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# A NOTE ON TWO CRITERIA FOR DEDEKIND DOMAINS

by Robert GILMER

Let  $A$  be an ideal of the commutative ring  $R$ . In case the residue class ring  $R/A$  is finite, we say that  $A$  has *finite norm*, and we set  $N(A) = |R/A|$ , where  $|S|$  denotes the cardinal number of the set  $S$ ;  $N(A)$  is called *the norm of  $A$* . We say that  $A$  has *finite length  $s$*  and we write  $L(A) = s$  if there is a chain  $A \subset A_1 \subset \dots \subset A_s = R$  of ideals of  $R$ , but no such chain of  $s+1$  ideals; therefore  $L(A)$  is the length of  $R/A$ , as an  $R/A$ -module, when this length is finite. If  $A$  has finite norm, then  $A$  has finite length. The converse fails; for example, each maximal ideal of  $R$  has finite length 1.

In [2], Butts and Wade have shown that each of the following conditions implies, in an integral domain  $R$  with identity, that  $R$  is a Dedekind domain:

(\*) Each nonzero ideal of  $R$  has finite norm, and  $N(AB) = N(A)N(B)$  for any nonzero ideals  $A, B$  of  $R$ .

(\*\*) Each nonzero ideal of  $R$  has finite length, and  $L(AB) = L(A) + L(B)$  for any nonzero ideals  $A, B$  of  $R$ .

Our purpose here is to show that in any commutative ring  $R$ , (\*) or (\*\*) implies the following condition ( $\sqrt{\phantom{x}}$ ).

( $\sqrt{\phantom{x}}$ )  $R$  is Noetherian, has an identity, and for any maximal ideal  $M$  of  $R$ , there are no ideals of  $R$  properly between  $M$  and  $M^2$ .

A result of Asano in [1] shows that ( $\sqrt{\phantom{x}}$ ) is equivalent to the condition that  $R$  be a *general ZPI-ring*—that is, each ideal of  $R$  is a finite product of prime ideals. Hence, by proving that (\*) or (\*\*) implies ( $\sqrt{\phantom{x}}$ ), we obtain a generalization of Butts and Wade's results already cited to the case of rings with zero divisors. In addition, our proof will be simpler than that given in [2] for the case of an integral domain with identity, so that we obtain both a generalization and a simplification of the results of [2].

**THEOREM 1.** *If condition (\*) or condition (\*\*) holds in the ring  $R$ , then  $R$  is Noetherian, contains an identity, and has the property that there are no ideals properly between  $M$  and  $M^2$  for any maximal ideal  $M$  of  $R$ .*

*Proof.* If (\*) or (\*\*) holds in  $R$ , it is clear that  $R/A$  is Noetherian for each nonzero ideal  $A$  of  $R$ . Therefore,  $R$  is also Noetherian. If (\*) holds,

then  $N(R^2) = [N(R)]^2 = 1$  so that  $R = R^2$ . And if  $(**)$  holds,  $L(R^2) = 2L(R) = 0$ , and again  $R = R^2$ . However, a finitely generated idempotent ideal of a commutative ring is principal and is generated by an idempotent element [1; 86]. Consequently,  $(*)$  or  $(**)$  implies that  $R$  has an identity.

We consider a maximal ideal  $M$  of  $R$ . If  $(**)$  is valid, then  $L(M^2) = 2L(M) = 2$ ; therefore, there are no ideals of  $R$  properly between  $M$  and  $M^2$ . If  $(*)$  holds in  $R$ , then  $N(M^2) = [N(M)]^2$ . But the isomorphism  $R/M \simeq (R/M^2)/(M/M^2)$  implies that  $N(M^2) = N(M) \cdot k$ , where  $k = |M/M^2|$ . Therefore,  $k = N(M)$ . This means that  $M/M^2$ , as a vector space over the field  $R/M$ , must have dimension 1. Consequently, there are no nontrivial  $R/M$ -subspaces of  $M/M^2$ , and therefore no ideals of  $R$  properly between  $M$  and  $M^2$ . This completes the proof of Theorem 1.

We remark that in our proof of Theorem 1 we have not used the full generality that  $N(AB) = N(A)N(B)$  or that  $L(AB) = L(A) + L(B)$ ; rather, we have only used the fact that,  $N(A^2) = [N(A)]^2$  and that  $L(A^2) = 2L(A)$  for any nonzero ideal  $A$  of  $R$ .

The remainder of this paper will be concerned with some results related to the converse of Theorem 1.

