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ON $L(p, q)$ SPACES [1])

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Section 0. INTRODUCTION

$L(p, q)$ spaces are function spaces which are closely related to L^p spaces. Recall that a complex-valued function f defined on a measure space (M, m) belongs to L^p if $\|f\|_p = (\int_E |f(x)|^p \, dm(x))^{1/p} < \infty$. From the definition of the above integral we have that $\|f\|_p^p$ is the least upper bound of finite sums $\Sigma y_n^p \, m(\{x \in M : y_n \leqq |f(x)| < y_{n+1}\})$ with $0 = y_1 < y_2 < \ldots$. It follows that $\|f\|_p$ is completely determined by the distribution function of f, $\lambda_f(y) = m(\{x \in M : |f(x)| > y\})$, $y > 0$. With each function $\lambda_f(y)$ we associate the function $f^*(t) = \inf\{y > 0 : \lambda_f(y) \leqq t\}$, $t > 0$. λ_f and f^* are non-negative and non-increasing. If $\lambda_f(y)$ is continuous and strictly decreasing f^* is the inverse function of λ_f. The most important property of f^* is that it has the same distribution function as f. It follows that

$$(\int_M |f(x)|^p \, dm(x))^{1/p} = (\int_0^\infty [f^*(t)]^p \, dt)^{1/p}.$$

Let us write this equation in a more suggestive form as

$$\|f\|_p = (\frac{p}{p} \int_0^\infty [t^{1/p} f^*(t)]^p \, dt/t)^{1/p}.$$

The Lorentz space $L(p, q)$ is the collection of all f such that $\|f\|_{pq}^* < \infty$, where

$$\|f\|_{pq}^* = \begin{cases} (\frac{q}{p} \int_0^\infty [t^{1/p} f^*(t)]^q \, dt/t)^{1/q}, & 0 < p < \infty, \quad 0 < q < \infty \\ \sup_{t>0} t^{1/p} f^*(t), & 0 < p \leqq \infty, \quad q = \infty. \end{cases}$$

We see that $\|f\|_p = \|f\|_{pp}^*$, so $L^p = L(p, p)$. We shall see that $\|f\|_{pq_2}^* \leqq \|f\|_{pq_1}^*$, $0 < q_1 \leqq q_2 \leqq \infty$. Hence, $L(p, q_1) \subset L(p, q_2)$ for $q_1 \leqq q_2$. In particular, $L(p, q_1) \subset L^p \subset L(p, q_2) \subset L(p, \infty)$ for $0 < q_1 \leqq p \leqq q_2 \leqq \infty$.

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In this sense the $L(p, q)$ spaces give a refinement of L^p and $L(p, \infty)$. $L(p, \infty)$ plays an important role in analysis and is sometimes called weak L^p.

The fact that $L(p, q)$ space theory provides an advantageous setting for L^p theory is best seen in results concerning the Marcinkiewicz interpolation theorem. (See [32, Vol. II, p. 112].) This theorem states:

If T belongs to a certain class (quasi-linear) of operators and $\| Tf \|^*_{q_i \infty} \leqq B_i \| f \|_{p_i}$, where $1 \leqq p_i \leqq q_i \leqq \infty, i = 0, 1$, $p_0 \neq p_1$ and $q_0 \neq q_1$, then $\| Tf \|_{q_\theta} \leqq B_\theta \| f \|_{p_\theta}$, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$, $0 < \theta < 1$.

Let us weaken the hypothesis of this theorem by requiring only that $\| Tf \|^*_{q_i \infty} \leqq B_i \| f \|_{p_i 1}$, $i = 0, 1$. We can then obtain the stronger conclusion $\| Tf \|^*_{q_\theta p_\theta} \leqq B_\theta \| f \|_{p_\theta}$ as a consequence of a well known inequality of Hardy. Hence, using elementary Lorentz space theory we weaken the hypothesis, strengthen the conclusion and shorten the proof of the L^p theorem (see [15]). Also, consideration of the Lorentz space analogue (the weak type theorem of Section 3) shows that the condition $q_\theta \geqq p_\theta$ is necessary in the L^p result (see [14]).

One of the purposes of this paper is to present, in one place, the basic properties of $L(p, q)$ spaces and some tools which are useful in their study. The behavior of operators on these spaces is also studied.

For the most part, the presentation presupposes only a knowledge of basic measure theory.

Section 1 of this paper contains a development of elementary properties and inequalities which are useful in the study of Lorentz spaces. In Section 2 we develop topological properties of the spaces. $\| \quad \|^*_{pq}$ gives a natural topology for $L(p, q)$ such that $L(p, q)$ is a topological vector space. The introduction of f^{**}, an analogue of f^*, leads to a metric on $L(p, q)$. ($f^{**}(t) = \sup\limits_{m(E) \geqq t} \left(\frac{1}{m(E)} \int_E | f(x) |^r dm(x)\right)^{1/r}$, $0 < r \leqq 1$.) $L(p, q)$ is seen to be a Frechet space and in some cases, a Banach space. The continuity of linear, sub-linear and quasi-linear operators is considered in terms of the above mentioned metric. Continuous linear functionals on the $L(p, q)$ spaces are discussed. Section 3 is devoted to the development of two interpolation theorems for Lorentz spaces. One of these is an analogue of the Marcinkiewicz theorem on the interpolation of operators acting on L^p spaces. The other is an analogue of the Riesz-Thorin convexity theorem. (See [32, Vol. II, p. 95].) The behavior of operators on $L(p, q)$ spaces is studied in Section 4.

This is done by considering in detail some classical L^p operators. Related references are contained in Section 5.

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Section 1. ELEMENTARY PROPERTIES AND INEQUALITIES

We consider only complex-valued, measurable functions defined on a measure space (M, m). The measure m is assumed to be non-negative and totally σ-finite. We assume the functions f are finite valued a.e. and, for some $y > 0$, $m(E_y) < \infty$, where $E_y = E_y[f] = \{x \in M : |f(x)| > y\}$. As usual, we identity functions which are equal a.e.

The *distribution function of f* is defined by $\lambda(y) = \lambda_f(y) = m(E_y)$, $y > 0$. $\lambda(y)$ is non-negative, non-increasing and continuous from the right. The *non-increasing rearrangement of* f *onto* $(0, \infty)$ is defined by $f^*(t) = \inf\{y > 0 : \lambda_f(y) \leqq t\}$, $t > 0$. Since $\lambda_f(y) < \infty$ for some $y > 0$ and f is finite valued a.e. we have that $\lambda_f(y) \to 0$ as $y \to \infty$. It follows that $f^*(t)$ is well defined for $t > 0$. $f^*(t)$ is clearly non-negative and non-increasing on $(0, \infty)$. If $\lambda_f(y)$ is continuous and strictly decreasing then $f^*(t)$ is the inverse function of $\lambda_f(y)$.

It follows immediately from the definition of $f^*(t)$ that

(1.1) $$f^*(\lambda_f(y)) \leqq y\,.$$

Since $\lambda_f(y)$ is continuous from the right we have

(1.2) $$\lambda_f(f^*(t)) \leqq t\,.$$

Inequalities (1.1) and (1.2) can be used to prove two elementary properties of f^*.

(1.3) *$f^*(t)$ is continuous from the right.*

Proof. We have $f^*(t) \geqq f^*(t+h)$ for all $h > 0$. If there exists y such that $f^*(t) > y > f^*(t+h)$ for all $h > 0$, then, using (1.2), we have $\lambda_f(y) \leqq \lambda_f(f^*(t+h)) \leqq t + h$ for all $h > 0$. That is, $\lambda_f(y) \leqq t$. It follows that $f^*(t) \leqq y$, which is a contradiction.

(1.4) $$\lambda_{f*}(y) = \lambda_f(y) \textit{ for all } y > 0\,.$$

Proof. $\lambda_{f*}(y)$ is the Lebesgue measure of the set of points $t > 0$ for which $f^*(t) > y$. Since f^* is non-increasing we have

(*) $$\lambda_{f*}(y) = \sup\{\, t > 0 : f^*(t) > y\,\}\,.$$

We see from (*) that $f^*(\lambda_f(y)) \leqq y$ implies $\lambda_f(y) \geqq \lambda_{f*}(y)$.

If $t > \lambda_{f*}(y)$, then (*) implies $f^*(t) \leqq y$. Hence, $\lambda_f(y) \leqq \lambda_f(f^*(t)) \leqq t$. It follows that $\lambda_f(y) \leqq \lambda_{f*}(y)$ and (1.4) is proved.

