

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 12 (1966)  
**Heft:** 4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON  $L(p,q)$  SPACES  
**Autor:** Hunt, Richard A.  
**Kapitel:** Section 0. Introduction  
**DOI:** <https://doi.org/10.5169/seals-40747>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 27.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# ON $L(p, q)$ SPACES <sup>1)</sup>

by Richard A. HUNT

## Section 0. INTRODUCTION

$L(p, q)$  spaces are function spaces which are closely related to  $L^p$  spaces. Recall that a complex-valued function  $f$  defined on a measure space  $(M, m)$  belongs to  $L^p$  if  $\|f\|_p = \left( \int_E |f(x)|^p dm(x) \right)^{1/p} < \infty$ . From the definition

of the above integral we have that  $\|f\|_p^p$  is the least upper bound of finite sums  $\sum y_n^p m(\{x \in M : y_n \leq |f(x)| < y_{n+1}\})$  with  $0 = y_1 < y_2 < \dots$ . It follows that  $\|f\|_p$  is completely determined by the distribution function of  $f$ ,  $\lambda_f(y) = m(\{x \in M : |f(x)| > y\})$ ,  $y > 0$ . With each function  $\lambda_f(y)$  we associate the function  $f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\}$ ,  $t > 0$ .  $\lambda_f$  and  $f^*$  are non-negative and non-increasing. If  $\lambda_f(y)$  is continuous and strictly decreasing  $f^*$  is the inverse function of  $\lambda_f$ . The most important property of  $f^*$  is that it has the same distribution function as  $f$ . It follows that

$$\left( \int_M |f(x)|^p dm(x) \right)^{1/p} = \left( \int_0^\infty [f^*(t)]^p dt \right)^{1/p}.$$

Let us write this equation in a more suggestive form as

$$\|f\|_p = \left( \frac{p}{p-1} \int_0^\infty [t^{1/p} f^*(t)]^p dt/t \right)^{1/p}.$$

The Lorentz space  $L(p, q)$  is the collection of all  $f$  such that  $\|f\|_{pq}^* < \infty$ , where

$$\|f\|_{pq}^* = \begin{cases} \left( \frac{q}{q-1} \int_0^\infty [t^{1/p} f^*(t)]^q dt/t \right)^{1/q}, & 0 < p < \infty, \quad 0 < q < \infty \\ \sup_{t>0} t^{1/p} f^*(t), & 0 < p \leq \infty, \quad q = \infty. \end{cases}$$

We see that  $\|f\|_p = \|f\|_{pp}^*$ , so  $L^p = L(p, p)$ . We shall see that  $\|f\|_{pq_2}^* \leq \|f\|_{pq_1}^*$ ,  $0 < q_1 \leq q_2 \leq \infty$ . Hence,  $L(p, q_1) \subset L(p, q_2)$  for  $q_1 \leq q_2$ . In particular,  $L(p, q_1) \subset L^p \subset L(p, q_2) \subset L(p, \infty)$  for  $0 < q_1 \leq p \leq q_2 \leq \infty$ .

<sup>1)</sup> This work was supported by the U.S. Army Contract DA-31-124-ARO (D)-58 and NSF Grant GP-5628.

In this sense the  $L(p, q)$  spaces give a refinement of  $L^p$  and  $L(p, \infty)$ .  $L(p, \infty)$  plays an important role in analysis and is sometimes called weak  $L^p$ .

The fact that  $L(p, q)$  space theory provides an advantageous setting for  $L^p$  theory is best seen in results concerning the Marcinkiewicz interpolation theorem. (See [32, Vol. II, p. 112].) This theorem states:

*If  $T$  belongs to a certain class (quasi-linear) of operators and  $\|Tf\|_{q_i}^* \leq B_i \|f\|_{p_i}$ , where  $1 \leq p_i \leq q_i \leq \infty, i = 0, 1, p_0 \neq p_1$  and  $q_0 \neq q_1$ , then  $\|Tf\|_{q_\theta} \leq B_\theta \|f\|_{p_\theta}$ , where  $1/p_\theta = (1-\theta)/p_0 + \theta/p_1, 1/q_\theta = (1-\theta)/q_0 + \theta/q_1, 0 < \theta < 1$ .*

Let us weaken the hypothesis of this theorem by requiring only that  $\|Tf\|_{q_i \infty}^* \leq B_i \|f\|_{p_i 1}, i = 0, 1$ . We can then obtain the stronger conclusion  $\|Tf\|_{q_\theta p_\theta}^* \leq B_\theta \|f\|_{p_\theta}$  as a consequence of a well known inequality of Hardy. Hence, using elementary Lorentz space theory we weaken the hypothesis, strengthen the conclusion and shorten the proof of the  $L^p$  theorem (see [15]). Also, consideration of the Lorentz space analogue (the weak type theorem of Section 3) shows that the condition  $q_\theta \geq p_\theta$  is necessary in the  $L^p$  result (see [14]).

