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# ON $L(p, q)$ SPACES<sup>1)</sup>

by Richard A. HUNT

## Section 0. INTRODUCTION

$L(p, q)$  spaces are function spaces which are closely related to  $L^p$  spaces. Recall that a complex-valued function  $f$  defined on a measure space  $(M, m)$  belongs to  $L^p$  if  $\|f\|_p = (\int_E |f(x)|^p dm(x))^{1/p} < \infty$ . From the definition of the above integral we have that  $\|f\|_p^p$  is the least upper bound of finite sums  $\sum y_n^p m(\{x \in M : y_n \leq |f(x)| < y_{n+1}\})$  with  $0 = y_1 < y_2 < \dots$ . It follows that  $\|f\|_p$  is completely determined by the distribution function of  $f$ ,  $\lambda_f(y) = m(\{x \in M : |f(x)| > y\})$ ,  $y > 0$ . With each function  $\lambda_f(y)$  we associate the function  $f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\}$ ,  $t > 0$ .  $\lambda_f$  and  $f^*$  are non-negative and non-increasing. If  $\lambda_f(y)$  is continuous and strictly decreasing  $f^*$  is the inverse function of  $\lambda_f$ . The most important property of  $f^*$  is that it has the same distribution function as  $f$ . It follows that

$$(\int_M |f(x)|^p dm(x))^{1/p} = (\int_0^\infty [f^*(t)]^p dt)^{1/p}.$$

Let us write this equation in a more suggestive form as

$$\|f\|_p = \left( \frac{p}{p} \int_0^\infty [t^{1/p} f^*(t)]^p dt / t \right)^{1/p}.$$

The Lorentz space  $L(p, q)$  is the collection of all  $f$  such that  $\|f\|_{pq}^* < \infty$ , where

$$\|f\|_{pq}^* = \begin{cases} \left( \frac{q}{p} \int_0^\infty [t^{1/p} f^*(t)]^q dt / t \right)^{1/q}, & 0 < p < \infty, \quad 0 < q < \infty \\ \sup_{t>0} t^{1/p} f^*(t), & 0 < p \leq \infty, \quad q = \infty. \end{cases}$$

We see that  $\|f\|_p = \|f\|_{pp}^*$ , so  $L^p = L(p, p)$ . We shall see that  $\|f\|_{pq_2}^* \leq \|f\|_{pq_1}^*$ ,  $0 < q_1 \leq q_2 \leq \infty$ . Hence,  $L(p, q_1) \subset L(p, q_2)$  for  $q_1 \leq q_2$ . In particular,  $L(p, q_1) \subset L^p \subset L(p, q_2) \subset L(p, \infty)$  for  $0 < q_1 \leq p \leq q_2 \leq \infty$ .

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In this sense the  $L(p, q)$  spaces give a refinement of  $L^p$  and  $L(p, \infty)$ .  $L(p, \infty)$  plays an important role in analysis and is sometimes called weak  $L^p$ .

The fact that  $L(p, q)$  space theory provides an advantageous setting for  $L^p$  theory is best seen in results concerning the Marcinkiewicz interpolation theorem. (See [32, Vol. II, p. 112].) This theorem states:

*If  $T$  belongs to a certain class (quasi-linear) of operators and  $\|Tf\|_{q_i}^* \leq B_i \|f\|_{p_i}$ , where  $1 \leq p_i \leq q_i \leq \infty$ ,  $i = 0, 1$ ,  $p_0 \neq p_1$  and  $q_0 \neq q_1$ , then  $\|Tf\|_{q_\theta} \leq B_\theta \|f\|_{p_\theta}$ , where  $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ ,  $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$ ,  $0 < \theta < 1$ .*

Let us weaken the hypothesis of this theorem by requiring only that  $\|Tf\|_{q_i}^* \leq B_i \|f\|_{p_i}$ ,  $i = 0, 1$ . We can then obtain the stronger conclusion  $\|Tf\|_{q_\theta p_\theta}^* \leq B_\theta \|f\|_{p_\theta}$  as a consequence of a well known inequality of Hardy. Hence, using elementary Lorentz space theory we weaken the hypothesis, strengthen the conclusion and shorten the proof of the  $L^p$  theorem (see [15]). Also, consideration of the Lorentz space analogue (the weak type theorem of Section 3) shows that the condition  $q_\theta \geq p_\theta$  is necessary in the  $L^p$  result (see [14]).