By a simple function we mean a function which can be written in the form

$$f(x) = \sum_{j=1}^{N} c_j\, \chi_{E_j}(x)\,,$$

where $c_1, \ldots, c_N$ are complex numbers, $E_1, \ldots, E_N$ are pairwise disjoint sets of finite measure and $\chi_E(x)$ denotes the characteristic function of the set E. For such a function let $c_1^*, \ldots, c_N^*$ be a rearrangement of the numbers $|c_1|, \ldots, |c_N|$ such that $c_1^* \geqq c_2^* \geqq \ldots \geqq c_N^* \geqq 0$. Then

$$f^*(t) = \begin{cases} c_1^* & 0 < t < m(E_1) \\ c^* & \sum_{k=1}^{j-1} m(E_k) \leqq t < \sum_{k=1}^{j} m(E_k), \quad j = 2, \ldots, N \\ 0 & t \geqq \sum_{k=1}^{N} m(E_k)\,. \end{cases}$$

It is very useful to note

(1.5) *If* $f(x)$ *is a non-negative simple function, then we can write* $f(x) = \sum_{j=1}^{N} f_j(x)$, *where* $f_j(x)$ *is a non-negative function with exactly one positive value and* $f^*(t) = \sum_{j=1}^{N} f_j^*(t)$.

Proof. Suppose $f(x) = \sum_{j=1}^{N} c_j\, \chi_{E_j}(x)$, where $E_1, \ldots, E_N$ are pairwise disjoint and $c_1 > \ldots > c_N > c_{N+1} = 0$. Let $F_j = \bigcup_{k=1}^{j} E_k$ and $\alpha_j = c_j - c_{j+1}$, $j = 1, \ldots, N$. Set $f_j(x) = \alpha_j\, \chi_{F_j}(x)$ and we are done.

Consideration of the functions $f(x) = 1-x$ and $g(x) = x$, $0 \leqq x \leqq 1$, shows that we do not always have $(f+g)^*(t) \leqq f^*(t) + g^*(t)$. However,

$$(f+g)^*(t_1+t_2) \leqq f^*(t_1) + g^*(t_2), \qquad t_1, t_2 > 0. \tag{1.6}$$

Proof. Since

$$\{x \in M : |f(x) + g(x)| > f^*(t_1) + g^*(t_2)\}$$
$$\subset \{x \in M : |f(x)| > f^*(t_1)\} \cup \{x \in M : |g(x)| > g^*(t_2)\}$$

we have $\lambda_{f+g}(f^*(t_1) + g^*(t_2)) \leqq \lambda_f(f^*(t_1)) + \lambda_g(g^*(t_2)) \leqq t_1 + t_2$. This implies (1.6).

The *Lorentz space* $L(p, q)$ is the collection of all f such that $\|f\|_{pq}^* < \infty$, where

$$\|f\|_{pq}^* = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^\infty [t^{1/p} f^*(t)]^q \dfrac{dt}{t}\right)^{1/q}, & 0 < p < \infty, \quad 0 < q < \infty \\ \sup\limits_{t>0} t^{1/p} f^*(t), & 0 < p \leqq \infty, \quad q = \infty. \end{cases}$$

The case $p = \infty$, $0 < q < \infty$ is not of interest since $\int_0^\infty [f^*(t)]^q \, dt/t < \infty$ implies $f = 0$ a.e.

Since f and f^* have the same distribution function we have $\|f\|_{pp}^* = (\int_M |f(x)|^p \, dm(x))^{1/p}$. Hence, L (p, p) is the familiar L^p space on (M, m).

Since f^* is essentially the inverse function of λ_f,

$$\sup_{t>0} t^{1/p} f^*(t) = \sup_{y>0} y [\lambda_f(y)]^{1/p}. \tag{1.7}$$

$L(p, \infty)$ plays an important role in analysis and is often called weak L^p. L^p and weak L^p, as well as all $L(p, q)$ which have the same first index p, are related by

$$\|f\|_{pq_2}^* \leqq \|f\|_{pq_1}^*, \qquad 0 < q_1 \leqq q_2 \leqq \infty. \tag{1.8}$$

Proof. In case $q_2 = \infty$ we have, since $f^*(t)$ is non-increasing,

$$t^{1/p} f^*(t) = f^*(t) \left(\frac{q_1}{p}\int_0^t y^{(q_1/p)-1} \, dy\right)^{1/q_1}$$

$$\leq \Big(\frac{q_1}{p}\int_0^t [y^{1/p}f^*(y)]^{q_1}\,dy/y\Big)^{1/q_1}.$$

The result follows immediately.

In case $q_2 < \infty$ it is sufficient to prove the inequality for simple functions since we can clearly find simple functions $f_n(t)$ such that $0 \leqq f_n \nearrow f^*$ and apply the monotone convergence theorem.

If f is a simple function we have $f^*(t) = c_k$ for $a_{k-1} \leqq t < a_k$, $k = 1, \ldots, N$, where $c_1 > c_2 > \ldots > c_N > 0$ and $0 = a_0 < a_1 < \ldots < a_N$. Then $\|f\|^*_{pq} = \big(\sum_{k=1}^{N} c_k^q\,(a_k^{q/p} - a_{k-1}^{q/p})\big)^{1/q}$. By setting $d_k = c_k^{q_2}$, $b_k = a_k^{q_2/p}$ and $\theta = q_1/q_2$ we see that (1.8) is a consequence of

$$(*)\qquad \sum_{k=1}^{N} d_k(b_k - b_{k-1}) \leqq \Big(\sum_{k=1}^{N} d_k^\theta(b_k^\theta - b_{k-1}^\theta)\Big)^{1/\theta},$$

for $\infty > d_1 > d_2 > \ldots > 0$, $0 = b_0 < b_1 < \ldots < \infty$ and $0 < \theta < 1$.

The proof of (*) is by finite induction. (*) is obviously true (with equality) for $N = 1$. Assume (*) is true for N and consider

$$\varphi(x) = \Big(\sum_{k=1}^{N} d_k^\theta(b_k^\theta - b_{k-1}^\theta) + x^\theta(b_{N+1} - b_N)\Big)^{1/\theta}$$
$$- \Big(\sum_{k=1}^{N} d_k(b_k - b_{k-1}) + x(b_{N+1} - b_N)\Big), \qquad 0 \leqq x \leqq d_N.$$

We must show that $\varphi(d_{N+1}) \geqq 0$. We have $\varphi(0) \geqq 0$ and $\varphi(d_N) \geqq 0$ by our induction hypothesis, since $\varphi(0) \geqq 0$ is exactly (*) and $\varphi(d_N)$ is (*) with b_N replaced by b_{N+1}. A simple calculation shows that $\varphi''(x) \leqq 0$ for $x > 0$. Hence, $\varphi(x) \geqq 0$ for $0 \leqq x \leqq d_N$. Since $0 < d_{N+1} < d_N$ this completes the proof.

If χ_E is the characteristic function of a set of finite measure then $\|\chi_E\|^*_{pq} = [m(E)]^{1/p}$ for all p, q. This implies that inequality (1.8) is best possible. Shorter proofs can be used to obtain $\|f\|^*_{pq_2} \leqq B\,\|f\|^*_{pq_1}$, $q_1 < q_2$. For example,

$$\Big(\frac{q_2}{p}\int_0^\infty [t^{1/p}f^*(t)]^{q_2}\frac{dt}{t}\Big)^{q_1/q_2} \leqq \Big(\sum_{k=-\infty}^{\infty} [f^*(2^{k-1})]^{q_2}\Big[\frac{q_2}{p}\int_{2^{k-1}}^{2^k} t^{(q_2/p)-1}\,dt\Big]\Big)^{q_1/q_2}$$

$$\leqq \sum_{k=-\infty}^{\infty} [f^*(2^{k-1})]^{q_1}\,2^{kq_1/p}$$

$$\leqq B \frac{q_1}{p} \sum_{k=-\infty}^{\infty} \int_{2^{k-2}}^{2^{k-1}} [t^{1/p} f^* (f)]^{q_1} \frac{dt}{t}$$

$$= B \left[\| f \|_{pq_1}^* \right]^{q_1}.$$

(1.8) clearly implies $L(p, q_1) \subset L(p, q_2)$, $0 < q_1 \leqq q_2 \leqq \infty$. If the measure space (M, m) contains a countably infinite collection of pairwise disjoint sets of finite non-zero measure it is easy to construct a simple function f which belongs to $L(p, q_1)$ but does not belong to $L(p, q_2)$ for any given p and $q_1 < q_2$.

