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This is done by considering in detail some classical $L^p$ operators. Related references are contained in Section 5.

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## Section 1. ELEMENTARY PROPERTIES AND INEQUALITIES

We consider only complex-valued, measurable functions defined on a measure space $(M, m)$. The measure $m$ is assumed to be non-negative and totally $\sigma$-finite. We assume the functions $f$ are finite valued a.e. and, for some $y > 0$, $m(E_y) < \infty$, where $E_y = E_y[f] = \{x \in M : |f(x)| > y\}$. As usual, we identity functions which are equal a.e.

The *distribution function of* $f$ is defined by $\lambda(y) = \lambda_f(y) = m(E_y)$, $y > 0$. $\lambda(y)$ is non-negative, non-increasing and continuous from the right. The *non-increasing rearrangement of* f *onto* $(0, \infty)$ is defined by $f^*(t) = \inf\{y > 0 : \lambda_f(y) \leqq t\}$, $t > 0$. Since $\lambda_f(y) < \infty$ for some $y > 0$ and $f$ is finite valued a.e. we have that $\lambda_f(y) \to 0$ as $y \to \infty$. It follows that $f^*(t)$ is well defined for $t > 0$. $f^*(t)$ is clearly non-negative and non-increasing on $(0, \infty)$. If $\lambda_f(y)$ is continuous and strictly decreasing then $f^*(t)$ is the inverse function of $\lambda_f(y)$.

It follows immediately from the definition of $f^*(t)$ that

$$f^*(\lambda_f(y)) \leqq y \, . \tag{1.1}$$

Since $\lambda_f(y)$ is continuous from the right we have

$$\lambda_f(f^*(t)) \leqq t \, . \tag{1.2}$$

Inequalities (1.1) and (1.2) can be used to prove two elementary properties of $f^*$.

(1.3) $\qquad$ *$f^*(t)$ is continuous from the right.*

*Proof.* We have $f^*(t) \geqq f^*(t+h)$ for all $h > 0$. If there exists $y$ such that $f^*(t) > y > f^*(t+h)$ for all $h > 0$, then, using (1.2), we have $\lambda_f(y) \leqq \lambda_f(f^*(t+h)) \leqq t + h$ for all $h > 0$. That is, $\lambda_f(y) \leqq t$. It follows that $f^*(t) \leqq y$, which is a contradiction.

(1.4) $$\lambda_{f*}(y) = \lambda_f(y) \textit{ for all } y > 0 .$$

*Proof.* $\lambda_{f*}(y)$ is the Lebesgue measure of the set of points $t > 0$ for which $f^*(t) > y$. Since $f^*$ is non-increasing we have

(*) $$\lambda_{f*}(y) = \sup \{ t > 0 : f^*(t) > y \} .$$

We see from (*) that $f^*(\lambda_f(y)) \leqq y$ implies $\lambda_f(y) \geqq \lambda_{f*}(y)$.

If $t > \lambda_{f*}(y)$, then (*) implies $f^*(t) \leqq y$. Hence, $\lambda_f(y) \leqq \lambda_f(f^*(t)) \leqq t$. It follows that $\lambda_f(y) \leqq \lambda_{f*}(y)$ and (1.4) is proved.

By a simple function we mean a function which can be written in the form

$$f(x) = \sum_{j=1}^{N} c_j \chi_{E_j}(x) ,$$

where $c_1, ..., c_N$ are complex numbers, $E_1, ..., E_N$ are pairwise disjoint sets of finite measure and $\chi_E(x)$ denotes the characteristic function of the set $E$. For such a function let $c_1^*, ..., c_N^*$ be a rearrangement of the numbers $|c_1|, ..., |c_N|$ such that $c_1^* \geqq c_2^* \geqq ... \geqq c_N^* \geqq 0$. Then

$$f^*(t) = \begin{cases} c_1^* & 0 < t < m(E_1) \\ c^* & \sum_{k=1}^{j-1} m(E_k) \leqq t < \sum_{k=1}^{j} m(E_k), \quad j = 2, ..., N \\ 0 & t \geqq \sum_{k=1}^{N} m(E_k) . \end{cases}$$

It is very useful to note

(1.5) *If* $f(x)$ *is a non-negative simple function, then we can write* $f(x) = \sum_{j=1}^{N} f_j(x)$, *where* $f_j(x)$ *is a non-negative function with exactly one positive value and* $f^*(t) = \sum_{j=1}^{N} f_j^*(t)$.

