A NOTE ON TWO CRITERIA FOR DEDEKIND DOMAINS

by Robert GILMER

Let A be an ideal of the commutative ring R. In case the residue class ring R/A is finite, we say that A *has finite norm*, and we set $N(A) = |R/A|$, where $|S|$ denotes the cardinal number of the set S; $N(A)$ is called *the norm of* A. We say that A *has finite length* s and we write $L(A) = s$ if there is a chain $A \subset A_1 \subset \ldots \subset A_s = R$ of ideals of R, but no such chain of $s+1$ ideals; therefore $L(A)$ is the length of R/A, as an R/A-module, when this length is finite. If A has finite norm, then A has finite length. The converse fails; for example, each maximal ideal of R has finite length 1.

In [2], Butts and Wade have shown that each of the following conditions implies, in an integral domain R with identity, that R is a Dedekind domain:

(*) Each nonzero ideal of R has finite norm, and $N(AB) = N(A)\,N(B)$ for any nonzero ideals A, B of R.

(**) Each nonzero ideal of R has finite length, and $L(AB) = L(A) + L(B)$ for any nonzero ideals A, B of R.

Our purpose here is to show that in any commutative ring R, (*) or (**) implies the following condition ($\surd$).

($\surd$) R is Noetherian, has an identity, and for any maximal ideal M of R, there are no ideals of R properly between M and M^2.

A result of Asano in [1] shows that ($\surd$) is equivalent to the condition that R be a *general ZPI-ring*—that is, each ideal of R is a finite product of prime ideals. Hence, by proving that (*) or (**) implies ($\surd$), we obtain a generalization of Butts and Wade's results already cited to the case of rings with zero divisors. In addition, our proof will be simpler than that given in [2] for the case of an integral domain with identity, so that we obtain both a generalization and a simplification of the results of [2].

THEOREM 1. *If condition (*) or condition (**) holds in the ring R, then R is Noetherian, contains an identity, and has the property that there are no ideals properly between M and M^2 for any maximal ideal M of R.*

Proof. If (*) or (**) holds in R, it is clear that R/A is Noetherian for each nonzero ideal A of R. Therefore, R is also Noetherian. If (*) holds, then $N(R^2) = [N(R)]^2 = 1$ so that $R = R^2$. And if (**) holds, $L(R^2) = 2L(R) = 0$, and again $R = R^2$. However, a finitely generated idempotent ideal of a commutative ring is principal and is generated by an idempotent element [1; 86]. Consequently, (*) or (**) implies that R has an identity.

We consider a maximal ideal M of R. If (**) is valid, then $L(M^2) = 2L(M) = 2$; therefore, there are no ideals of R properly between M and M^2. If (*) holds in R, then $N(M^2) = [N(M)]^2$. But the isomorphism $R/M \simeq (R/M^2)/(M/M^2)$ implies that $N(M^2) = N(M) \cdot k$, where $k = |M/M^2|$. Therefore, $k = N(M)$. This means that M/M^2, as a vector space over the field R/M, must have dimension 1. Consequently, there are no nontrivial R/M-subspaces of M/M^2, and therefore no ideals of R properly between M and M^2. This completes the proof of Theorem 1.

We remark that in our proof of Theorem 1 we have not used the full generality that $N(AB) = N(A)\, N(B)$ or that $L(AB) = L(A) + L(B)$; rather, we have only used the fact that, $N(A^2) = [N(A)]^2$ and that $L(A^2) = 2L(A)$ for any nonzero ideal A of R.

The remainder of this paper will be concerned with some results related to the converse of Theorem 1.