$L(p, q)$ spaces with different first indices are related only in special cases. For example, if $m(M) < \infty$, $L(p_2, q_2) \subset L(p_2, \infty) \subset L(p_1, q_1)$ for $p_1 \leqq p_2$. If $m(E) \geqq 1$ for every measurable set $E \subset M$ with $m(E) > 0$, then $L(p_1, q_1) \subset L(p_1, \infty) \subset L(p_2, q_2)$ for $p_1 \leqq p_2$.

(1.8) and the following inequalities are fundamental to the study of $L(p, q)$ spaces.

A function $\varphi(x)$ defined on an interval of the real line is said to be *convex* if for every pair of points P_1, P_2 on the curve $y = \varphi(x)$ the points of the arc $P_1 P_2$ are below, or on, the chord $P_1 P_2$. For example, x^r, $r \geqq 1$, is convex in $(0, \infty)$ and e^x is convex in $(-\infty, \infty)$. We will need Jensen's integral inequality. (See [32, Vol. I, p. 24].)

THEOREM. (Jensen): *Suppose* $\varphi(u)$ *is convex in an interval* $\alpha \leqq u \leqq \beta$, $\alpha \leqq f(x) \leqq \beta$ *in* $a \leqq x \leqq b$ *and that* $p(x)$ *is non-negative with* $\int_a^b p(x)\,dx \neq 0$. *Then*

$$\varphi\left(\frac{\int_a^b f(x)\,p(x)\,dx}{\int_a^b p(x)\,dx} \right) \leqq \frac{\int_a^b \varphi(f(x))\,p(x)\,dx}{\int_a^b p(x)\,dx},$$

where all integrals in question are assumed to exist and be finite.

Proof. Let $\gamma = \int_a^b fp\,dx / \int_a^b p\,dx$. Then $\alpha \leqq \gamma \leqq \beta$. Let us first suppose that $\alpha < \gamma < \beta$, and let k be the slope of a supporting line of φ through the point $(\gamma, \varphi(\gamma))$. Then since φ is convex, we have

$$(*) \qquad \varphi(u) - \varphi(\gamma) \geqq k(u - \gamma), \quad \alpha \leqq u \leqq \beta.$$

Replacing u by $f(x)$ in (*), multiplying both sides by $p(x)$, and integrating over $a \leqq x \leqq b$, we obtain

$$\int_a^b \varphi(f(x))\, p(x)\, dx - \varphi(\gamma) \int_a^b p(x)\, dx \geqq k \left\{ \int_a^b f(x) p(x)\, dx - \gamma \int_a^b p(x)\, dx \right\} = 0,$$

which is the desired inequality. If $\gamma = \beta$, then $f(x) = \beta$ at a.e. point at which $p(x) > 0$ and the inequality is obvious. Similarly if $\gamma = \alpha$.

THEOREM (Hardy): *If* $q \geqq 1$, $r > 0$ *and* $f \geqq 0$, *then*

$$\left(\int_0^\infty \left[\int_0^t f(y)\, dy\right]^q t^{-r-1}\, dt\right)^{1/q} \leqq \frac{q}{r} \left(\int_0^\infty [y f(y)]^q y^{-r-1}\, dy\right)^{1/q}$$

and

$$\left(\int_0^\infty \left[\int_t^\infty f(y)\, dy\right]^q t^{r-1}\, dt\right)^{1/q} \leqq \frac{q}{r} \left(\int_0^\infty [y f(y)]^q y^{r-1}\, dy\right)^{1/q}.$$

Proof. The technique of the proof is to write $[\int_0^t f(x)\, dy]^q$ as $[\int_0^t f(x)\, y^{-\alpha} y^\alpha\, dy]^q$ and apply Jensen's inequality to the measure $y^\alpha\, dy$. We obtain an inequality of the form

$$\left(\int_0^\infty \left[\int_0^t f(y)\, dy\right]^q t^{-r-1}\, dt\right)^{1/q} \leqq C(\alpha) \left(\int_0^\infty [y f(y)]^q y^{-r-1}\, dy\right)^{1/q}.$$

α is then chosen so that $C(\alpha)$ is minimal. In this case $\alpha = (r/q) - 1$ is the best choice.

$$\left(\int_0^\infty \left[\int_0^t f(y)\, dy\right]^q t^{-r-1}\, dt\right)^{1/q}$$

$$= \frac{q}{r} \left(\int_0^\infty \left[\frac{r}{q} t^{-r/q} \int_0^t f(y)\, y^{-(r/q)+1}\, y^{(r/q)-1}\, dy\right]^q t^{-1}\, dt\right)^{1/q}$$

which, by Jensen's inequality, is majorized by

$$\left(\frac{q}{r}\right)^{1-1/q} \left(\int_0^\infty \left[\int_0^t \left(f(y)\, y^{-(r/q)+1}\right)^q y^{(r/q)-1}\, dy\right] t^{-(r/q)-1}\, dt\right)^{1/q}.$$

After applying Fubini's Theorem we see that the last expression is equal to

$$\frac{q}{r}\left(\int_0^\infty [y f(y)]^q y^{-r-1}\, dy\right)^{1/q}.$$

The proof of the second inequality is the same except that r is replaced by $-r$.

$$(1.9)\qquad \int_E |f(x) g(x)|\, dm(x) \leqq \int_0^{m(E)} f^*(t) g^*(t)\, dt .$$

Proof. We may assume f and g are non-negative simple functions. We then write $f = \Sigma f_j$ and $g = \Sigma g_k$ as in (1.5). (1.9) is clearly true for the functions $f_j g_k$ and the result follows.

Finally, let us note

$$(1.10)\qquad \frac{1}{y}\int_0^y g(t)\, dt \leqq \frac{1}{x}\int_a^x g(t)\, dt \quad \text{for } 0 < x \leqq y,$$

where g (t) *is non-negative and non-increasing on* t > 0.

(1.10) is geometrically obvious.

Section 2. Topological properties

(1.6) implies that $f + g \in L(p, q)$ if $f, g \in L(p, q)$. Since $\|\,.\,\|_{pq}^*$ is positive homogeneous we see that $L(p, q)$ is a linear space. $\|\,.\,\|_{pq}^*$ leads to a topology on $L(p, q)$ such that $L(p, q)$ is a topological vector space. $f_n \to f \in L(p, q)$ in this topology if and only if $\|f - f_n\|_{pq}^* \to 0$. We shall see that this space is metrizable.

For p, q fixed we define two analogues of f^*. Choose r such that $0 < r \leqq 1$, $r \leqq q$ and $r < p$. Let

$$f^{**}(t) = f^{**}(t, r) = \begin{cases} \sup\limits_{m(E) \geqq t} \left(\dfrac{1}{m(E)} \displaystyle\int_E |f(x)|^r\, dm(x)\right)^{1/r}, & t \leqq m(M) \\[2ex] \left(\dfrac{1}{t}\displaystyle\int_M |f(x)|^r\, dm(x)\right)^{1/r}, & t > m(M). \end{cases}$$

Consider $(f^*)^{**}(t)$. Since any g^{**} is non-negative and non-increasing we can use (1.9) and (1.10) to see that

$$(f^*)^{**}(t) = \left(\frac{1}{t}\int_0^t [f^*(y)]^r\, dy\right)^{1/r}.$$

f^{**} leads easily to a metric on $L(p, q)$, avoiding technical difficulties which might occur when the measure m is atomic. f^{**} is also useful because for some purposes it is more closely related to f than is f^*. $(f^*)^{**}$ is especially suited for applications of Hardy's inequality.

$$(2.1) \qquad f^* \leqq f^{**}(t) \leqq (f^*)^{**}(t).$$

The first inequality in (2.1) follows from the fact that if $E = \{x \in M : |f(x)| \geqq f^*(t)\}$ then $m(E) \geqq t$. The second inequality follows from (1.9) and (1.10).

Let $\|f\|_{pq} = \|f^{**}\|^*_{pq}$.

f^*, f^{**} and $(f^*)^{**}$ are further related by

$$(2.2) \qquad \|f\|^*_{pq} \leqq \|f\|_{pq} \leqq \|f^*\|_{pq} \leqq (p/(p-r))^{1/r}\, \|f\|^*_{pq}.$$

(2.2) follows immediately from (2.1) and Hardy's inequality.

It is clear that

$$[(f+g)^{**}(t)]^r \leqq [f^{**}(t)]^r + [g^{**}(t)]^r,$$

so that $\rho(f, g) = \|f-g\|^r_{pq}$ is a metric on $L(p, q)$. (2.2) implies the topology of $L(p, q)$ given by $\|.\|^*_{pq}$ is equivalent to the metric topology given by $\|.\|^r_{pq}$.

