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This is done by considering in detail some classical  $L^p$  operators. Related references are contained in Section 5.

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### Section 1. ELEMENTARY PROPERTIES AND INEQUALITIES

We consider only complex-valued, measurable functions defined on a measure space  $(M, m)$ . The measure  $m$  is assumed to be non-negative and totally  $\sigma$ -finite. We assume the functions  $f$  are finite valued a.e. and, for some  $y > 0$ ,  $m(E_y) < \infty$ , where  $E_y = E_y[f] = \{x \in M : |f(x)| > y\}$ . As usual, we identify functions which are equal a.e.

The *distribution function* of  $f$  is defined by  $\lambda_f(y) = m(E_y)$ ,  $y > 0$ .  $\lambda_f(y)$  is non-negative, non-increasing and continuous from the right. The *non-increasing rearrangement of  $f$  onto  $(0, \infty)$*  is defined by  $f^*(t) = \inf \{y > 0 : \lambda_f(y) \leq t\}$ ,  $t > 0$ . Since  $\lambda_f(y) < \infty$  for some  $y > 0$  and  $f$  is finite valued a.e. we have that  $\lambda_f(y) \rightarrow 0$  as  $y \rightarrow \infty$ . It follows that  $f^*(t)$  is well defined for  $t > 0$ .  $f^*(t)$  is clearly non-negative and non-increasing on  $(0, \infty)$ . If  $\lambda_f(y)$  is continuous and strictly decreasing then  $f^*(t)$  is the inverse function of  $\lambda_f(y)$ .

It follows immediately from the definition of  $f^*(t)$  that

$$(1.1) \quad f^*(\lambda_f(y)) \leq y.$$

Since  $\lambda_f(y)$  is continuous from the right we have

$$(1.2) \quad \lambda_f(f^*(t)) \leq t.$$

Inequalities (1.1) and (1.2) can be used to prove two elementary properties of  $f^*$ .

$$(1.3) \quad f^*(t) \text{ is continuous from the right.}$$

*Proof.* We have  $f^*(t) \geq f^*(t+h)$  for all  $h > 0$ . If there exists  $y$  such that  $f^*(t) > y > f^*(t+h)$  for all  $h > 0$ , then, using (1.2), we have  $\lambda_f(y) \leq \lambda_f(f^*(t+h)) \leq t+h$  for all  $h > 0$ . That is,  $\lambda_f(y) \leq t$ . It follows that  $f^*(t) \leq y$ , which is a contradiction.

$$(1.4) \quad \lambda_{f^*}(y) = \lambda_f(y) \text{ for all } y > 0.$$

*Proof.*  $\lambda_{f^*}(y)$  is the Lebesgue measure of the set of points  $t > 0$  for which  $f^*(t) > y$ . Since  $f^*$  is non-increasing we have

$$(*) \quad \lambda_{f^*}(y) = \sup \{ t > 0 : f^*(t) > y \}.$$

We see from  $(*)$  that  $f^*(\lambda_f(y)) \leq y$  implies  $\lambda_f(y) \geq \lambda_{f^*}(y)$ .

If  $t > \lambda_{f^*}(y)$ , then  $(*)$  implies  $f^*(t) \leq y$ . Hence,  $\lambda_f(y) \leq \lambda_f(f^*(t)) \leq t$ . It follows that  $\lambda_f(y) \leq \lambda_{f^*}(y)$  and  $(1.4)$  is proved.

