTIONAL DEVIATION NORMS AND SURFACE AREA

Autor: TORALBALLA, L. V.

Kapitel: Introduction

DOI: https://doi.org/10.5169/seals-41533

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 16.10.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

DIRECTIONAL DEVIATION NORMS AND SURFACE AREA

by L. V. TORALBALLA

Introduction

One of the earlier attempts at giving a general definition of surface area was made by J. A. Serret[1] in 1868. Following the quite adequate definition of arc length, he defined the area of a surface to be the L.U.B. of the areas of polyhedra inscribed in it. The inadequacy of this definition was made apparent in 1882 by H. A. Schwarz [2] when he showed that by this definition even such a simple and smooth surface as a circular cylinder has no area. This discovery prompted a vigorous search for a definition of surface area that would have adequate generality. In 1902 Henri Lebesgue [3] proposed that surface area be defined as the G.L.B. of the set of the limit inferiors of the sequences of areas of polyhedral surfaces which converge uniformly to the given surface. An enormous literature [see 4, 5, 6, 7] has grown using Lebesgue's definition as a basis.

However, some mathematicians came to feel that while Lebesgue's definition is quite general, it lacks geometric simplicity. They initiated a return to a presentation by means of inscribed triangular polyhedra. The idea is to limit the class of the inscribed triangular polyhedra in such a manner as to preclude the occurrence of the Schwarz phenomenon. Thus, M. W. H. Young [8], for continuously differentiable surfaces, requires that the angles of the triangles on the xy plane have an upper bound less than π . Rademacher [9] for surfaces satisfying the Lipschitz condition, requires that these angles have a positive lower bound. Kempisty [7] limits consideration to right triangles having the ratio of the base to the altitude between 1/2 and 2. In a certain sense these latter definitions are *ad hoc* and thus seem to lack naturalness.

The present note is restricted to continuously differentiable surfaces. However, it makes use of a simple geometric idea which, as far as this writer knows, has not been considered in the literature.

If a polyhedron, inscribed on a continuously differentiable surface is to be thought of, in a good geometric sense, as an approximation to the surface, one might expect that the direction of the normal to each face of the polyhedron should not differ very much from the directions of the normals to the part of the surface which is subtended by the particular face. One sees that the polyhedra constructed by Schwarz do not have this property; that in fact, as the norms of the polyhedra converge to zero, the angle between the normal to each face and the normals to the surface subtended by the particular face approaches $\pi/2$. It is this pleating effect that produces a set of polyhedral areas that is unbounded.

The present paper is an attempt to take into consideration the angular or directional deviation of the faces of the polyhedra.

We shall here confine ourselves to non-paramatric surfaces. Such a surface is the locus in E^3 of an equation z = f(x, y) where the domain is the closure of a bounded, open, and connected set E in E^2 , and f is continuous on \overline{E} .

THE BASIS

We shall make use of the following properties of E^3 .

1) Let U and V be any two vectors in E^3 such that $|\cos(U, V)| < k$, where 0 < k < 1. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if U_1 and V_1 are any two vectors such that $|\sin(U_1, U)| < \delta$ and $|\sin(V_1, V)| < \delta$ then $|\sin(U \times V, U_1 \times V_1)| < \epsilon$.

Let E be an open, bounded, and connected set on the xy plane. Let f(x, y) be defined and continuously differentiable on \overline{E} . Then

2) The directional derivative of f is uniformly continuous on \overline{E} , i.e. for every $\in > 0$ there exists $\delta > 0$ such that if (x_1, y_1) and (x_2, y_2) are in \overline{E} and $0 < \rho\left((x_1, y_1), (x_2, y_2)\right) < \delta$ then $D_{x_1, y_1; x_2, y_2} f(x_1, y_1) - D_{x_1, y_1; x_2, y_2} f(x_2, y_2) < \epsilon$. Here $\rho\left((x_1, y_1), (x_2, y_2)\right)$ is the distance between (x_1, y_1) and (x_2, y_2) . $D_{x_1, y_1; x_2, y_2} (x_1, y_1)$ is the directional derivative of f at (x_1, y_1) in the direction of the vector from (x_1, y_1) to (x_2, y_2) .

The directional derivative is uniformly Lipschitzian over \overline{E} .

- 3) There exist positive numbers k and δ , k < 1 such that if P, P_1 , and P_2 are any three distinct points of \overline{E} such that
 - a) $\rho(P, P_1) < \delta$
 - b) $\rho(P, P_2) < \delta$ and
 - c) $\cos(\overrightarrow{PP_1}, \overrightarrow{PP_2}) = 0$, then

 $|\cos(QQ_1, QQ_2)| < k$, where Q = f(P), $Q_1 = f(P_1)$ and $Q_2 = f(P_2)$.