LEMMA 1. *Suppose that A and B are relatively prime ideals of the ring R with identity.*

a) *If A and B have finite norm, then AB has finite norm and $N(AB) = N(A)\, N(B)$.*

b) *If A and B have finite length, then AB has finite length and $L(AB) = L(A) + L(B)$.*

Proof. *a)* is immediate from the fact that R/AB is isomorphic to $R/A \oplus R/B$, the direct sum of R/A and R/B [4; 178]. To prove *b)*, we need only note that if R_1 and R_2 are rings of which the zero ideals have finite lengths n_1 and n_2, then the zero ideal of $R_1 \oplus R_2$ has finite length $n_1 + n_2$. This is immediate from the fact, however, that each ideal of $R_1 \oplus R_2$ is of the form $A_1 \oplus A_2$, where A_i is an ideal of R_i [4; 175].

LEMMA 2. *Let M be a maximal ideal of a commutative ring R with identity such that there are no ideals properly between M and M^2, and let k be a positive integer such that $M^k \subset M^{k-1}$.*

a) *If M has finite norm, then* $N(M^k) = [N(M)]^k$.

b) *If M has finite length, then* $L(M^k) = kL(M) = k$.

Proof. If r is a positive integer such that $M^r \supset M^{r+1}$, it is known that M^r/M^{r+1} and R/M are, as vector spaces over R/M, isomorphic [1; 83]. Hence, $|R/M| = |M^r/M^{r+1}|$. If we assume that $N(M^{k-1}) = [N(M)]^{k-1}$, then the isomorphism $R/M^{k-1} \simeq (R/M^k)/(M^{k-1}/M^k)$ implies that $N(M^k) = N(M^{k-1})|M^{k-1}/M^k| = [N(M)]^{k-1} N(M) = [N(M)]^k$. This establishes *a)*.

To prove *b)*, we note that if A is an ideal of R containing M^k, then $\sqrt{M^k} = M \subseteq \sqrt{A}$. Therefore, $\sqrt{A} = M$ or $\sqrt{A} = R$. In the first case $M^k \subseteq A \subseteq M$, so that A is a power of M since there are no ideals properly between M and M^2 [3; 45]. And in the second case, $A = R$. Therefore, $\{M^i\}_{i=0}^{k-1}$ is the set of ideals of R properly containing M^k, and $L(M^k) = k = kL(M)$.

Corollary 1. *Let A and B be ideals of the Dedekind domain D.*

a) *If A and B have finite norm, then AB has finite norm and* $N(AB) = N(A)N(B)$.

b) *If A and B have finite length, then AB has finite length and* $L(AB) = L(A) + L(B)$.

Proof. Since D is Dedekind, there is a set $\{M_i\}_1^n$ of maximal ideals of D and sets $\{e_i\}_1^n$, $\{f_i\}_1^n$ of nonnegative integers such that $A = M_1^{e1} \dots M_n^{e_n}$ and such that $B = M_1^{f1} \dots M_n^{f_n}$. Further, we may assume that for each i, either e_i or f_i is positive. The fact that D is Dedekind implies that for each i, there are no ideals properly between M_i and M_i^2 and the powers of M_i properly descend.

If *a)* holds, then each M_i has finite norm and Lemmas 1 and 2 show that $N(AB) = N(M_1^{e1+f1} \dots M_n^{e_n+f_n}) = N(M_1^{e1+f1}) \dots N(M_n^{e_n+f_n}) = N(M_1)^{e1+f1} \dots N(M_n)^{e_n+f_n} = N(M_1)^{e1} \dots N(M_n)^{e_n} N(M_1)^{f1} \dots N(M_n)^{f_n} = N(A)N(B)$. Similarly, if *b)* holds, then each M_i has finite length, and Lemmas 1 and 2 imply that $L(AB) = L(A) + L(B)$.

Remark. Corollary 1 need not hold in a general *ZPI*-ring R, for in such a ring, a nonzero maximal ideal of finite norm may be idempotent. This occurs, for example, if R is the direct sum of two finite fields.

A close reading of [2] will indicate that some of our results were likely known to Butts and Wade, especially Corollary 1 and our proof that (**) implies ($\sqrt{}$) in a ring with identity (cf. pages 18 and 20 of [2]).

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