*Proof.* Suppose $f(x) = \sum_{j=1}^{N} c_j \chi_{E_j}(x)$, where $E_1, ..., E_N$ are pairwise disjoint and $c_1 > ... > c_N > c_{N+1} = 0$. Let $F_j = \bigcup_{k=1}^{j} E_k$ and $\alpha_j = c_j - c_{j+1}$, $j = 1, ..., N$. Set $f_j(x) = \alpha_j \chi_{F_j}(x)$ and we are done.

Consideration of the functions $f(x) = 1-x$ and $g(x) = x$, $0 \leqq x \leqq 1$, shows that we do not always have $(f+g)^*(t) \leqq f^*(t) + g^*(t)$. However,

$$(f+g)^*(t_1+t_2) \leqq f^*(t_1) + g^*(t_2), \qquad t_1, t_2 > 0. \tag{1.6}$$

*Proof.* Since

$$\{x \in M : |f(x) + g(x)| > f^*(t_1) + g^*(t_2)\}$$
$$\subset \{x \in M : |f(x)| > f^*(t_1)\} \cup \{x \in M : |g(x)| > g^*(t_2)\}$$

we have $\lambda_{f+g}(f^*(t_1) + g^*(t_2)) \leqq \lambda_f(f^*(t_1)) + \lambda_g(g^*(t_2)) \leqq t_1 + t_2$. This implies (1.6).

The *Lorentz space* $L(p, q)$ is the collection of all $f$ such that $\|f\|^*_{pq} < \infty$, where

$$\|f\|^*_{pq} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^\infty [t^{1/p} f^*(t)]^q \dfrac{dt}{t}\right)^{1/q}, & 0 < p < \infty, \quad 0 < q < \infty \\ \sup\limits_{t>0} t^{1/p} f^*(t), & 0 < p \leqq \infty, \quad q = \infty. \end{cases}$$

The case $p = \infty$, $0 < q < \infty$ is not of interest since $\int_0^\infty [f^*(t)]^q \, dt/t < \infty$ implies $f = 0$ a.e.

Since $f$ and $f^*$ have the same distribution function we have $\|f\|^*_{pp} = (\int_M |f(x)|^p \, dm(x))^{1/p}$. Hence, $L$ (p, p) is the familiar $L^p$ space on $(M, m)$.

Since $f^*$ is essentially the inverse function of $\lambda_f$,

$$\sup_{t>0} t^{1/p} f^*(t) = \sup_{y>0} y [\lambda_f(y)]^{1/p}. \tag{1.7}$$

$L(p, \infty)$ plays an important role in analysis and is often called weak $L^p$. $L^p$ and weak $L^p$, as well as all $L(p, q)$ which have the same first index $p$, are related by

$$\|f\|^*_{pq_2} \leqq \|f\|^*_{pq_1}, \qquad 0 < q_1 \leqq q_2 \leqq \infty. \tag{1.8}$$

*Proof.* In case $q_2 = \infty$ we have, since $f^*(t)$ is non-increasing,

$$t^{1/p} f^*(t) = f^*(t) \left(\frac{q_1}{p}\int_0^t y^{(q_1/p)-1} \, dy\right)^{1/q_1}$$

$$\leq \Big(\frac{q_1}{p}\int_0^t [y^{1/p} f^*(y)]^{q_1}\, dy/y\Big)^{1/q_1}.$$

The result follows immediately.

