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# A NOTE ON TWO CRITERIA FOR DEDEKIND DOMAINS

## by Robert GILMER

Let A be an ideal of the commutative ring R. In case the residue class ring R/A is finite, we say that A has finite norm, and we set N(A) = |R/A|, where |S| denotes the cardinal number of the set S; N(A) is called the norm of A. We say that A has finite length S and we write L(A) = S if there is a chain  $A \subset A_1 \subset ... \subset A_S = R$  of ideals of R, but no such chain of S+1 ideals; therefore L(A) is the length of R/A, as an R/A-module, when this length is finite. If A has finite norm, then A has finite length. The converse fails; for example, each maximal ideal of R has finite length 1.

In [2], Butts and Wade have shown that each of the following conditions implies, in an integral domain R with identity, that R is a Dedekind domain:

- (\*) Each nonzero ideal of R has finite norm, and N(AB) = N(A) N(B) for any nonzero ideals A, B of R.
- (\*\*) Each nonzero ideal of R has finite length, and L(AB) = L(A) + L(B) for any nonzero ideals A, B of R.

Our purpose here is to show that in any commutative ring R, (\*) or (\*\*) implies the following condition  $(\sqrt{})$ .

 $(\sqrt{})$  R is Noetherian, has an identity, and for any maximal ideal M of R, there are no ideals of R properly between M and  $M^2$ .

A result of Asano in [1] shows that  $(\sqrt{})$  is equivalent to the condition that R be a general ZPI-ring—that is, each ideal of R is a finite product of prime ideals. Hence, by proving that (\*) or (\*\*) implies  $(\sqrt{})$ , we obtain a generalization of Butts and Wade's results already cited to the case of rings with zero divisors. In addition, our proof will be simpler than that given in [2] for the case of an integral domain with identity, so that we obtain both a generalization and a simplification of the results of [2].

THEOREM 1. If condition (\*) or condition (\*\*) holds in the ring R, then R is Noetherian, contains an identity, and has the property that there are no ideals properly between M and  $M^2$  for any maximal ideal M of R.

*Proof.* If (\*) or (\*\*) holds in R, it is clear that R/A is Noetherian for each nonzero ideal A of R. Therefore, R is also Noetherian. If (\*) holds,

then  $N(R^2) = [N(R)]^2 = 1$  so that  $R = R^2$ . And if (\*\*) holds,  $L(R^2) = 2L(R) = 0$ , and again  $R = R^2$ . However, a finitely generated idempotent ideal of a commutative ring is principal and is generated by an idempotent element [1; 86]. Consequently, (\*) or (\*\*) implies that R has an identity.

We consider a maximal ideal M of R. If (\*\*) is valid, then  $L(M^2) = 2L(M) = 2$ ; therefore, there are no ideals of R properly between M and  $M^2$ . If (\*) holds in R, then  $N(M^2) = [N(M)]^2$ . But the isomorphism  $R/M \simeq (R/M^2)/(M/M^2)$  implies that  $N(M^2) = N(M) \cdot k$ , where  $k = |M/M^2|$ . Therefore, k = N(M). This means that  $M/M^2$ , as a vector space over the field R/M, must have dimension 1. Consequently, there are no nontrivial R/M-subspaces of  $M/M^2$ , and therefore no ideals of R properly between M and  $M^2$ . This completes the proof of Theorem 1.

We remark that in our proof of Theorem 1 we have not used the full generality that N(AB) = N(A) N(B) or that L(AB) = L(A) + L(B); rather, we have only used the fact that,  $N(A^2) = [N(A)]^2$  and that  $L(A^2) = 2L(A)$  for any nonzero ideal A of R.

The remainder of this paper will be concerned with some results related to the converse of Theorem 1.