By a simple function we mean a function which can be written in the form

$$f(x) = \sum_{j=1}^N c_j \chi_{E_j}(x),$$

where  $c_1, \dots, c_N$  are complex numbers,  $E_1, \dots, E_N$  are pairwise disjoint sets of finite measure and  $\chi_E(x)$  denotes the characteristic function of the set  $E$ . For such a function let  $c_1^*, \dots, c_N^*$  be a rearrangement of the numbers  $|c_1|, \dots, |c_N|$  such that  $c_1^* \geq c_2^* \geq \dots \geq c_N^* \geq 0$ . Then

$$f^*(t) = \begin{cases} c_1^* & 0 < t < m(E_1) \\ c^* & \sum_{k=1}^{j-1} m(E_k) \leq t < \sum_{k=1}^j m(E_k), \quad j = 2, \dots, N \\ 0 & t \geq \sum_{k=1}^N m(E_k). \end{cases}$$

It is very useful to note

$$(1.5) \quad \text{If } f(x) \text{ is a non-negative simple function, then we can write } f(x) = \sum_{j=1}^N f_j(x), \text{ where } f_j(x) \text{ is a non-negative function with exactly one positive value and } f^*(t) = \sum_{j=1}^N f_j^*(t).$$

*Proof.* Suppose  $f(x) = \sum_{j=1}^N c_j \chi_{E_j}(x)$ , where  $E_1, \dots, E_N$  are pairwise disjoint and  $c_1 > \dots > c_N > c_{N+1} = 0$ . Let  $F_j = \bigcup_{k=1}^j E_k$  and  $\alpha_j = c_j - c_{j+1}$ ,  $j = 1, \dots, N$ . Set  $f_j(x) = \alpha_j \chi_{F_j}(x)$  and we are done.

Consideration of the functions  $f(x) = 1-x$  and  $g(x) = x$ ,  $0 \leq x \leq 1$ , shows that we do not always have  $(f+g)^*(t) \leq f^*(t) + g^*(t)$ . However,

$$(1.6) \quad (f+g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2), \quad t_1, t_2 > 0.$$

*Proof.* Since

$$\begin{aligned} & \{x \in M : |f(x) + g(x)| > f^*(t_1) + g^*(t_2)\} \\ & \subset \{x \in M : |f(x)| > f^*(t_1)\} \cup \{x \in M : |g(x)| > g^*(t_2)\} \end{aligned}$$

we have  $\lambda_{f+g}(f^*(t_1) + g^*(t_2)) \leq \lambda_f(f^*(t_1)) + \lambda_g(g^*(t_2)) \leq t_1 + t_2$ . This implies (1.6).

The *Lorentz space*  $L(p, q)$  is the collection of all  $f$  such that  $\|f\|_{pq}^* < \infty$ , where

$$\|f\|_{pq}^* = \begin{cases} \left( \frac{q}{p} \int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t} \right)^{1/q}, & 0 < p < \infty, \quad 0 < q < \infty \\ \sup_{t>0} t^{1/p} f^*(t), & 0 < p \leq \infty, \quad q = \infty. \end{cases}$$

The case  $p = \infty$ ,  $0 < q < \infty$  is not of interest since  $\int_0^\infty [f^*(t)]^q dt/t < \infty$

implies  $f = 0$  a.e.

Since  $f$  and  $f^*$  have the same distribution function we have  $\|f\|_{pp}^* = (\int_M |f(x)|^p dm(x))^{1/p}$ . Hence,  $L(p, p)$  is the familiar  $L^p$  space on  $(M, m)$ .

Since  $f^*$  is essentially the inverse function of  $\lambda_f$ ,

$$(1.7) \quad \sup_{t>0} t^{1/p} f^*(t) = \sup_{y>0} y [\lambda_f(y)]^{1/p}.$$

$L(p, \infty)$  plays an important role in analysis and is often called weak  $L^p$ .  $L^p$  and weak  $L^p$ , as well as all  $L(p, q)$  which have the same first index  $p$ , are related by

$$(1.8) \quad \|f\|_{pq_2}^* \leq \|f\|_{pq_1}^*, \quad 0 < q_1 \leq q_2 \leq \infty.$$

*Proof.* In case  $q_2 = \infty$  we have, since  $f^*(t)$  is non-increasing,

$$t^{1/p} f^*(t) = f^*(t) \left( \frac{q_1}{p} \int_0^t y^{(q_1/p)-1} dy \right)^{1/q_1}$$

$$\leq \left( \frac{q_1}{p} \int_0^t [y^{1/p} f^*(y)]^{q_1} dy/y \right)^{1/q_1}.$$

The result follows immediately.