One of the purposes of this paper is to present, in one place, the basic properties of  $L(p, q)$  spaces and some tools which are useful in their study. The behavior of operators on these spaces is also studied.

For the most part, the presentation presupposes only a knowledge of basic measure theory.

Section 1 of this paper contains a development of elementary properties and inequalities which are useful in the study of Lorentz spaces. In Section 2 we develop topological properties of the spaces.  $\|\cdot\|_{pq}^*$  gives a natural topology for  $L(p, q)$  such that  $L(p, q)$  is a topological vector space. The introduction of  $f^{**}$ , an analogue of  $f^*$ , leads to a metric on  $L(p, q)$ .

$$(f^{**}(t) = \sup_{m(E) \geq t} \left( \frac{1}{m(E)} \int_E |f(x)|^r dm(x) \right)^{1/r}, \quad 0 < r \leq 1.)$$

$L(p, q)$  is seen

to be a Frechet space and in some cases, a Banach space. The continuity of linear, sub-linear and quasi-linear operators is considered in terms of the above mentioned metric. Continuous linear functionals on the  $L(p, q)$  spaces are discussed. Section 3 is devoted to the development of two interpolation theorems for Lorentz spaces. One of these is an analogue of the Marcinkiewicz theorem on the interpolation of operators acting on  $L^p$  spaces. The other is an analogue of the Riesz-Thorin convexity theorem. (See [32, Vol. II, p. 95].) The behavior of operators on  $L(p, q)$  spaces is studied in Section 4.

This is done by considering in detail some classical  $L^p$  operators. Related references are contained in Section 5.

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### Section 1. ELEMENTARY PROPERTIES AND INEQUALITIES

We consider only complex-valued, measurable functions defined on a measure space  $(M, m)$ . The measure  $m$  is assumed to be non-negative and totally  $\sigma$ -finite. We assume the functions  $f$  are finite valued a.e. and, for some  $y > 0$ ,  $m(E_y) < \infty$ , where  $E_y = E_y[f] = \{x \in M : |f(x)| > y\}$ . As usual, we identify functions which are equal a.e.

The *distribution function* of  $f$  is defined by  $\lambda_f(y) = m(E_y)$ ,  $y > 0$ .  $\lambda_f(y)$  is non-negative, non-increasing and continuous from the right. The *non-increasing rearrangement of  $f$  onto  $(0, \infty)$*  is defined by  $f^*(t) = \inf \{y > 0 : \lambda_f(y) \leq t\}$ ,  $t > 0$ . Since  $\lambda_f(y) < \infty$  for some  $y > 0$  and  $f$  is finite valued a.e. we have that  $\lambda_f(y) \rightarrow 0$  as  $y \rightarrow \infty$ . It follows that  $f^*(t)$  is well defined for  $t > 0$ .  $f^*(t)$  is clearly non-negative and non-increasing on  $(0, \infty)$ . If  $\lambda_f(y)$  is continuous and strictly decreasing then  $f^*(t)$  is the inverse function of  $\lambda_f(y)$ .

It follows immediately from the definition of  $f^*(t)$  that

$$(1.1) \quad f^*(\lambda_f(y)) \leq y.$$

Since  $\lambda_f(y)$  is continuous from the right we have

$$(1.2) \quad \lambda_f(f^*(t)) \leq t.$$

Inequalities (1.1) and (1.2) can be used to prove two elementary properties of  $f^*$ .

$$(1.3) \quad f^*(t) \text{ is continuous from the right.}$$

*Proof.* We have  $f^*(t) \geq f^*(t+h)$  for all  $h > 0$ . If there exists  $y$  such that  $f^*(t) > y > f^*(t+h)$  for all  $h > 0$ , then, using (1.2), we have  $\lambda_f(y) \leq \lambda_f(f^*(t+h)) \leq t+h$  for all  $h > 0$ . That is,  $\lambda_f(y) \leq t$ . It follows that  $f^*(t) \leq y$ , which is a contradiction.