- LEMMA 1. Suppose that A and B are relatively prime ideals of the ring R with identity.
- a) If A and B have finite norm, then AB has finite norm and N(AB) = N(A) N(B).
- b) If A and B have finite length, then AB has finite length and L(AB) = L(A) + L(B).
- *Proof.* a) is immediate from the fact that R/AB is isomorphic to  $R/A \oplus R/B$ , the direct sum of R/A and R/B [4; 178]. To prove b), we need only note that if  $R_1$  and  $R_2$  are rings of which the zero ideals have finite lengths  $n_1$  and  $n_2$ , then the zero ideal of  $R_1 \oplus R_2$  has finite length  $n_1 + n_2$ . This is immediate from the fact, however, that each ideal of  $R_1 \oplus R_2$  is of the form  $A_1 \oplus A_2$ , where  $A_i$  is an ideal of  $R_i$  [4; 175].
- LEMMA 2. Let M be a maximal ideal of a commutative ring R with identity such that there are no ideals properly between M and  $M^2$ , and let k be a positive integer such that  $M^k \subset M^{k-1}$ .

- a) If M has finite norm, then  $N(M^k) = [N(M)]^k$ .
- b) If M has finite length, then  $L(M^k) = kL(M) = k$ .

*Proof.* If r is a positive integer such that  $M^r \supset M^{r+1}$ , it is known that  $M^r/M^{r+1}$  and R/M are, as vector spaces over R/M, isomorphic [1; 83]. Hence,  $|R/M| = |M^r/M^{r+1}|$ . If we assume that  $N(M^{k-1}) = [N(M)]^{k-1}$ , then the isomorphism  $R/M^{k-1} \simeq (R/M^k)/(M^{k-1}/M^k)$  implies that  $N(M^k) = N(M^{k-1})|M^{k-1}/M^k| = [N(M)]^{k-1} N(M) = [N(M)]^k$ . This establishes a).

To prove b), we note that if A is an ideal of R containing  $M^k$ , then  $\sqrt{M^k} = M \subseteq \sqrt{A}$ . Therefore,  $\sqrt{A} = M$  or  $\sqrt{A} = R$ . In the first case  $M^k \subseteq A \subseteq M$ , so that A is a power of M since there are no ideals properly between M and  $M^2$  [3; 45]. And in the second case, A = R. Therefore,  $\{M^i\}_{i=0}^{k-1}$  is the set of ideals of R properly containing  $M^k$ , and  $L(M^k) = k = kL(M)$ .

COROLLARY 1. Let A and B be ideals of the Dedekind domain D.

- a) If A and B have finite norm, then AB has finite norm and N(AB) = N(A) N(B).
- b) If A and B have finite length, then AB has finite length and L(AB) = L(A) + L(B).

*Proof.* Since D is Dedekind, there is a set  $\{M_i\}_1^n$  of maximal ideals of D and sets  $\{e_i\}_1^n$ ,  $\{f_i\}_1^n$  of nonnegative integers such that  $A = M_1^{e_1} \dots M_n^{e_n}$  and such that  $B = M_1^{f_1} \dots M_n^{f_n}$ . Further, we may assume that for each i, either  $e_i$  or  $f_i$  is positive. The fact that D is Dedekind implies that for each i, there are no ideals properly between  $M_i$  and  $M_i^2$  and the powers of  $M_i$  properly descend.

If a) holds, then each  $M_i$  has finite norm and Lemmas 1 and 2 show that  $N(AB) = N(M_1^{e_1+f_1} \dots M_n^{e_n+f_n}) = N(M_1^{e_1+f_1}) \dots N(M_n^{e_n+f_n}) = N(M_1)^{e_1+f_1} \dots N(M_n)^{e_n+f_n} = N(M_1)^{e_1} \dots N(M_n)^{e_n} N(M_1)^{f_1} \dots N(M_n)^{f_n} = N(A) N(B)$ . Similarly, if b) holds, then each  $M_i$  has finite length, and Lemmas 1 and 2 imply that L(AB) = L(A) + L(B).

Remark. Corollary 1 need not hold in a general ZPI-ring R, for in such a ring, a nonzero maximal ideal of finite norm may be idempotent. This occurs, for example, if R is the direct sum of two finite fields.

A close reading of [2] will indicate that some of our results were likely known to Butts and Wade, especially Corollary 1 and our proof that (\*\*) implies  $(\sqrt{})$  in a ring with identity (cf. pages 18 and 20 of [2]).

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