(2.3) L (p, q) *is complete with respect to the metric* ρ (f, g) $= \|\mathrm{f}-\mathrm{g}\|^r_{pq}$.

Proof. Suppose $\rho(f_m, f_n) \to 0$ as $m, n \to \infty$, where $f_n \in L(p, q)$, $n \geqq 1$. We have $\|f\|^*_{p\infty} \leqq \|f\|^*_{pq} \leqq \|f\|_{pq}$. It follows from (1.7) that the sequence $\{f_n\}$ is fundamental in measure and, hence, the exists a subsequence $\{f_{n_k}\}$ which converges almost uniformly to a function f. (See [7, p. 93].)

Fix L such that $\rho(f_n, f_L) < \varepsilon$ for $n \geqq N(\varepsilon)$. Let $\varphi_k = f_{n_k} - f_L$ and $\varphi = f - f_L$. Then φ_k converges almost uniformly to φ and by Fatou's lemma, $\varphi^{**}(t) \leqq \liminf\limits_{k \to \infty} \varphi_k^{**}(t)$, and $\|\varphi\|^r_{pq} \leqq \liminf \|\varphi_k\|^r_{pq}$. That is, $\rho(f, f_L) < \varepsilon$. Hence, $f \in L(p, q)$ and $\rho(f, f_L) \to 0$ as $L \to \infty$.

(2.4) *Simple functions are dense in* L (p, q), $q \neq \infty$.

Proof. Suppose $f \in L(p, q)$, $p \neq \infty$. We may assume that $f \geqq 0$. We show that given any $\varepsilon, \delta > 0$ there exists a simple function f_n such that $0 \leqq f_n \leqq f$ and $(f-f_n)^*(t) \leqq \varepsilon$ for all $t \geqq \delta$. Note that $f^*(t) \to 0$ as $t \to \infty$. It follows that $m(E_\varepsilon[f]) < \infty$. Hence, we can find a simple function $f_n \geqq 0$

such that $f_n(x) = 0$ for $x \notin E_\varepsilon[f]$ and $0 \leqq f(x) - f_n(x) < \varepsilon$ for all $x \in E_\varepsilon[f]$ except for a set of measure less than δ. Then $m(\{x \in M : |f(x) - f_n(x)| > \varepsilon\}) < \delta$, so $(f-f_n)^*(t) \leqq \varepsilon$ for $t \geqq \delta$. We obtain a sequence of simple functions f_n such that $(f-f_n)^*(t) \to 0$ as $n \to \infty$ and $f_n^*(t) \leqq f^*(t)$, for each $t > 0$. We have $(f-f_n)^*(t) \leqq f^*(t/2) + f^*(t/2) \leqq 2f^*(t/2)$ and Lebesgue's Theorem on dominated convergence implies $\|f-f_n\|_{pq}^* \to 0$ as $n \to \alpha$ for $q \neq \infty$.

It is well known that a linear mapping of one Frechet space into another is continuous if and only if it maps bounded sets into bounded sets. (See [6, p. 54].) Since $\|f\|_{pq}^*$ is positive homogeneous, a linear operator T which maps $L(p, q)$ into $L(p', q')$ is continuous if and only if there exists a positive number c such that $\|Tf\|_{p'q'}^* \leqq c \|f\|_{pq}^*$, where c is independent of $f \in L(p, q)$.

Let us note the following interesting and useful result:

(2.5) *Suppose* T *is a linear operator which maps characteristic functions* χ_E, $m(E) < \infty$, *into a Banach space* B *and* $\|T\chi_E\| \leqq c \|\chi_E\|_{p1}^*$, *where* c *is independent of* χ_E. *Then there exists a unique linear extension of* T *to a continuous mapping of* L (p, 1) *into* B.

Proof. Suppose $f \geqq 0$ is a simple function. According to (1.5) we write $f = \Sigma f_n$, where $f_n = c_n \chi_{F_n}$ and $f^* = \Sigma f_n^*$. Then

$$\|Tf\| = \|T(\Sigma f_n)\| \leqq \Sigma \|Tf_n\| \leqq c\, \Sigma \|f_n\|_{p1}^* = c \|f\|_{p1}^* .$$

Then $\|Tf\| \leqq c' \|f\|_{p1}^*$ for any complex-valued simple function f. Since the simple functions are dense in $L(p, 1)$ we can then extend T uniquely to a bounded operator of $L(p, 1)$ into B.

It is of interest to know which of the $L(p, q)$ spaces may be considered to be Banach spaces.

(2.6) L (1, 1) *and* L (p, q), $1 < p \leqq \infty$, $1 \leqq q \leqq \infty$, *are Banach spaces for any measure space* (M, m). *For any other* p, q *there are measure spaces such that* L (p, q) *cannot be considered to be a Banach space in such a way that the topology corresponding to the norm is comparable to the metric topology.*

Proof. It is immediate that $\|.\|_{pq}^r$, with $r = 1$ is a norm. This norm is applicable to the spaces $L(p, q)$, $1 < p \leqq \infty, 1 \leqq q \leqq \infty$. $\|.\|_{11}^*$ is already a norm for $L(1, 1)$. Also, note that $\|.\|_{11}^* = \|.\|_{1\infty}^{\times}$.

Let $M = (0\ \infty)$ and m be real Lebesgue measure. Since $L(p, q)$ is a Frechet space, (2.6) follows from the fact that none of the remaining spaces contain a bounded convex open set. (See [16].) This is easily seen from the following constructions:

In case $0 < q < 1$ let

$$f_k(t) = \begin{cases} 2^k & 0 < t < 2^{-kp} \\ 0 & t \geqq 2^{-kp}, \quad k \geqq 1 . \end{cases}$$

Then $\|f_k\|^*_{pq} = 1$, but $\|\frac{1}{n} \sum_{k=1}^{n} f_k\|^*_{pq} \to \infty$ as $n \to \infty$.

In case $0 < p < 1$ choose ε such that $1 < \varepsilon < \frac{1}{p}$ and let

$$f_k(t) = \begin{cases} k^{-(1/p)+\varepsilon} & k < t \leqq k+1 \\ 0 & \text{otherwise}, \ k \geqq 1 . \end{cases}$$

Then $\|f_k\|^*_{pq} \leqq 1$, but $\|\frac{1}{n} \sum_{k=1}^{n} f_k\|^*_{pq} \to \infty$ as $n \to \infty$.

In the cases where $p = 1$ divide $(0, \infty)$ into pairs of intervals

$$I_{k0}, I_{k1}, \text{ where } I_{00} = (0, 1], \quad I_{01} = (1,2]$$

$$I_{k0} = (2^{k-1}(3) + (k-1)\,2^{k-1}, \quad 2^k(3) + (k-1)\,2^{k-1}] \quad \text{and}$$

$$I_{k1} = (2^{k-1}(3) + (k-1)\,2^{k-1}, \quad 2^k(3) + (k-1)\,2^k], \quad k \geqq 1. \quad \text{Let}$$

$$J_{k0} = (\bigcup_{i=0}^{k} I_{i0}) \cup (\bigcup_{i=0}^{k-1} I_{i1}) \quad \text{and } J_{k1} = I_{k1}. \text{ Note that } |J_{k0}| = |J_{k1}| .$$

If f_{k0} is zero on J_{k1} define f_{k1} by

$$f_{k1}(t) = \begin{cases} 0 & t \in J_{k0} \\ f_{k0}(t - |J_{k0}|) & t \in J_{k1} \\ f_{k0} & \text{otherwise} . \end{cases}$$

In case $q = \infty$ let

$$f_{00} = \begin{cases} 2^{k-1} & t \in I_{k0} \\ 0 & t \in I_{k1}, \quad k \geqq 0 . \end{cases}$$

In case $1 < q < \infty$ choose $\frac{1}{q} < \alpha < 1$ and let

$$f_{00}(t) = \begin{cases} 2^{-k} \cdot k^{-\alpha} & t \in I_{k0} \\ 0 & t \in I_{k1}, \quad k \geqq 0 . \end{cases}$$

The result is then seen by considering sums of the form $f_{k+1,0} = (f_{k0} + f_{k1})/2$, $k = 0, 1, \ldots$.

For the remainder of this section let us consider continuous linear functionals l on $L(p, q)$. We have $|\, l(f) \,| \leqq B \,\|f\|_{pq}^{*}$ for all $f \in L(p, q)$.

