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DIRECTIONAL DEVIATION NORMS AND SURFACE AREA

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Introduction

One of the earlier attempts at giving a general definition of surface area was made by J. A. Serret[1] in 1868. Following the quite adequate definition of arc length, he defined the area of a surface to be the L.U.B. of the areas of polyhedra inscribed in it. The inadequacy of this definition was made apparent in 1882 by H. A. Schwarz [2] when he showed that by this definition even such a simple and smooth surface as a circular cylinder has no area. This discovery prompted a vigorous search for a definition of surface area that would have adequate generality. In 1902 Henri Lebesgue [3] proposed that surface area be defined as the G.L.B. of the set of the limit inferiors of the sequences of areas of polyhedral surfaces which converge uniformly to the given surface. An enormous literature [see 4, 5, 6, 7] has grown using Lebesgue's definition as a basis.

However, some mathematicians came to feel that while Lebesgue's definition is quite general, it lacks geometric simplicity. They initiated a return to a presentation by means of inscribed triangular polyhedra. The idea is to limit the class of the inscribed triangular polyhedra in such a manner as to preclude the occurrence of the Schwarz phenomenon. Thus, M. W. H. Young [8], for continuously differentiable surfaces, requires that the angles of the triangles on the xy plane have an upper bound less than π. Rademacher [9] for surfaces satisfying the Lipschitz condition, requires that these angles have a positive lower bound. Kempisty [7] limits consideration to right triangles having the ratio of the base to the altitude between 1/2 and 2. In a certain sense these latter definitions are *ad hoc* and thus seem to lack naturalness.

The present note is restricted to continuously differentiable surfaces. However, it makes use of a simple geometric idea which, as far as this writer knows, has not been considered in the literature.

If a polyhedron, inscribed on a continuously differentiable surface is to be thought of, in a good geometric sense, as an approximation to the surface, one might expect that the direction of the normal to each face of the polyhedron should not differ very much from the directions of the normals to the part of the surface which is subtended by the particular face. One sees that the polyhedra constructed by Schwarz do not have this property; that in fact, as the norms of the polyhedra converge to zero, the angle between the normal to each face and the normals to the surface subtended by the particular face approaches $\pi/2$. It is this pleating effect that produces a set of polyhedral areas that is unbounded.

The present paper is an attempt to take into consideration the angular or directional deviation of the faces of the polyhedra.

We shall here confine ourselves to non-paramatric surfaces. Such a surface is the locus in E^3 of an equation $z = f(x, y)$ where the domain is the closure of a bounded, open, and connected set E in E^2, and f is continupus on $\bar{E}$.

The basis

We shall make use of the following properties of E^3.

1) Let U and V be any two vectors in E^3 such that $|\cos(U, V)| < k$, where $0 < k < 1$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that if U_1 and V_1 are any two vectors such that $|\sin(U_1, U)| < \delta$ and $|\sin(V_1, V)| < \delta$ then $|\sin(U \times V, U_1 \times V_1)| < \epsilon$.

Let E be an open, bounded, and connected set on the xy plane. Let $f(x, y)$ be defined and continuously differentiable on $\bar{E}$. Then

2) The directional derivative of f is uniformly continuous on $\bar{E}$, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that if (x_1, y_1) and (x_2, y_2) are in $\bar{E}$ and $0 < \rho((x_1, y_1), (x_2, y_2)) < \delta$ then $|D_{x_1,y_1;x_2,y_2} f(x_1, y_1) - D_{x_1,y_1;x_2,y_2} f(x_2, y_2)| < \epsilon$. Here $\rho((x_1, y_1), (x_2, y_2))$ is the distance between (x_1, y_1) and (x_2, y_2). $D_{x_1,y_1;x_2,y_2}(x_1, y_1)$ is the directional derivative of f at (x_1, y_1) in the direction of the vector from (x_1, y_1) to (x_2, y_2).

The directional derivative is uniformly Lipschitzian over $\bar{E}$.

3) There exist positive numbers k and δ, $k < 1$ such that if P, P_1, and P_2 are any three distinct points of $\bar{E}$ such that

a) $\rho(P, P_1) < \delta$

b) $\rho(P, P_2) < \delta$ and

c) $\cos(\overrightarrow{PP_1}, \overrightarrow{PP_2}) = 0$, then

$|\cos(\overrightarrow{QQ_1}, \overrightarrow{QQ_2})| < k$, where $Q = f(P)$, $Q_1 = f(P_1)$ and $Q_2 = f(P_2)$.