One of the purposes of this paper is to present, in one place, the basic properties of  $L(p, q)$  spaces and some tools which are useful in their study. The behavior of operators on these spaces is also studied.

For the most part, the presentation presupposes only a knowledge of basic measure theory.

Section 1 of this paper contains a development of elementary properties and inequalities which are useful in the study of Lorentz spaces. In Section 2 we develop topological properties of the spaces.  $\|\cdot\|_{pq}^*$  gives a natural topology for  $L(p, q)$  such that  $L(p, q)$  is a topological vector space. The introduction of  $f^{**}$ , an analogue of  $f^*$ , leads to a metric on  $L(p, q)$ .

$(f^{**}(t) = \sup_{m(E) \geq t} \left( \frac{1}{m(E)} \int_E |f(x)|^r dm(x) \right)^{1/r}, 0 < r \leq 1.)$   $L(p, q)$  is seen

to be a Frechet space and in some cases, a Banach space. The continuity of linear, sub-linear and quasi-linear operators is considered in terms of the above mentioned metric. Continuous linear functionals on the  $L(p, q)$  spaces are discussed. Section 3 is devoted to the development of two interpolation theorems for Lorentz spaces. One of these is an analogue of the Marcinkiewicz theorem on the interpolation of operators acting on  $L^p$  spaces. The other is an analogue of the Riesz-Thorin convexity theorem. (See [32, Vol. II, p. 95].) The behavior of operators on  $L(p, q)$  spaces is studied in Section 4.

This is done by considering in detail some classical  $L^p$  operators. Related references are contained in Section 5.

I would like to acknowledge that much of this development was contained in my Ph. D. Thesis obtained at Washington University in St. Louis under the direction of Professor Guido Weiss. My thanks go to Professor Weiss and to Professor Mitchell Taibleson for their many helpful suggestions in the preparation of this paper. Professor Antoni Zygmund suggested the present expository form of the  $L(p, q)$  space results.

### Section 1. ELEMENTARY PROPERTIES AND INEQUALITIES

We consider only complex-valued, measurable functions defined on a measure space  $(M, m)$ . The measure  $m$  is assumed to be non-negative and totally  $\sigma$ -finite. We assume the functions  $f$  are finite valued a.e. and, for some  $y > 0$ ,  $m(E_y) < \infty$ , where  $E_y = E_y[f] = \{x \in M : |f(x)| > y\}$ . As usual, we identify functions which are equal a.e.

The *distribution function of  $f$*  is defined by  $\lambda(y) = \lambda_f(y) = m(E_y)$ ,  $y > 0$ .  $\lambda(y)$  is non-negative, non-increasing and continuous from the right. The *non-increasing rearrangement of  $f$  onto  $(0, \infty)$*  is defined by  $f^*(t) = \inf \{y > 0 : \lambda_f(y) \leq t\}$ ,  $t > 0$ . Since  $\lambda_f(y) < \infty$  for some  $y > 0$  and  $f$  is finite valued a.e. we have that  $\lambda_f(y) \rightarrow 0$  as  $y \rightarrow \infty$ . It follows that  $f^*(t)$  is well defined for  $t > 0$ .  $f^*(t)$  is clearly non-negative and non-increasing on  $(0, \infty)$ . If  $\lambda_f(y)$  is continuous and strictly decreasing then  $f^*(t)$  is the inverse function of  $\lambda_f(y)$ .

It follows immediately from the definition of  $f^*(t)$  that

$$(1.1) \quad f^*(\lambda_f(y)) \leq y.$$

Since  $\lambda_f(y)$  is continuous from the right we have

$$(1.2) \quad \lambda_f(f^*(t)) \leq t.$$

Inequalities (1.1) and (1.2) can be used to prove two elementary properties of  $f^*$ .

$$(1.3) \quad f^*(t) \text{ is continuous from the right.}$$

*Proof.* We have  $f^*(t) \geq f^*(t+h)$  for all  $h > 0$ . If there exists  $y$  such that  $f^*(t) > y > f^*(t+h)$  for all  $h > 0$ , then, using (1.2), we have  $\lambda_f(y) \leq \lambda_f(f^*(t+h)) \leq t+h$  for all  $h > 0$ . That is,  $\lambda_f(y) \leq t$ . It follows that  $f^*(t) \leq y$ , which is a contradiction.