Consider $L(p, 1)$, $1 \leqq p < \infty$. Define $\mu(E) = l(\chi_E)$. $\mu(E)$ is a measure and $|\, \mu(E) \,| \leqq B \,\|\chi_E\|_{p1}^{*} = B\,[m(E)]^{1/p}$. Hence, μ is absolutely continuous with respect to m. The Radon-Nikadyn Theorem (see [7, p. 138]) then gives a function $g(x)$ such that $\mu(E) = l(\chi_E) = \int_M \chi_E(x)\, g(x)\, dm(x)$. This leads to $l(f) = \int_M f(x)\, g(x)\, dm(x)$ and hence $\left| \int_M f(x)\, g(x)\, dm(x) \right| \leqq B \,\|f\|_{p1}^{*}$ for all $f \in L(p, 1)$. Setting $f(x) = [\exp(-i \arg g(x))] \cdot \chi_E(x)$ we obtain $\int_E |\, g(x) \,|\, dm(x) \leqq B\,[m(E)]^{1/p}$. Therefore,

$$\frac{1}{m(E)} \int_E |\, g(x) \,|\, dm(x) \leqq B\,[m(E)]^{-1/p'} \leqq B t^{-1/p'}$$

for $t \leqq m(E)$, where $1/p + 1/p' = 1$. It follows that $g^{**}(t) \leqq B t^{-1/p'}$, so $g \in L(p', \infty)$ and $\|g\|_{p'\infty}^{*} \leqq B$. (It is interesting to note how naturally g^{**} appeared in the above discussion.) Conversely, for any $g \in L(p', \infty)$, $l(f) = \int_M g(x)\, f(x)\, dm(x)$ defines a continuous linear functional on $L(p, 1)$. Since

$$\left| \int_M g(x) f(x)\, dm(x) \right| \leqq \int_0^\infty g^*(t) f^*(t)\, dt \leqq \|g\|_{p'\infty}^{*} \int_0^\infty t^{-1/p'} f^*(t)\, dt$$

$$= p \,\|g\|_{p'\infty}^{*} \,\|f\|_{p1}^{*} .$$

This proves that $L(p', \infty)$ is the conjugate space of $L(p, 1)$. For the same reasons that L^1 is not the conjugate space of L^∞ we cannot expect $L(p, 1)$ to be the conjugate space of $L(p', \infty)$.

Suppose now that l is a continuous linear functional on $L(p, q)$, $1<p<\infty$, $1<q<\infty$. Since $\|f\|^*_{pq} \leqq \|f\|^*_{p1}$, l is also a continuous linear function on $L(p, 1)$. Hence, there exists a function $g \in L(p', \infty)$ such that

$$(*) \qquad l(f) = \int_M f(x)\, g(x)\, dm(x) \quad \text{for all} \quad f \in L(p, 1).$$

In particular, (*) holds for all simple functions. Using (*) and $|l(f)| \leqq B\|f\|^*_{pq}$ it can be shown that $g \in L(p', q')$, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, and (*) holds for all $f \in L(p, q)$. Conversely, for any $g \in L(p', q')$, (*) defines a continuous linear functional on $L(p, q)$. We have obtained

(2.7) *The conjugate space of* L (p, 1) *is* L (p′, ∞), *where* $\frac{1}{p}+\frac{1}{p'}=1$.
The conjugate space of L (p, q), $1<p<\infty$, $1<q<\infty$, *is* L (p′, q′), *where* $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, *and hence, these spaces are reflexive.*

According to (2.5) any continuous linear functional on $L(p, q)$, $1 \leqq p < \infty$, $q<1$, can be extended to a continuous linear functional on $L(p, 1)$.

Suppose l is a continuous linear functional on $L(p, 1)$, $0<p<1$. Let us assume that $m(M)<\infty$. Since (M, m) is σ-finite this will result in no loss of generality in the following argument. We have

$$|l(\chi_E)| \leqq B\,\|\chi_E\|^*_{p1} = B[m(E)]^{1/p} \leqq B[m(M)]^{(1/p)-1}\,\|\chi_E\|^*_{11}.$$

Hence, by (2.5), l can be extended to a continuous linear functional on $L(1, 1) = L^1$. Then there exists a function $g \in L^\infty$ such that $l(f) = \int_M f(x)\, g(x)\, dm(x)$ for all $f \in L^1$. Also, $\left|\int_M g(x) f(x)\, dm(x)\right| \leqq B\|f\|^*_{p1}$. As before, we have

$$\frac{1}{m(E)}\int_E |g(x)|\, dm(x) \leqq B[m(E)]^{(1/p)-1}.$$

In case (M, m) is non-atomic this implies that $g(x) = 0$ a.e. and, hence $l \equiv 0$ on $L(p, 1)$. It follows that *the trivial functional* $l \equiv 0$ *is the only continuous linear functional on the spaces* $L(p, q)$, $0<p<1$, $0<q<\infty$.

If l is a continuous linear functional on $L(1, q)$, $1<q$, then l is a continuous linear functional on $L(1, 1) = L^1$, so there exists a function $g \in L^\infty$

such that $l(f) = \int_M f(x)\, g(x)\, dm(x)$ for all $f \in L(1,1)$ and $\left| \int_M f(x)\, dm(x) \right| \leqq B \, \|f\|^*_{1q}$. If (M, m) is non-atomic we can use this to show that $g = 0$ a.e. and, hence *the trivial functional* $l \equiv 0$ *is the only continuous linear functional on* L (1, q), $1 < q < \infty$.

Section 3. INTERPOLATION THEOREMS

Suppose T is an operator which maps $L(p_i, q_i)$ boundedly into $L(p'_i, q'_i)$, $i = 0, 1$. An interpolation theorem for $L(p, q)$ spaces can then be described as a method which leads to inequalities of the form $\| Tf \|^*_{p'q'} \leqq B \, \| f \|^*_{pq}$, B independent of $f \in L(p, q)$. The intermediate spaces $L(p, q)$ and $L(p', q')$ and the corresponding constant B are determined by the method of interpolation.

Interpolation theorems can generally be classified as either weak type or strong type. The two types of theorems are easily characterized. The weak type theorems are proved by real variable methods which utilize only minimal hypotheses. Since the weak hypotheses are characteristic of the real method of proof, the conclusions are limited. In the case of Lorentz spaces the essential part of the weak type hypothesis is that the range spaces of the given end point conditions are weak L^p spaces. We can then conclude only that an intermediate space $L(p, q)$ is mapped boundedly into an appropriate space $L(p', q')$, where $q' \geqq q$. In order to utilize a stronger hypothesis to arrive at a stronger conclusion, we must go to the complex methods of proof which are characteristic of the strong type theorems. The two methods also differ in the intermediate spaces obtained and in the behavior of the corresponding constants B. In general, we obtain more intermediate spaces by the weak type methods. However, the constants corresponding to the weak type methods are, in some sense, not as satisfactory. This is seen in the prototypes of the weak and strong type theorems, the interpolation theorem of Marcinkiewicz and the Riesz-Thorin convexity theorem.

An operator T mapping functions on a measure space into functions on another measure space is called *quasi-linear* if $T(f+g)$ is defined whenever Tf and Tg are defined and if $|T(f+g)| \leqq K(|Tf| + |Tg|)$ a.e., where K is independent of f and g. An argument similar to that which led to (1.6) gives

$$(T(f+g))^*(t) \leqq K((Tf)^*(t/2) + (Tg)^*(t/2)). \tag{3.1}$$

Our weak type theorem is a consequence of Hardy's inequality.

WEAK TYPE THEOREM: *If* T *is quasi-linear and*

H) $\| Tf \|^*_{p_i' q_i'} \leqq B_i \| f \|^*_{p_i q_i}, \quad i = 0, 1, \quad p_o < p_1, \quad p_o' \neq p_1',$

then

C) $$\| Tf \|^*_{p_\theta' s} \leqq B_\theta \| f \|^*_{p_\theta q},$$

where $q \leqq s$ *and, for* $0 < \theta < 1$, $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$, $1/p_\theta' = (1-\theta)/p_0' + \theta/p_1'$. *If* $r = \min(q, q_0, q_1)$, *then* $B_\theta = O([\theta(1-\theta]^{-1/r})$.

Proof. Let $p = p_\theta$ and $p' = p_\theta'$. Since $q \leqq s$ implies $\| Tf \|^*_{p's} \leqq \| Tf \|^*_{p'q'}$, it is sufficient to prove C) with $s = q$. Similarly, we assume that $q_0' = q_1' = \infty$ and that $q_o, q_1 \leqq q$, except when $p_1 = q_1 = \infty$. Put

$$f^t(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| > f^*(t^\gamma) \\ 0 & \text{otherwise} \end{cases}$$

and $f_t(x) = f(x) - f(x)$, where $\gamma = \dfrac{1/p_0' - 1/p'}{1/p_0 - 1/p} = \dfrac{1/p' - 1/p_1'}{1/p - 1/p_1}$.