In case  $q_2 < \infty$  it is sufficient to prove the inequality for simple functions since we can clearly find simple functions  $f_n(t)$  such that  $0 \leq f_n \nearrow f^*$  and apply the monotone convergence theorem.

If  $f$  is a simple function we have  $f^*(t) = c_k$  for  $a_{k-1} \leq t < a_k$ ,  $k = 1, \dots, N$ , where  $c_1 > c_2 > \dots > c_N > 0$  and  $0 = a_0 < a_1 < \dots < a_N$ . Then  $\|f\|_{pq}^* = \left( \sum_{k=1}^N c_k^q (a_k^{q/p} - a_{k-1}^{q/p}) \right)^{1/q}$ . By setting  $d_k = c_k^{q_2}$ ,  $b_k = a_k^{q_2/p}$  and  $\theta = q_1/q_2$  we see that (1.8) is a consequence of

$$(*) \quad \sum_{k=1}^N d_k (b_k - b_{k-1}) \leq \left( \sum_{k=1}^N d_k^\theta (b_k^\theta - b_{k-1}^\theta) \right)^{1/\theta},$$

for  $\infty > d_1 > d_2 > \dots > 0$ ,  $0 = b_0 < b_1 < \dots < \infty$  and  $0 < \theta < 1$ .

The proof of (\*) is by finite induction. (\*) is obviously true (with equality) for  $N = 1$ . Assume (\*) is true for  $N$  and consider

$$\begin{aligned} \varphi(x) &= \left( \sum_{k=1}^N d_k^\theta (b_k^\theta - b_{k-1}^\theta) + x^\theta (b_{N+1} - b_N) \right)^{1/\theta} \\ &- \left( \sum_{k=1}^N d_k (b_k - b_{k-1}) + x (b_{N+1} - b_N) \right), \quad 0 \leq x \leq d_N. \end{aligned}$$

We must show that  $\varphi(d_{N+1}) \geq 0$ . We have  $\varphi(0) \geq 0$  and  $\varphi(d_N) \geq 0$  by our induction hypothesis, since  $\varphi(0) \geq 0$  is exactly (\*) and  $\varphi(d_N)$  is (\*) with  $b_N$  replaced by  $b_{N+1}$ . A simple calculation shows that  $\varphi''(x) \leq 0$  for  $x > 0$ . Hence,  $\varphi(x) \geq 0$  for  $0 \leq x \leq d_N$ . Since  $0 < d_{N+1} < d_N$  this completes the proof.

If  $\chi_E$  is the characteristic function of a set of finite measure then  $\|\chi_E\|_{pq}^* = [m(E)]^{1/p}$  for all  $p, q$ . This implies that inequality (1.8) is best possible. Shorter proofs can be used to obtain  $\|f\|_{pq_2}^* \leq B \|f\|_{pq_1}^*$ ,  $q_1 < q_2$ . For example,

$$\begin{aligned} \left( \frac{q_2}{p} \int_0^\infty [t^{1/p} f^*(t)]^{q_2} \frac{dt}{t} \right)^{q_1/q_2} &\leq \left( \sum_{k=-\infty}^\infty [f^*(2^{k-1})]^{q_2} \left[ \frac{q_2}{p} \int_{2^{k-1}}^{2^k} t^{(q_2/p)-1} dt \right] \right)^{q_1/q_2} \\ &\leq \sum_{k=-\infty}^\infty [f^*(2^{k-1})]^{q_1} 2^{kq_1/p} \end{aligned}$$

$$\begin{aligned} &\leq B \frac{q_1}{p} \sum_{k=-\infty}^{\infty} \int_{2^{k-2}}^{2^{k-1}} [t^{1/p} f^*(f)]^{q_1} \frac{dt}{t} \\ &= B [\|f\|_{pq_1}^*]^{q_1}. \end{aligned}$$

(1.8) clearly implies  $L(p, q_1) \subset L(p, q_2)$ ,  $0 < q_1 \leq q_2 \leq \infty$ . If the measure space  $(M, m)$  contains a countably infinite collection of pairwise disjoint sets of finite non-zero measure it is easy to construct a simple function  $f$  which belongs to  $L(p, q_1)$  but does not belong to  $L(p, q_2)$  for any given  $p$  and  $q_1 < q_2$ .