In case $q_2 < \infty$ it is sufficient to prove the inequality for simple functions since we can clearly find simple functions $f_n(t)$ such that $0 \leqq f_n \nearrow f^*$ and apply the monotone convergence theorem.

If $f$ is a simple function we have $f^*(t) = c_k$ for $a_{k-1} \leqq t < a_k$, $k = 1, \ldots, N$, where $c_1 > c_2 > \ldots > c_N > 0$ and $0 = a_0 < a_1 < \ldots < a_N$. Then $\|f\|_{pq}^* = \Big(\sum_{k=1}^{N} c_k^q \,(a_k^{q/p} - a_{k-1}^{q/p})\Big)^{1/q}$. By setting $d_k = c_k^{q_2}$, $b_k = a_k^{q_2/p}$ and $\theta = q_1/q_2$ we see that (1.8) is a consequence of

$$(*)\qquad \sum_{k=1}^{N} d_k (b_k - b_{k-1}) \leqq \Big(\sum_{k=1}^{N} d_k^\theta (b_k^\theta - b_{k-1}^\theta)\Big)^{1/\theta},$$

for $\infty > d_1 > d_2 > \ldots > 0$, $0 = b_0 < b_1 < \ldots < \infty$ and $0 < \theta < 1$.

The proof of (*) is by finite induction. (*) is obviously true (with equality) for $N = 1$. Assume (*) is true for $N$ and consider

$$\varphi(x) = \Big(\sum_{k=1}^{N} d_k^\theta (b_k^\theta - b_{k-1}^\theta) + x^\theta (b_{N+1} - b_N)\Big)^{1/\theta}$$

$$- \Big(\sum_{k=1}^{N} d_k (b_k - b_{k-1}) + x (b_{N+1} - b_N)\Big), \qquad 0 \leqq x \leqq d_N.$$

We must show that $\varphi(d_{N+1}) \geqq 0$. We have $\varphi(0) \geqq 0$ and $\varphi(d_N) \geqq 0$ by our induction hypothesis, since $\varphi(0) \geqq 0$ is exactly (*) and $\varphi(d_N)$ is (*) with $b_N$ replaced by $b_{N+1}$. A simple calculation shows that $\varphi''(x) \leqq 0$ for $x > 0$. Hence, $\varphi(x) \geqq 0$ for $0 \leqq x \leqq d_N$. Since $0 < d_{N+1} < d_N$ this completes the proof.

If $\chi_E$ is the characteristic function of a set of finite measure then $\|\chi_E\|_{pq}^* = [m(E)]^{1/p}$ for all $p, q$. This implies that inequality (1.8) is best possible. Shorter proofs can be used to obtain $\|f\|_{pq_2}^* \leqq B \|f\|_{pq_1}^*$, $q_1 < q_2$. For example,

$$\Big(\frac{q_2}{p}\int_0^\infty [t^{1/p} f^*(t)]^{q_2} \frac{dt}{t}\Big)^{q_1/q_2} \leqq \Big(\sum_{k=-\infty}^{\infty} [f^*(2^{k-1})]^{q_2} \Big[\frac{q_2}{p}\int_{2^{k-1}}^{2^k} t^{(q_2/p)-1}\, dt\Big]\Big)^{q_1/q_2}$$

$$\leqq \sum_{k=-\infty}^{\infty} [f^*(2^{k-1})]^{q_1}\, 2^{kq_1/p}$$

$$\leqq B\frac{q_1}{p}\sum_{k=-\infty}^{\infty}\int_{2^{k-2}}^{2^{k-1}}\left[t^{1/p}f^*(f)\right]^{q_1}\frac{dt}{t}$$

$$= B\left[\|f\|^*_{pq_1}\right]^{q_1}.$$

(1.8) clearly implies $L(p, q_1) \subset L(p, q_2)$, $0 < q_1 \leqq q_2 \leqq \infty$. If the measure space $(M, m)$ contains a countably infinite collection of pairwise disjoint sets of finite non-zero measure it is easy to construct a simple function $f$ which belongs to $L(p, q_1)$ but does not belong to $L(p, q_2)$ for any given $p$ and $q_1 < q_2$.

