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A CONDITION FOR EXISTENCE OF A SMALLEST BOREL ALGEBRA CONTAINING A GIVEN COLLECTION OF SETS

by Arthur B. BROWN and Gerald FREILICH

The origin of this note lies in an oversight appearing in [1] and [2], a difficulty that was already realized by the translators of [1]. (See Translators' Note in [1], page 16. Since situations arise in which B-algebras with different units are used, Theorem 4 on page 19 of [1] and on page 25 of [2] requires an additional hypothesis. See the theorem below.) It is hoped that the present note will be of independent interest.

DEFINITIONS. *A σ-ring (of sets) is a non-empty collection of sets closed under the operations of difference (of a pair of sets) and countable union.*

A Borel algebra, or B*-algebra, is a σ-ring which has an element that contains every other element of the σ-ring. The (unique) maximal element is called the unit of the* B*-algebra.*

A member of a collection of sets is called the smallest member if it is contained in every other member of the collection.

LEMMA. *If* S *is a non-empty collection of sets each contained in a set* X*, then there exists a smallest* B*-algebra* B (S) *with unit* X *containing* S.

Proof. Take $B(S)$ to be the intersection of all B-algebras with unit X that contain S.

If we want now to generalize the lemma by omitting the requirement that the B-algebras under consideration have the same unit X, an additional hypothesis is necessary.

THEOREM. *Let* S *be a non-empty collection of sets whose union is* X*. Then there is a smallest* B*-algebra containing* S *if and only if* X *is the union of some countable collection of sets of* S*. If there is a smallest* B*-algebra containing* S*, then that algebra has unit* X *and is the algebra* B (S) *of the Lemma.*

Proof. Suppose that X is the union of a countable collection of sets of S. Let W be any B-algebra containing S, where sets of W are not restricted to be subsets of X, and let $D = W \cap B(S)$, where $B(S)$ is the smallest

B-algebra with unit X and containing S. (See Lemma.) Since W is a σ-ring, $X \in W$; hence $X \in D$. Since $D \subseteq B(S)$, the sets in D are subsets of X. Since W and $B(S)$ are σ-rings, so is D. Hence D is a B-algebra with unit X. Since $B(S)$ is the smallest B-algebra with unit X, we infer that $B(S) \subseteq D$, and hence $B(S) = D$. Thus $B(S) = D \subseteq W$, so $B(S)$ is the smallest B-algebra containing S, as was to be proved.

Now suppose X is not the union of any countable collection of sets of S. Choose $\alpha \notin X$ and let $Y = X \cup \{\alpha\}$. Let $S' = \{A: A \subseteq \cup_{i=1}^{\infty} S_i, S_i \in S\}$, $S'' = \{A: A \in S'$ or $(Y-A) \in S'\}$. Then any subset of a member of S' is a member of S', and S' is clearly a σ-ring. It is obvious that S'' contains S. We now prove that S'' is a B-algebra.

Since $\varnothing \in S'$, $Y \in S''$, so Y will be the unit. Let $A_1, A_2, \ldots$ be members of S''. If each $A_j \in S'$, with $A_j \subseteq \cup_{i=1}^{\infty} S_{ij}$, $S_{ij} \in S$, then $\cup_{j=1}^{\infty} A_j \subseteq \cup_{j=1}^{\infty} \cup_{i=1}^{\infty} S_{ij}$, so $\cup_{j=1}^{\infty} S_j \in S' \subseteq S''$. If some $A_k \notin S'$, then $Y-A_k \in S'$. Hence $Y-\cup_{j=1}^{\infty} A_j = \cap_{j=1}^{\infty} (Y-A_j) \subseteq Y-A_k \in S'$, so that $Y-\cup_{j=1}^{\infty} A_j \in S'$ and consequently $\cup_{j=1}^{\infty} A_j \in S''$. Thus it is proved that S'' is closed under countable unions. We now consider differences.

Suppose $\{A, B\} \subseteq S''$. If $A \in S'$ then $A-B \subseteq A \in S'$, so $A-B \in S'$ and hence $A-B \in S''$. If $A \notin S'$ then $Y-A \in S'$, and if $B \in S'$ we have $Y-(A-B) \subseteq (Y-A) \cup B \in S'$, so $Y-(A-B) \in S'$; hence $A-B \in S''$. If $A \notin S'$ and $B \notin S'$, then $A-B = (Y-B) - (Y-A) \in S' \subseteq S''$. This completes the proof that S'' is a B-algebra.

Since X is not the union of any countable collection of sets of S, it is clear that $X \notin S'$. Consequently $X \notin S''$, for if $X = Y-A$ with $A \in S'$, we would have $\alpha \in X$, contrary to the choice of α. We are now in a position to complete the proof.

If there were a smallest B-algebra V containing S, then by the definition of unit, the unit E of V would contain X. Furthermore, V would be contained in the set of all subsets of X (the latter being a B-algebra containing S), so $E \subseteq X$. Hence X would be the unit E of V. Then, from $X \in V$ and $X \notin S''$, it would follow that $V \nsubseteq S''$, contrary to the fact that S'' is a B-algebra containing S. Hence there is no smallest B-algebra containing S.

Examples. Let X be an uncountable set and let S be the set of all countable subsets of X. By the Theorem, there is no smallest B-algebra containing S.

Note that the Theorem implies that if S is a non-empty collection of sets such that there is no smallest B-algebra containing S, then the union of the members of S must be uncountable.

REFERENCES

1. Kolmogorov, A. N. and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Vol. 2 (translated by H. Kamel and H. Komm), Graylock Press, Albany, N. Y., 1961. Translators' Note, page 16: "This definition leads to difficulties in the statements and proofs of certain theorems in the sequel..."

2. —— and S. V. Fomin, *Measure, Lebesgue Integrals, and Hilbert Space* (translated by N. A. Brunswick and A. Jeffrey), Academic Press, New York, 1961.

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