$L(p, q)$  spaces with different first indices are related only in special cases. For example, if  $m(M) < \infty$ ,  $L(p_2, q_2) \subset L(p_2, \infty) \subset L(p_1, q_1)$  for  $p_1 \leq p_2$ . If  $m(E) \geq 1$  for every measurable set  $E \subset M$  with  $m(E) > 0$ , then  $L(p_1, q_1) \subset L(p_1, \infty) \subset L(p_2, q_2)$  for  $p_1 \leq p_2$ .

(1.8) and the following inequalities are fundamental to the study of  $L(p, q)$  spaces.

A function  $\varphi(x)$  defined on an interval of the real line is said to be *convex* if for every pair of points  $P_1, P_2$  on the curve  $y = \varphi(x)$  the points of the arc  $P_1 P_2$  are below, or on, the chord  $P_1 P_2$ . For example,  $x^r$ ,  $r \geq 1$ , is convex in  $(0, \infty)$  and  $e^x$  is convex in  $(-\infty, \infty)$ . We will need Jensen's integral inequality. (See [32, Vol. I, p. 24].)

**THEOREM. (Jensen):** Suppose  $\varphi(u)$  is convex in an interval  $\alpha \leq u \leq \beta$ ,  $\alpha \leq f(x) \leq \beta$  in  $a \leq x \leq b$  and that  $p(x)$  is non-negative with  $\int_a^b p(x) dx \neq 0$ . Then

$$\varphi\left(\frac{\int_a^b f(x) p(x) dx}{\int_a^b p(x) dx}\right) \leq \frac{\int_a^b \varphi(f(x)) p(x) dx}{\int_a^b p(x) dx},$$

where all integrals in question are assumed to exist and be finite.

*Proof.* Let  $\gamma = \int_a^b f(x) p(x) dx / \int_a^b p(x) dx$ . Then  $\alpha \leq \gamma \leq \beta$ . Let us first suppose that  $\alpha < \gamma < \beta$ , and let  $k$  be the slope of a supporting line of  $\varphi$  through the point  $(\gamma, \varphi(\gamma))$ . Then since  $\varphi$  is convex, we have

$$(*) \quad \varphi(u) - \varphi(\gamma) \geq k(u - \gamma), \quad \alpha \leq u \leq \beta.$$

Replacing  $u$  by  $f(x)$  in (\*), multiplying both sides by  $p(x)$ , and integrating over  $a \leq x \leq b$ , we obtain

$$\int_a^b \varphi(f(x)) p(x) dx - \varphi(\gamma) \int_a^b p(x) dx \geq k \left\{ \int_a^b f(x) p(x) dx - \gamma \int_a^b p(x) dx \right\} = 0,$$

which is the desired inequality. If  $\gamma = \beta$ , then  $f(x) = \beta$  at a.e. point at which  $p(x) > 0$  and the inequality is obvious. Similarly if  $\gamma = \alpha$ .

