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Proof. From (2.7), we have

$$\| f * g \|^*_{p1} = \sup_h B \left| \int f * g(x)\, h(x)\, dm(x) \right| ,$$

where $h^*(t) \leqq t^{(1/p)-1}$. Let $I(h) = \int f * g(x)\, h(x)\, dm(x)$.

$$| I(h) | \leqq \int \left(\int | f(y) | \cdot | g(xy^{-1}) | \, dm(y) \, | h(x) | \, dm(x) \right.$$
$$= \int | f(y) | \left(\int | g(xy^{-1}) | \cdot | h(x) | \, dm(x) \right) dm(y) .$$

Hence, $| I(h) | \leqq \| fk \|^*_{11}$, where $k(y) = \int | g(xy^{-1}) | . | h(x) | \, dm(x)$. By the multiplication theorem it follows that $| I(h) | \leqq B \| f \|^*_{p_0 q_0} \| k \|^*_{p'_0 q_1}$, where $1/p_0 + 1/p'_0 = 1$. But $k = | \bar{g} | * | h |$, where $\bar{g}(x) = g(x^{-1})$. Hence by Lemma 4.8,

$$\| k \|^*_{p'_0 q_1} \leqq B \| \bar{g} \|^*_{p_1 q_1} \| h \|^*_{p' \infty} \leqq B \| \bar{g} \|^*_{p_1 q_1} .$$

Since (G, dm) is unimodular, we have $(\bar{g})^*(t) = g^*(t)$ and the lemma follows.

By applying the strong type interpolation theorem to the end point results of Lemma 4.8 and Lemma 4.9, we obtain

THEOREM 4.10. (Convolution theorem):

$$\| f * g \|^*_{pq} \leqq B \| f \|^*_{p_0 q_0} \| g \|^*_{p_1 q_1} ,$$

where $0 < 1/p = 1/p_0 + 1/p_1 - 1 < 1$, $1 < p_0, p_1 < \infty$ *and* $0 \leqq 1/q = 1/q_0 + 1/q_1 \leqq 1$.

Section 5. REFERENCES

Various properties of $L(p, q)$ spaces have appeared in many places, often as special cases of a more general theory. We will mention several places where related results and applications are found. The references given are not necessarily the first or the only place where the indicated result appears.

The principal references are [19] and [20], where G. G. Lorentz defines special cases of $L(p, q)$ spaces and proves many of their properties. The notion of a non-increasing rearrangement of a function was used by Hardy Littlewood and Payley. (See [32].) A simple proof of the inequality $\| f \|^*_{pq_2} \leqq B \| f \|^*_{pq_1}$, $q_1 \leqq q_2$, is found in O'Neil [22]. The technique used in the proof of Hardy's inequality is well known. A different proof of Hardy's inequality is found in [9, p. 245] or [32, Vol. I, p. 20]. The author learned the simple proof of inequality (1.9) from class notes from a course given by A. Zygmund in Chicago. A similar proof is given in [11, p. 278].

The $L(p, q)$ spaces which are Banach spaces appear as intermediate spaces in the general interpolation theory of Calderón [1]. Peetre [24] identifies $L(p, q)$ spaces as intermediate spaces for the interpolation theory of Lions and Peetre [18]. Many $L(p, q)$ results are then contained in these general theories. In particular, there are results concerning density, inclusion, separability and duality of the spaces. A ** norm is used in these results. Riviere [25] generalized the results of Calderón [1] to include $L(p, q)$, $p, q > 0$. Similarly, P. Kree and J. Peetre generalized the results of Lions and Peetre [18].

(2.5) is proved by Krein and Semenov [18] and is contained implicitly in Stein and Weiss [30]. Halperin [8] and [9] obtains general results on conjugate spaces and reflexitivity. Results on uniform convexity of some related spaces are found in Halperin [10]. The results concerning linear functionals on $L(p, q)$, $p < 1$, correspond to results of Day [5] for L^p spaces $0 < p < 1$.

The weak type theorem of Section 3 restricted to linear operators on the $L(p, q)$ spaces which are Banach spaces was proved by A. P. Calderón [2]. We learned that E. M Stein also obtain these results. Proof of these cases is found in Lions and Peetre [18], together with Peetre [24]. Also see Calderón [1] and Oklander [21]. Krein and Semonov [17] prove some special cases. The theorem is closely related to results of Stein and Weiss [29] and [30]. The weak type theorem for $L(p, q)$, $p, q > 0$, is proved in Hunt [14]. These cases are also contained in work of P. Kree and J. Peetre.

The strong type theorem of Section 3 for linear operators on the $L(p, q)$ spaces which are Banach spaces is found in Calderón [1]. These results are related to results of Hirschman [13], Stein [26] and Stein and Weiss [27]. The result for sublinear operators follows ideas found in Calderón [1], Calderón and Zygmund [3] and Weiss [31]. Rivière [25] obtains results for linear operators acting on $L(p, q)$ spaces, $p, q > 0$.

Stein [26] proves an analogue of Theorem 4.3 for Fourier coefficients. The multiplication and convolution theorems are proved by a different method in O'Neil [22]. E. M. Stein also obtained these results. (See [22].)