It follows from the definitions that

$$(3.2) \qquad \begin{cases} f^{t*}(y) \leqq \begin{cases} f^*(y) & 0 < y < t^\gamma \\ 0 & y \geqq t^\gamma \end{cases} \\ \\ f_t^*(y) \leqq \begin{cases} f^*(t) & 0 < y < t^\gamma \\ f^*(y) & y \geqq t^\gamma . \end{cases} \end{cases} \quad \textit{and}$$

Case 1: $p_1 < \infty$, $q < \infty$.

We use (3.1), a change of variables and Minkowski's inequality (or, if $q < 1$, an obvious substitute which introduces an additional factor of $2^{1/q}$) to obtain

$$\| Tf \|^*_{p'q} \leqq K 2^{1/p'} (q/p')^{1/q} \left\{ \left(\int_0^\infty [t^{1/p'} (Tf^t)^*(t)]^q \frac{dt}{t} \right)^{1/q} + \left(\int_0^\infty [t^{1/p'} (Tf_t)^*(t)]^q \frac{dt}{t} \right)^{1/q} \right\}.$$

By H), this sum is majorized by

$$K2^{1/p'}(q/p')^{1/q}\left\{\left(\int_0^\infty [B_0\, t^{1/p'-1/p_0'}\, \| f^t \|^*_{p_0 q_0}]^q \frac{dt}{t}\right)^{1/q} + \left(\int_0^\infty [B_1\, t^{1/p'-1/p_1'}\, \| f_t \|^*_{p_1 q_1}]^q \frac{dt}{t}\right)^{1/q}\right\}.$$

By using (3.2) and Minkowski's inequality again, we dominate this by

$$\{K2^{1/p'}(q/p')^{1/q}\}\cdot$$

$$\left\{B_0\left(\int_0^\infty t^{-q(1/p_0'-1/p')}\left[\frac{q_0}{p_0}\int_0^{t^\gamma}[f^*(y)]^{q_0}\, y^{(q_0/p_0)-1}\,dy\right]^{q/q_0}\frac{dt}{t}\right)^{1/q}\right.$$

$$+ B_1\left(\int_0^\infty t^{q(1/p'-1/p_1')}\left[\frac{q_1}{p_1}\int_{t^\gamma}^\infty[f^*(y)]^{q_1} y^{(q_1/p_1)-1}\,dy\right]^{q/q_1}\frac{dt}{t}\right)^{1/q}$$

$$\left.+ B_1\left(\int_0^\infty t^{q(1/p'-1/p_1)}\left[\frac{q_1}{p_1}\int_0^{t^\gamma}[f^*(t^\gamma)]^{q_1}\, y^{(q_1/p_1)-1}\,dy\right]^{q/q_1}\frac{dt}{t}\right)^{1/q}\right\}.$$

Again changing variables and then using Hardy's inequality, we majorize the last sum by

$$K2^{1/p'}\,|\gamma|^{-1/q}(p/p')^{1/q}\left\{\frac{B_0}{(1-(p_0/p))^{1/q_1}} + \frac{B_1}{((p_1/p)-1)^{1/q_1}} + B_1\right\}\|f\|^*_{pq}\,.$$

(Note that in order to apply Hardy's inequality it was necessary to weaken the hypothesis so that $q/q_i \geqq 1, i = 0, 1$.)

Case 2: $p_1 < \infty$, $q = \infty$.

Following the proof of case 1, we obtain

$$t^{1/p'}(Tf)^*(t) \leqq K\cdot 2^{1/p'}\left\{B_0\, t^{1/p'-1/p_0'}\left(\frac{q_0}{p_0}\int_0^{t^\gamma}[f^*(y)]^{q_0}\, y^{(q_0/p_0)-1}\,dy\right)^{1/q_0}\right.$$

$$+ B_1\, t^{1/p'-1/p'_1}\left(\frac{q_1}{p_1}\int_{t^\gamma}^\infty[f^*(y)]^{q_1}\, y^{(q_1/p_1)-1}\,dy\right)^{1/q_1}$$

$$\left.+ B_1\, t^{1/p'-1/p_1'}\left(\frac{q_1}{p_1}\int_0^{t^\gamma}[f^*(t)]^{q_1}\, y^{(q_1/p_1)-1}\,dy\right)^{1/q_1}\right\}.$$

Then, after use of the estimate $y^{1/p}f^*(y) \leqq \|f\|^*_{p\infty}$, the proof of case 2 is clear.

The remaining cases are

Case 3: $p_1 = q_1 = \infty$, $q < \infty$,
and
Case 4: $p_1 = q_1 = q = \infty$.

The proofs of these cases follows the proofs of cases 1 and 2, except we now use the estimate $\|f_t\|^*_{\infty\infty} \leqq f^*(t^\gamma)$.

An operator T which maps functions on a measure space into functions on another measure space is called *sublinear* if whenever Tf and Tg are defined and c is a constant, then $T(f+g)$ and $T(cf)$ are defined with

$$(3.3)\qquad \begin{cases} |T(f+g)| \leqq |Tf| + |Tg| \quad \text{and} \\ |T(cf)| = |c| \cdot |Tf| . \end{cases}$$

It follows that

$$(3.4)\qquad ||Tf| - |Tg|| \leqq |T(f-g)| .$$

Our analogue of the Riesz-Thorin convexity theorem depends on harmonic majorization of subharmonic functions.

STRONG TYPE THEOREM: *Suppose* T *is a sublinear operator and*

$$\| Tf \|_{p_i' q_i'} \leqq B_i \|f\|^*_{p_i q_i}, \qquad i = 0, 1 .$$

Then $\| \mathrm{Tf} \|^*_{p_\theta' q_\theta'} \leqq \mathrm{B} \mathrm{B}_0^{1-\theta} \mathrm{B}_1^\theta \| \mathrm{f} \|^*_{p_\theta q_\theta}$, *where* $1/\mathrm{p}_\theta = (1-\theta)/\mathrm{p}_0 + \theta/\mathrm{p}_1$, $1/\mathrm{p}_\theta' = (1-\theta)/\mathrm{p}_0' + \theta/\mathrm{p}_1'$, $1/\mathrm{q}_\theta = (1-\theta)/q_0 + \theta/\mathrm{q}_1$ and $1/\mathrm{q}_\theta' = (1-\theta)/\mathrm{q}_\theta' + \theta/\mathrm{q}_1'$, $0 < \theta < 1$.

Proof. Let $p_\theta = p$, $q_\theta = q$, $p_\theta' = p'$, and $q_\theta' = q'$.

Suppose that f is a simple function. Then f can be written in the form

$$f(x) = e^{i \arg f(x)} (G_0(x))^{1-\theta} (G_1(x))^\theta ,$$

where G_i is a non-negative simple function such that

$$(3.5)\qquad \| G_i \|^*_{p_i q_i} \leqq B(\|f\|^*_{pq})^{q/q_i} , \qquad i = 0, 1 .$$

To see this, consider $(f^*)^{**}$, $0 < r < \min(p_0, p_1, q_0, q_1, q_0', q_1')$. We have $(f^*)^{**}(t) = (h_0(t))^{1-\theta} (h_1(t))^\theta$, where

$$h_i(t) = [(f^*)^{**}(t)]^{q/q_i} t^{(q/q_i)(q/p - q_i/p_i)} , \qquad i = 0, 1 .$$

If $Sh(u) = (\int_u^\infty [h(t)]^r \frac{dt}{t})^{1/r}$, it is not difficult to see that $f^*(u) \leqq S((f^*)^{**})(u)$, and hence, by Holder's inequality, that $f^*(u) \leqq (Sh_0(u))^{1-\theta} (Sh_1(u))^\theta$. G_i is obtained by choosing values smaller than Sh_i. (3.5) follows from Hardy's inequality.

Let $F(x, z) = e^{i \arg f(x)} [G_0(x)]^{1-z} [G_1(x)]^z$, z complex, $0 \leqq Re\, z \leqq 1$.