$L(p, q)$ spaces with different first indices are related only in special cases. For example, if $m(M) < \infty$, $L(p_2, q_2) \subset L(p_2, \infty) \subset L(p_1, q_1)$ for $p_1 \leqq p_2$. If $m(E) \geqq 1$ for every measurable set $E \subset M$ with $m(E) > 0$, then $L(p_1, q_1) \subset L(p_1, \infty) \subset L(p_2, q_2)$ for $p_1 \leqq p_2$.

(1.8) and the following inequalities are fundamental to the study of $L(p, q)$ spaces.

A function $\varphi(x)$ defined on an interval of the real line is said to be *convex* if for every pair of points $P_1, P_2$ on the curve $y = \varphi(x)$ the points of the arc $P_1 P_2$ are below, or on, the chord $P_1 P_2$. For example, $x^r$, $r \geqq 1$, is convex in $(0, \infty)$ and $e^x$ is convex in $(-\infty, \infty)$. We will need Jensen's integral inequality. (See [32, Vol. I, p. 24].)

THEOREM. (Jensen): *Suppose* $\varphi(u)$ *is convex in an interval* $\alpha \leqq u \leqq \beta$, $\alpha \leqq f(x) \leqq \beta$ *in* $a \leqq x \leqq b$ *and that* $p(x)$ *is non-negative with* $\int_a^b p(x)\,dx \neq 0$. *Then*

$$\varphi\left(\frac{\int_a^b f(x)\,p(x)\,dx}{\int_a^b p(x)\,dx}\right) \leqq \frac{\int_a^b \varphi(f(x))\,p(x)\,dx}{\int_a^b p(x)\,dx},$$

*where all integrals in question are assumed to exist and be finite.*

*Proof.* Let $\gamma = \int_a^b fp\,dx / \int_a^b p\,dx$. Then $\alpha \leqq \gamma \leqq \beta$. Let us first suppose that $\alpha < \gamma < \beta$, and let $k$ be the slope of a supporting line of $\varphi$ through the point $(\gamma, \varphi(\gamma))$. Then since $\varphi$ is convex, we have

$$(*) \qquad \varphi(u) - \varphi(\gamma) \geqq k(u - \gamma), \quad \alpha \leqq u \leqq \beta.$$

Replacing $u$ by $f(x)$ in (*), multiplying both sides by $p(x)$, and integrating over $a \leqq x \leqq b$, we obtain

$$\int_a^b \varphi(f(x))\, p(x)\, dx - \varphi(\gamma) \int_a^b p(x)\, dx \geqq k \left\{ \int_a^b f(x) p(x)\, dx - \gamma \int_a^b p(x)\, dx \right\} = 0,$$

which is the desired inequality. If $\gamma = \beta$, then $f(x) = \beta$ at a.e. point at which $p(x) > 0$ and the inequality is obvious. Similarly if $\gamma = \alpha$.

THEOREM (Hardy): *If* $q \geqq 1$, $r > 0$ *and* $f \geqq 0$, *then*

$$\left(\int_0^\infty \left[\int_0^t f(y)\, dy\right]^q t^{-r-1}\, dt\right)^{1/q} \leqq \frac{q}{r} \left(\int_0^\infty [y f(y)]^q y^{-r-1}\, dy\right)^{1/q}$$

*and*

$$\left(\int_0^\infty \left[\int_t^\infty f(y)\, dy\right]^q t^{r-1}\, dt\right)^{1/q} \leqq \frac{q}{r} \left(\int_0^\infty [y f(y)]^q y^{r-1}\, dy\right)^{1/q}.$$

