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A CONDITION FOR EXISTENCE OF A SMALLEST BOREL ALGEBRA CONTAINING A GIVEN COLLECTION OF SETS

by Arthur B. BROWN and Gerald FREILICH

The origin of this note lies in an oversight appearing in [1] and [2], a difficulty that was already realized by the translators of [1]. (See Translators' Note in [1], page 16. Since situations arise in which *B*-algebras with different units are used, Theorem 4 on page 19 of [1] and on page 25 of [2] requires an additional hypothesis. See the theorem below.) It is hoped that the present note will be of independent interest.

DEFINITIONS. A σ -ring (of sets) is a non-empty collection of sets closed under the operations of difference (of a pair of sets) and countable union. A Borel algebra, or B-algebra, is a σ -ring which has an element that contains every other element of the σ -ring. The (unique) maximal element is called the unit of the B-algebra.

A member of a collection of sets is called the smallest member if it is contained in every other member of the collection.

LEMMA. If S is a non-empty collection of sets each contained in a set X, then there exists a smallest B-algebra B (S) with unit X containing S.

Proof. Take B(S) to be the intersection of all *B*-algebras with unit *X* that contain *S*.

If we want now to generalize the lemma by omitting the requirement that the B-algebras under consideration have the same unit X, an additional hypothesis is necessary.

THEOREM. Let S be a non-empty collection of sets whose union is X. Then there is a smallest B-algebra containing S if and only if X is the union of some countable collection of sets of S. If there is a smallest B-algebra containing S, then that algebra has unit X and is the algebra B (S) of the Lemma.

Proof. Suppose that X is the union of a countable collection of sets of S. Let W be any B-algebra containing S, where sets of W are not restricted to be subsets of X, and let $D = W \cap B(S)$, where B(S) is the smallest

B-algebra with unit X and containing S. (See Lemma.) Since W is a σ -ring, $X \in W$; hence $X \in D$. Since $D \subseteq B(S)$, the sets in D are subsets of X. Since W and B(S) are σ -rings, so is D. Hence D is a B-algebra with unit X. Since B(S) is the smallest B-algebra with unit X, we infer that $B(S) \subseteq D$, and hence B(S) = D. Thus $B(S) = D \subseteq W$, so B(S) is the smallest B-algebra containing S, as was to be proved.

Now suppose X is not the union of any countable collection of sets of S. Choose $\alpha \notin X$ and let $Y = X \cup \{\alpha\}$. Let $S' = \{A : A \subseteq \bigcup_{i=1}^{\infty} S_i, S_i \in S\}$, $S'' = \{A : A \in S' \text{ or } (Y-A) \in S'\}$. Then any subset of a member of S' is a member of S', and S' is clearly a σ -ring. It is obvious that S'' contains S. We now prove that S'' is a B-algebra.

Since $\emptyset \in S'$, $Y \in S''$, so Y will be the unit. Let $A_1, A_2, ...$ be members of S''. If each $A_j \in S'$, with $A_j \subseteq \bigcup_{i=1}^{\infty} S_{ij}$, $S_{ij} \in S$, then $\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{\infty} S_{ij}$, so $\bigcup_{j=1}^{\infty} S_j \in S' \subseteq S''$. If some $A_k \notin S'$, then $Y - A_k \in S'$. Hence $Y - \bigcup_{j=1}^{\infty} A_j = \bigcap_{j=1}^{\infty} (Y - A_j) \subseteq Y - A_k \in S'$, so that $Y - \bigcup_{j=1}^{\infty} A_j \in S'$ and consequently $\bigcup_{j=1}^{\infty} A_j \in S''$. Thus it is proved that S'' is closed under countable unions. We now consider differences.

Suppose $\{A, B\} \subseteq S''$. If $A \in S'$ then $A - B \subseteq A \in S'$, so $A - B \in S'$ and hence $A - B \in S''$. If $A \notin S'$ then $Y - A \in S'$, and if $B \in S'$ we have $Y - (A - B) \subseteq (Y - A) \cup B \in S'$, so $Y - (A - B) \in S'$; hence $A - B \in S''$. If $A \notin S'$ and $B \notin S'$, then $A - B = (Y - B) - (Y - A) \in S' \subseteq S''$. This completes the proof that S'' is a *B*-algebra.

Since X is not the union of any countable collection of sets of S, it is clear that $X \notin S'$. Consequently $X \notin S''$, for if X = Y - A with $A \in S'$, we would have $\alpha \in X$, contrary to the choice of α . We are now in a position to complete the proof.

If there were a smallest *B*-algebra *V* containing *S*, then by the definition of unit, the unit *E* of *V* would contain *X*. Furthermore, *V* would be contained in the set of all subsets of *X* (the latter being a *B*-algebra containing *S*), so $E \subseteq X$. Hence *X* would be the unit *E* of *V*. Then, from $X \in V$ and $X \notin S''$, it would follow that $V \notin S''$, contrary to the fact that S'' is a *B*-algebra containing *S*. Hence there is no smallest *B*-algebra containing *S*.

EXAMPLES. Let X be an uncountable set and let S be the set of all countable subsets of X. By the Theorem, there is no smallest B-algebra containing S.

Note that the Theorem implies that if S is a non-empty collection of sets such that there is no smallest *B*-algebra containing S, then the union of the members of S must be uncountable.

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