Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 13 (1967)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CONDITION FOR EXISTENCE OF A SMALLEST BOREL ALGEBRA CONTAINING A GIVEN COLLECTION OF SETS

Autor: Brown, Arthur B. / Freilich, Gerald

DOI: https://doi.org/10.5169/seals-41532

A CONDITION FOR EXISTENCE OF A SMALLEST BOREL ALGEBRA CONTAINING A GIVEN COLLECTION OF SETS

by Arthur B. Brown and Gerald Freilich

The origin of this note lies in an oversight appearing in [1] and [2], a difficulty that was already realized by the translators of [1]. (See Translators' Note in [1], page 16. Since situations arise in which B-algebras with different units are used, Theorem 4 on page 19 of [1] and on page 25 of [2] requires an additional hypothesis. See the theorem below.) It is hoped that the present note will be of independent interest.

DEFINITIONS. *A σ-ring (of sets) is a non-empty collection of sets closed under the operations of difference (of a pair of sets) and countable union.*

A Borel algebra, or B-*algebra, is a σ-ring which has an element that contains every other element of the σ-ring. The (unique) maximal element is called the unit of the* B-*algebra.*

A member of a collection of sets is called the smallest member if it is contained in every other member of the collection.

LEMMA. *If* S *is a non-empty collection of sets each contained in a set* X, *then there exists a smallest* B-*algebra* B (S) *with unit* X *containing* S.

Proof. Take $B(S)$ to be the intersection of all B-algebras with unit X that contain S.

If we want now to generalize the lemma by omitting the requirement that the B-algebras under consideration have the same unit X, an additional hypothesis is necessary.

THEOREM. *Let* S *be a non-empty collection of sets whose union is* X. *Then there is a smallest* B-*algebra containing* S *if and only if* X *is the union of some countable collection of sets of* S. *If there is a smallest* B-*algebra containing* S, *then that algebra has unit* X *and is the algebra* B (S) *of the Lemma.*

Proof. Suppose that X is the union of a countable collection of sets of S. Let W be any B-algebra containing S, where sets of W are not restricted to be subsets of X, and let $D = W \cap B(S)$, where $B(S)$ is the smallest

B-algebra with unit X and containing S. (See Lemma.) Since W is a σ-ring, $X \in W$; hence $X \in D$. Since $D \subseteq B(S)$, the sets in D are subsets of X. Since W and $B(S)$ are σ-rings, so is D. Hence D is a B-algebra with unit X. Since $B(S)$ is the smallest B-algebra with unit X, we infer that $B(S) \subseteq D$, and hence $B(S) = D$. Thus $B(S) = D \subseteq W$, so $B(S)$ is the smallest B-algebra containing S, as was to be proved.

Now suppose X is not the union of any countable collection of sets of S. Choose $\alpha \notin X$ and let $Y = X \cup \{\alpha\}$. Let $S' = \{A : A \subseteq \cup_{i=1}^{\infty} S_i, S_i \in S\}$, $S'' = \{A : A \in S'$ or $(Y - A) \in S'\}$. Then any subset of a member of S' is a member of S', and S' is clearly a σ-ring. It is obvious that S'' contains S. We now prove that S'' is a B-algebra.

Since $\varnothing \in S'$, $Y \in S''$, so Y will be the unit. Let $A_1, A_2, \ldots$ be members of S''. If each $A_j \in S'$, with $A_j \subseteq \cup_{i=1}^{\infty} S_{ij}$, $S_{ij} \in S$, then $\cup_{j=1}^{\infty} A_j \subseteq \cup_{j=1}^{\infty} \cup_{i=1}^{\infty} S_{ij}$, so $\cup_{j=1}^{\infty} S_j \in S' \subseteq S''$. If some $A_k \notin S'$, then $Y - A_k \in S'$. Hence $Y - \cup_{j=1}^{\infty} A_j = \cap_{j=1}^{\infty} (Y - A_j) \subseteq Y - A_k \in S'$, so that $Y - \cup_{j=1}^{\infty} A_j \in S'$ and consequently $\cup_{j=1}^{\infty} A_j \in S''$. Thus it is proved that S'' is closed under countable unions. We now consider differences.

Suppose $\{A, B\} \subseteq S''$. If $A \in S'$ then $A - B \subseteq A \in S'$, so $A - B \in S'$ and hence $A - B \in S''$. If $A \notin S'$ then $Y - A \in S'$, and if $B \in S'$ we have $Y - (A - B) \subseteq (Y - A) \cup B \in S'$, so $Y - (A - B) \in S'$; hence $A - B \in S''$. If $A \notin S'$ and $B \notin S'$, then $A - B = (Y - B) - (Y - A) \in S' \subseteq S''$. This completes the proof that S'' is a B-algebra.

Since X is not the union of any countable collection of sets of S, it is clear that $X \notin S'$. Consequently $X \notin S''$, for if $X = Y - A$ with $A \in S'$, we would have $\alpha \in X$, contrary to the choice of α. We are now in a position to complete the proof.

If there were a smallest B-algebra V containing S, then by the definition of unit, the unit E of V would contain X. Furthermore, V would be contained in the set of all subsets of X (the latter being a B-algebra containing S), so $E \subseteq X$. Hence X would be the unit E of V. Then, from $X \in V$ and $X \notin S''$, it would follow that $V \nsubseteq S''$, contrary to the fact that S'' is a B-algebra containing S. Hence there is no smallest B-algebra containing S.

Examples. Let X be an uncountable set and let S be the set of all countable subsets of X. By the Theorem, there is no smallest B-algebra containing S.

Note that the Theorem implies that if S is a non-empty collection of sets such that there is no smallest B-algebra containing S, then the union of the members of S must be uncountable.

REFERENCES

1. Kolmogorov, A. N. and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Vol. 2 (translated by H. Kamel and H. Komm), Graylock Press, Albany, N. Y., 1961. Translators' Note, page 16: " This definition leads to difficulties in the statements and proofs of certain theorems in the sequel... "

2. —— and S. V. Fomin, *Measure, Lebesgue Integrals, and Hilbert Space* (translated by N. A. Brunswick and A. Jeffrey), Academic Press, New York, 1961.

(Reçu le 20 décembre 1966)

Arthur B. Brown
Queens College of CUNY
Flushing, N.Y.

Gerald Freilich
Clty College of CUNY
New York, N.Y.