LEMMA 1. *Suppose that  $A$  and  $B$  are relatively prime ideals of the ring  $R$  with identity.*

a) *If  $A$  and  $B$  have finite norm, then  $AB$  has finite norm and  $N(AB) = N(A)N(B)$ .*

b) *If  $A$  and  $B$  have finite length, then  $AB$  has finite length and  $L(AB) = L(A) + L(B)$ .*

*Proof.* a) is immediate from the fact that  $R/AB$  is isomorphic to  $R/A \oplus R/B$ , the direct sum of  $R/A$  and  $R/B$  [4; 178]. To prove b), we need only note that if  $R_1$  and  $R_2$  are rings of which the zero ideals have finite lengths  $n_1$  and  $n_2$ , then the zero ideal of  $R_1 \oplus R_2$  has finite length  $n_1 + n_2$ . This is immediate from the fact, however, that each ideal of  $R_1 \oplus R_2$  is of the form  $A_1 \oplus A_2$ , where  $A_i$  is an ideal of  $R_i$  [4; 175].

LEMMA 2. *Let  $M$  be a maximal ideal of a commutative ring  $R$  with identity such that there are no ideals properly between  $M$  and  $M^2$ , and let  $k$  be a positive integer such that  $M^k \subset M^{k-1}$ .*

a) If  $M$  has finite norm, then  $N(M^k) = [N(M)]^k$ .

b) If  $M$  has finite length, then  $L(M^k) = kL(M) = k$ .

*Proof.* If  $r$  is a positive integer such that  $M^r \supset M^{r+1}$ , it is known that  $M^r/M^{r+1}$  and  $R/M$  are, as vector spaces over  $R/M$ , isomorphic [1; 83]. Hence,  $|R/M| = |M^r/M^{r+1}|$ . If we assume that  $N(M^{k-1}) = [N(M)]^{k-1}$ , then the isomorphism  $R/M^{k-1} \simeq (R/M^k)/(M^{k-1}/M^k)$  implies that  $N(M^k) = N(M^{k-1})|M^{k-1}/M^k| = [N(M)]^{k-1} N(M) = [N(M)]^k$ . This establishes *a*).

To prove *b*), we note that if  $A$  is an ideal of  $R$  containing  $M^k$ , then  $\sqrt{M^k} = M \subseteq \sqrt{A}$ . Therefore,  $\sqrt{A} = M$  or  $\sqrt{A} = R$ . In the first case  $M^k \subseteq A \subseteq M$ , so that  $A$  is a power of  $M$  since there are no ideals properly between  $M$  and  $M^2$  [3; 45]. And in the second case,  $A = R$ . Therefore,  $\{M^i\}_{i=0}^{k-1}$  is the set of ideals of  $R$  properly containing  $M^k$ , and  $L(M^k) = k = kL(M)$ .

**COROLLARY 1.** Let  $A$  and  $B$  be ideals of the Dedekind domain  $D$ .

a) If  $A$  and  $B$  have finite norm, then  $AB$  has finite norm and  $N(AB) = N(A)N(B)$ .

b) If  $A$  and  $B$  have finite length, then  $AB$  has finite length and  $L(AB) = L(A) + L(B)$ .

*Proof.* Since  $D$  is Dedekind, there is a set  $\{M_i\}_1^n$  of maximal ideals of  $D$  and sets  $\{e_i\}_1^n, \{f_i\}_1^n$  of nonnegative integers such that  $A = M_1^{e_1} \dots M_n^{e_n}$  and such that  $B = M_1^{f_1} \dots M_n^{f_n}$ . Further, we may assume that for each  $i$ , either  $e_i$  or  $f_i$  is positive. The fact that  $D$  is Dedekind implies that for each  $i$ , there are no ideals properly between  $M_i$  and  $M_i^2$  and the powers of  $M_i$  properly descend.

If *a*) holds, then each  $M_i$  has finite norm and Lemmas 1 and 2 show that  $N(AB) = N(M_1^{e_1+f_1} \dots M_n^{e_n+f_n}) = N(M_1^{e_1+f_1}) \dots N(M_n^{e_n+f_n}) = N(M_1)^{e_1+f_1} \dots N(M_n)^{e_n+f_n} = N(M_1)^{e_1} \dots N(M_n)^{e_n} N(M_1)^{f_1} \dots N(M_n)^{f_n} = N(A)N(B)$ . Similarly, if *b*) holds, then each  $M_i$  has finite length, and Lemmas 1 and 2 imply that  $L(AB) = L(A) + L(B)$ .

*Remark.* Corollary 1 need not hold in a general ZPI-ring  $R$ , for in such a ring, a nonzero maximal ideal of finite norm may be idempotent. This occurs, for example, if  $R$  is the direct sum of two finite fields.

A close reading of [2] will indicate that some of our results were likely known to Butts and Wade, especially Corollary 1 and our proof that (\*\*) implies  $(\sqrt{\phantom{x}})$  in a ring with identity (cf. pages 18 and 20 of [2]).

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