Since G_i is simple and non-negative, $i = 0, 1$, $TF(\ , z)$ is defined for z fixed. By considering first a countable dense set $\{z_k\}_{k \geqq 1}$ and then.extending by continuity to all z, we may assume that except for a set of measure zero $|TF(y, z)|$ is defined for all z and y fixed and (3.3) and (.3.4) are true pointwise in y. Fix such a point y. (3.3) and (3.4) imply that $|TF(y, z)|$ is a bounded and continuous function of z, $0 \leqq Re\, z \leqq 1$. We need that $\log |TF(y, z)|$ is subharmonic in $0 < Re\, z < 1$. This follows from the fact that $|TF(y, z)|\, e^{h(z)}$ is subharmonic for every harmonic function $h(z)$. That is, let $H(z)$ be analytic with real part $h(z)$. For a fixed point z let z_{km}, $k = 1, \ldots, m$, be points which are evenly distributed over the circle with radius r and center z, $m \geqq 1$. If $D(x, m, z)$ is defined by

$$e^{H(z)} F(x, z) = \frac{1}{m} \sum_{k=1}^{m} F(x, z_{km})\, e^{H(z_{km})} + D(x, m, z),$$

then

$$e^{h(z)} |TF(y, z)| \leqq \frac{1}{m} \sum_{k=1}^{m} e^{h(z_{km})} |TF(x, z_{km})| + |TD(y, m, z)|.$$

Since $D(x, m, z)$ is of the form $\sum_{j=1}^{N} (\varphi_j(z) - \frac{1}{n} \sum_{k=1}^{m} \varphi_j(z_{km}))\, \chi_{E_j}(x)$, with φ_j analytic, we may again assume that (3.3) holds pointwise in y, so $|TD(y, m, z)| \to 0$ as $m \to \infty$. Then

$$e^{h(z)} |TF(y, z)| \leqq \frac{1}{2\pi} \int_0^{2\pi} e^{h(z + re^{i\theta})} |TF(y, z + re^{i\theta})|\, d\theta,$$

so $\log |TF(v, z)|$ is subharmonic.

The preceding paragraph implies that $\log |TF(y, z)|$ is majorized in $0 < Re\, z < 1$ by the Poisson integral of its boundary values. In particular,

$$\log |TF(y,\theta)| \leqq \int_{-\infty}^{\infty} P_0(\theta,t) \log |TF(y, it)|\, dt + \int_{-\infty}^{\infty} P_1(\theta,t) \log |TF(y,$$

$1+it) \mid dt$, where $P_0(\theta, t)$ and $P_1(\theta\ t)$ and $P_1(\theta, t)$ are positive, $\int_{-\infty}^{\infty} P_0(\theta, t)\, dt = 1-\theta$ and $\int_{-\infty}^{\infty} P_1(\theta, t)\, dt = \theta$. We then obtain

$$| TF(y, \theta) |^r \leqq \left\{ \exp\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta, t) \log | TF(y, it) |^r dt\right)\right\}^{1-\theta}$$

$$\cdot \left\{ \exp\left(\frac{1}{\theta}\int_{-\infty}^{\infty} P_1(\theta, t) \log | TF(y, 1+it) |^r dt\right)\right\}^{\theta}.$$

Noting that $TF(y, \theta) = Tf(y)$, we use Jensen's inequality to obtain

$$| Tf(y) | \leqq H_0(y)^{1-\theta} \cdot H_1(y)^{\theta},$$

where

$$H_0(y) = \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta, t) \,| TF(y, it) |^r dt\right)^{1/r}$$

and

$$H_1(y) = \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_1(\theta, t) \,| TF(y, 1+it) /^r dt\right)^{1/r}.$$

Holder's inequality implies $(Tf)^{**}(y) \leqq \{H_0^{**}(y)\}^{1-\theta}\{H_1^{**}(y)\}^{\theta}$ and then $\| Tf \|^*_{p'q'} \leqq B \,\| H_0 \|^{1-\theta}_{p_0'q_0'} \,\| H_1 \|^{\theta}_{p_1'q_1'}$.

By Fubini's theorem, $H_0^{**}(y) \leqq \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P(\theta, t)\, [TF^{**}(y, it)]^r dt\right)^{1/r}$.

Hence

$$\| H_0 \|_{p_0'q_0'} \leqq \left(\frac{q_0'}{p_0'}\int_0^{\infty}\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta,t)\left[TF^{**}(y,it)\right]^r dt\right)^{q_0/r} y^{(q_0'/p_0')-1} dy\right)^{1/q}.$$

By Jensen's inequality the right hand term is dominated by

$$\left(\frac{q_0'}{p_0'}\int_0^{\infty}\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta, t)\left[TF^{**}(y, it)\right]^{q_0'} dt\right) y^{(q_0'/p_0')-1} dy\right)^{1/q_0'}.$$

Thus, using Fubini's theorem, our hypothesis and (3.5), we have

$$\| H_0 \|_{p_0'q_0'} \leqq \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta, t) \,\| TF(\cdot, it) \|^{q_0'}_{p_0'q_0'} dt\right)^{1/q_0'}$$

$$\leqq BB_0\Big(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta, t) \, \| F(\cdot, it) \|_{p_0q_0}^{q_0'} dt\Big)^{1/q_0'}$$

$$= BB_0 \, \| G_0 \|_{p_0q_0}^{*} \leqq BB_0 \, [\| f \|_{pq}^{*}]^{q/q_0}.$$

Similarly, $\| H_1 \|_{p_1'q_1'} \leqq BB_1 \, [\| f \|_{pq}^{*}]^{q/q_1}$.

We now have

$$\| Tf \|_{p'q'}^{*} \leqq BB_0^{1-\theta} B_1^{\theta} \, \| f \|_{pq}^{*} \tag{3.6}$$

where f is any simple function.

For any $f \in L(p, q)$ we find a sequence of simple functions f_n such that $\| f_n \|_{pq} \to \| f \|_{pq}$ and $| Tf_n | \to | Tf |$ a.e. Then, using Fatou's lemma, we have $(Tf)^{**}(t) \leqq \liminf (Tf_n)^{**}(t)$ and $\| Tf \|_{p'q'} \leqq \liminf \| TF_n \|_{p'q'}$. (3.6) then implies that $\| Tf \|_{p'q'}^{*} \leqq BB_0^{1-\theta} B_1^{\theta} \| f \|_{pq}^{*}$.

Note that in case $p_0 = q_0$, $p_1 = q_1$, $p_0' = q_0'$ and $p_1' = q_1'$ the proof is simpler and the constant B may be omitted from the conclusion so the constant $B_0^{1-\theta} B_1^{\theta}$ of the Riesz-Thorin convexity theorem is retained.

Section 4. Applications

Many classical operators are known to map L^p boundedly into $L^{p'}$, where the points $(1/p, 1/p')$ form a non-degenerate line segment and $p \leqq p'$. Operators of this type are, for example, the Fourier transform [32, Vol. I, p. 254], the Hilbert transform [23], the Hardy-Littlewood maximal function operator [32, Vol. I, p. 32], singular integral operators [4] and fractional integral operators [12] and [28]. We see from the weak type interpolation theorem that operators of this type map $L(p, q)$ boundedly into $L(p', q')$, $0 < q \leqq \infty$. Hence, we know the behavior of the operators acting on some additional spaces. If $p = p'$, this is the only extension of the L^p results. However, if $p < p'$, the L^p result is improved, since we see that L^p is mapped boundedly into $L(p', p)$, a space which is continuously contained in $L^{p'}$.

The germ of the weak type theorem can be seen in a theorem of Hardy and Littlewood on the rearrangement of Fourier coefficients. (See [32, Vol. II, p. 130].) Let us develop an $L(p, q)$ version of this result for the Fourier integral transform.

We write the Fourier transform of a function $f \in L^1(E_n)$ as

$$f^\wedge(x) = \int_{E_n} f(y)\, e^{-2\pi i x \cdot y}\, dy \,. \tag{4.1}$$

Recall that if s is a simple function then (4.1) defines $s^\wedge$ and we have $\| s^\wedge \|_2 = \| s \|_2$. The Fourier transform can then be uniquely extended such that $\| f^\wedge \|_2 = \| f \|_2$ for all $f \in L^2(E_n)$. Suppose $f \in L(p, q)$, $1 < p < 2$, $1 \leqq q \leqq \infty$. Then $f \in L^1 + L^2$ and, hence, $f^\wedge$ is defined. We have (4.1) and

$$f(x) = \int_{E_n} f^\wedge(y)\, e^{2\pi i x \cdot y}\, dy = (f^\wedge)^\vee(x)\,, \tag{4.2}$$

in the sense that

$$\int_{|y| \leqq R} f(y)\, e^{-2\pi i x \cdot y}\, dy \to f^\wedge(x) \quad \text{and} \quad \int_{|x| \leqq R} f^\wedge(x)\, e^{2\pi i x \cdot y}\, dx \to f(y)$$

in the appropriate $L(p, q)$ norm as $R \to \infty$.

THEOREM 4.3 *Suppose* $1 \leqq q \leqq \infty$, $1 < p < 2$ *and* $1/p + 1/p' = 1$.