*Proof.* The technique of the proof is to write $[\int_0^t f(x)\, dy]^q$ as $[\int_0^t f(x)\, y^{-\alpha} y^{\alpha}\, dy]^q$ and apply Jensen's inequality to the measure $y^{\alpha}\, dy$. We obtain an inequality of the form

$$\left(\int_0^\infty \left[\int_0^t f(y)\, dy\right]^q t^{-r-1}\, dt\right)^{1/q} \leqq C(\alpha) \left(\int_0^\infty [y f(y)]^q y^{-r-1}\, dy\right)^{1/q}.$$

$\alpha$ is then chosen so that $C(\alpha)$ is minimal. In this case $\alpha = (r/q) - 1$ is the best choice.

$$\left(\int_0^\infty \left[\int_0^t f(y)\, dy\right]^q t^{-r-1}\, dt\right)^{1/q}$$

$$= \frac{q}{r} \left(\int_0^\infty \left[\frac{r}{q} t^{-r/q} \int_0^t f(y)\, y^{-(r/q)+1}\, y^{(r/q)-1}\, dy\right]^q t^{-1}\, dt\right)^{1/q}$$

which, by Jensen's inequality, is majorized by

$$\left(\frac{q}{r}\right)^{1-1/q} \left(\int_0^\infty \left[\int_0^t (f(y)\, y^{-(r/q)+1})^q\, y^{(r/q)-1}\, dy\right] t^{-(r/q)-1}\, dt\right)^{1/q}.$$

After applying Fubini's Theorem we see that the last expression is equal to

$$\frac{q}{r}\Big(\int_0^\infty [yf(y)]^q y^{-r-1}\,dy\Big)^{1/q}.$$

The proof of the second inequality is the same except that $r$ is replaced by $-r$.

$$(1.9)\qquad \int_E |f(x)g(x)|\,dm(x) \leqq \int_0^{m(E)} f^*(t)\,g^*(t)\,dt\,.$$

*Proof.* We may assume $f$ and $g$ are non-negative simple functions. We then write $f = \Sigma f_j$ and $g = \Sigma g_k$ as in (1.5). (1.9) is clearly true for the functions $f_j\,g_k$ and the result follows.

Finally, let us note

$$(1.10)\qquad \frac{1}{y}\int_0^y g(t)\,dt \leqq \frac{1}{x}\int_a^x g(t)\,dt \quad \text{for } 0 < x \leqq y\,,$$

*where* g (t) *is non-negative and non-increasing on* t > 0.

(1.10) is geometrically obvious.

## Section 2. Topological properties

(1.6) implies that $f + g \in L(p, q)$ if $f, g \in L(p, q)$. Since $||\,.\,||_{pq}^*$ is positive homogeneous we see that $L(p, q)$ is a linear space. $||\,.\,||_{pq}^*$ leads to a topology on $L(p, q)$ such that $L(p, q)$ is a topological vector space. $f_n \to f \in L(p, q)$ in this topology if and only if $||f - f_n||_{pq}^* \to 0$. We shall see that this space is metrizable.

For $p, q$ fixed we define two analogues of $f^*$. Choose $r$ such that $0 < r \leqq 1$, $r \leqq q$ and $r < p$. Let

$$f^{**}(t) = f^{**}(t, r) = \begin{cases} \sup\limits_{m(E) \geqq t} \Big(\dfrac{1}{m(E)}\displaystyle\int_E |f(x)|^r\,dm(x)\Big)^{1/r}, & t \leqq m(M) \\[2ex] \Big(\dfrac{1}{t}\displaystyle\int_M |f(x)|^r\,dm(x)\Big)^{1/r}, & t > m(M)\,. \end{cases}$$

Consider $(f^*)^{**}(t)$. Since any $g^{**}$ is non-negative and non-increasing we can use (1.9) and (1.10) to see that