**THEOREM (Hardy):** *If  $q \geq 1$ ,  $r > 0$  and  $f \geq 0$ , then*

$$\left( \int_0^\infty \left[ \int_0^t f(y) dy \right]^q t^{-r-1} dt \right)^{1/q} \leq \frac{q}{r} \left( \int_0^\infty [y f(y)]^q y^{-r-1} dy \right)^{1/q}$$

and

$$\left( \int_0^\infty \left[ \int_t^\infty f(y) dy \right]^q t^{r-1} dt \right)^{1/q} \leq \frac{q}{r} \left( \int_0^\infty [y f(y)]^q y^{r-1} dy \right)^{1/q}.$$

*Proof.* The technique of the proof is to write  $\left[ \int_0^t f(x) dy \right]^q$  as  $\left[ \int_0^t f(x) y^{-\alpha} y^\alpha dy \right]^q$  and apply Jensen's inequality to the measure  $y^\alpha dy$ . We obtain an inequality of the form

$$\left( \int_0^\infty \left[ \int_0^t f(y) dy \right]^q t^{-r-1} dt \right)^{1/q} \leq C(\alpha) \left( \int_0^\infty [y f(y)]^q y^{-r-1} dy \right)^{1/q}.$$

$\alpha$  is then chosen so that  $C(\alpha)$  is minimal. In this case  $\alpha = (r/q) - 1$  is the best choice.

$$\begin{aligned} & \left( \int_0^\infty \left[ \int_0^t f(y) dy \right]^q t^{-r-1} dt \right)^{1/q} \\ &= \frac{q}{r} \left( \int_0^\infty \left[ \frac{r}{q} t^{-r/q} \int_0^t f(y) y^{-(r/q)+1} y^{(r/q)-1} dy \right]^q t^{-1} dt \right)^{1/q} \end{aligned}$$

which, by Jensen's inequality, is majorized by

$$\left( \frac{q}{r} \right)^{1-1/q} \left( \int_0^\infty \left[ \int_0^t (f(y) y^{-(r/q)+1})^q y^{(r/q)-1} dy \right] t^{-(r/q)-1} dt \right)^{1/q}.$$

After applying Fubini's Theorem we see that the last expression is equal to

$$\frac{q}{r} \left( \int_0^\infty [yf(y)]^q y^{-r-1} dy \right)^{1/q}.$$

The proof of the second inequality is the same except that  $r$  is replaced by  $-r$ .

$$(1.9) \quad \int_E |f(x)g(x)| dm(x) \leq \int_0^{m(E)} f^*(t)g^*(t) dt.$$

*Proof.* We may assume  $f$  and  $g$  are non-negative simple functions. We then write  $f = \sum f_j$  and  $g = \sum g_k$  as in (1.5). (1.9) is clearly true for the functions  $f_j g_k$  and the result follows.

Finally, let us note

$$(1.10) \quad \frac{1}{y} \int_0^y g(t) dt \leq \frac{1}{x} \int_a^x g(t) dt \quad \text{for } 0 < x \leq y,$$

where  $g(t)$  is non-negative and non-increasing on  $t > 0$ .

(1.10) is geometrically obvious.

## Section 2. TOPOLOGICAL PROPERTIES

(1.6) implies that  $f + g \in L(p, q)$  if  $f, g \in L(p, q)$ . Since  $\|\cdot\|_{pq}^*$  is positive homogeneous we see that  $L(p, q)$  is a linear space.  $\|\cdot\|_{pq}^*$  leads to a topology on  $L(p, q)$  such that  $L(p, q)$  is a topological vector space.  $f_n \rightarrow f \in L(p, q)$  in this topology if and only if  $\|f - f_n\|_{pq}^* \rightarrow 0$ . We shall see that this space is metrizable.

For  $p, q$  fixed we define two analogues of  $f^*$ . Choose  $r$  such that  $0 < r \leq 1$ ,  $r \leq q$  and  $r < p$ . Let

$$f^{**}(t) = f^{**}(t, r) = \begin{cases} \sup_{m(E) \geq t} \left( \frac{1}{m(E)} \int_E |f(x)|^r dm(x) \right)^{1/r}, & t \leq m(M) \\ \left( \frac{1}{t} \int_M |f(x)|^r dm(x) \right)^{1/r}, & t > m(M). \end{cases}$$

Consider  $(f^*)^{**}(t)$ . Since any  $g^{**}$  is non-negative and non-increasing we can use (1.9) and (1.10) to see that