(a) $f \in L(p, q)$ *if and only if for all* F *such that* $F^* = f^*$, *there exists* $F^\wedge = g \in L(p', q)$. *Furthermore,* $g^\wedge = F$ a.e. *and* $\| g \|^*_{p'q} \leqq B \| f \|^*_{pq}$;

(b) $g \in L(p', q)$ *if and only if, for some* G *such that* $G^* = g^*$, *there exists* $G^\vee = f \in L(p,q)$. *Furthermore,* $f^\vee = G$ q.e. *and* $\| g \|^*_{p'q} \leqq B \| f \|^*_{pq}$.

The proof of Theorem 4.3 depends on a result which is a slight extension of a lemma found in [32, Vol. II, p. 129]:

LEMMA 4.4. *Suppose* $f(t)$ *is non-negative, locally integrable and an even function of* t, $-\infty < t < \infty$. *Further, suppose* $f(t)$ *is non-increasing on* $(0, \infty)$ *and* $f(t) \to 0$ *as* $t \to \infty$. *Then* $g(x) = \int_0^\infty f(t) \cos xt\, dt \in L(r, q)$ *if and only if* $f \in L(r', q)$, *where* $1 \leqq q \leqq \infty$, $1 < r < \infty$ *and* $1/r + 1/r' = 1$.

Proof. Suppose $g \in L(r, q)$. Let $G(x) = \int_0^x g(y)\, dy$. Then $| G(x) | \leqq | x | g^{**}(| x |)$. Elementary arguments show that

$$G(x) = \int_0^\infty f(t) \sin xt \frac{dt}{t}\,,$$

and then $| G(x) | \geqq B f(1/| x |)$. (See [32, Vol. II, p. 129].) It follows that

$f^*(t) \leqq B(1/t)\, g^{**}(1/t)$, $t > 0$. A change of variables and Hardy's inequality then show that $\|f\|^*_{r'q} \leqq B\|g\|^*_{rq}$.

Conversely, suppose $f \in L(r', q)$. We have

$$|g(x)| \leqq B \int_0^{1/|x|} f(y)\, dy\,.$$

(See [32, Vol. II, p. 129].) Hence, $|g(x)|$ is majorized by $(1/|x|) f^{**}(1/|x|)$. It follows that $g^*(t) \leqq B(1/t) f^{**}(1/t)$, $t > 0$. As above, this implies that $\|g\|^*_{rq} \leqq B\|f\|^*_{r'q}$.

Proof of Theorem 4.3. $F^\wedge$ and $G^\vee$ are given by (4.1) and (4.2). The inequalities are obtained from the weak type interpolation theorem and the end point results $\|f^\wedge\|_2 = \|f\|_2$ and $\|f^\wedge\|_\infty \leqq \|f\|_1$. The theorem is then clear for $n = 1$, since $2g = f^\wedge = f^\vee$ for functions of the type described in Lemma 4.4. For $n > 1$ use special functions of the form

$$f(x_1) \cdot \chi_{[0,1]}(x_2) \cdots \chi_{[0,1]}(x_n)\,,$$

where $x = (x_1, \ldots, x_n)$ and f is as in Lemma 4.4.

We prove a multiplication theorem for functions belonging to $L(p, q)$ spaces. This result is used to prove a convolution theorem for the $L(p, q)$ spaces which are Banach spaces. Note that functions which do not belong to one of the Banach spaces are not appropriate for convolution since they are not necessarily locally integrable.

THEOREM 4.5. (Multiplication theorem):

$$\|fg\|^*_{pq} \leqq B\|f\|_{p_0 q_0}\|g\|_{p_1 q_1}\,,$$

where $1/p = 1/p_0 + 1/p_1$ *and* $1/q = 1/q_0 + 1/q_1$.

Proof. Applying Holder's inequality twice, we obtain $(fg)^{**}(t, r) \leqq f^{**}(t, 2r)\, g^{**}(t, 2r)$ and then the theorem.

Suppose (G, dm) is a locally compact unimodular topological group, where dm is Haar measure on the group G. The convolution of two functions is then defined by $f * g(x) = \int f(y)\, g(xy^{-1})\, dm(y)$, provided the integral exists. We develop a convolution theorem for $L(p, q)$ spaces by interpolating certain end point results.

LEMMA 4.6. $\|f * g\|^*_{p_1\infty} \leqq B\|f\|^*_{11}\|g\|^*_{p_1\infty}$, $1 < p_1 < \infty$.

Proof. $$(f * g)^*(t) \leqq \sup_{m(E) \geqq t} \frac{1}{m(E)} \int_E |f * g(x)|\, dm(x)\,.$$

By Fubini's theorem

$$\frac{1}{m(E)}\int_E |f*g(x)|\,dm(x) \leqq \frac{1}{m(E)}\int_E \Big(\int |f(y)|\cdot|g(xy^{-1})|\,dm(y)\Big)dm(x)$$

$$= \int\Big(\frac{1}{m(E)}\int_E |g(xy^{-1})|\,dm(x)\Big)|f(y)|\,dm(y),$$

but

$$\frac{1}{m(E)}\int_E |g(xy^{-1})|\,dm(x) \leqq g^{**}(t) \leqq Bt^{-1/p_1}\,||\,g\,||^*_{p_1\infty}\,.$$

It follows that $t^{1/p_1}(f*g)^*(t) \leqq B\,||f||^*_{11}\cdot||\,g\,||^*_{p_1\infty}$.

LEMMA 4.7. $||\,f*g\,||^*_{\infty\infty} \leqq B\,||\,f\,||^*_{p_1'1}\,||\,g\,||^*_{p_1\infty}$, *where* $1 < p_1 < \infty$ *and* $1/p_1 + 1/p_1' = 1$.

Proof. $|f*g(x)| = |\int f(y)\,g(xy^{-1})\,dm(y)|$. By (1.9) this is majorized by $\int_0^\infty f^*(t)\,g^*(t)\,dt$, which is dominated by $||\,g\,||^*_{p_1\infty}\int_0^\infty f^*(t)^{-1/p_1}\,dt$.

By applying the weak type interpolation theorem to the end point results of Lemma 4.6 and Lemma 4.7 we obtain

LEMMA 4.8. $||\,f*g\,||^*_{pq} \leqq B\,||\,f\,||^*_{p_0q}\,||\,g\,||^*_{p_1\infty}$, *where* $0 < 1/p = 1/p_0 + 1/p_1 - 1 < 1$, $1 \leqq q \leqq \infty$ *and* $1 < p_0, p_1 < \infty$.

Lemma 4.8 contains the fractional integration theorem of Hardy and Littlewood [12] and Stein and Weiss [20]. It is interesting to note that it is not true that

$$(*)\qquad ||\,f*g\,||^*_{\infty\infty} \leqq B\,||\,f\,||^*_{p_0p_0}\,||\,g\,||^*_{p_1\infty},$$

B independent of f and g, $1/p_0 + 1/p_1 = 1$. (See [12].) Hence, the classical Maerinkiewicz interpolation theorem for L^p spaces does not apply directly to obtain Lemma 4.8. The Stein-Weiss extension of the Marcinkiewicz theorem does apply directly. (See [30].) Their theorem uses the end point result that (*) is true if f is restricted to the class of characteristic functions of measurable sets of finite measure. According to (2.5) this is equivalent to the end point result of Lemma 4.7.

LEMMA 4.9. $||\,f*g\,||^*_{p1} \leqq B\,||\,f\,||^*_{p_0q_0}\,||\,g\,||^*_{p_1q_1}$, *where* $0 < 1/p = 1/p_0 + 1/p_1 - 1 < 1$, $1/q_0 + 1/q_1 = 1$ *and* $1 < p_0, p_1 < \infty$.

Proof. From (2.7), we have

$$||f^* g||^*_{p1} = \sup_h B \,|\int f^* g(x) h(x)\, dm(x)|\,,$$

where $h^*(t) \leqq t^{(1/p)-1}$. Let $I(h) = \int f^* g(x) h(x)\, dm(x)$.

$$|I(h)| \leqq \int (\int |f(y)| \cdot |g(xy^{-1})|\, dm(y)\, |h(x)|\, dm(x)$$
$$= \int |f(y)| (\int |g(xy^{-1})| \cdot |h(x)|\, dm(x)) dm(y)\,.$$

Hence, $|I(h)| \leqq ||fk||^*_{11}$, where $k(y) = \int |g(xy^{-1})| . |h(x)|\, dm(x)$. By the multiplication theorem it follows that $|I(h)| \leqq B\, ||f||^*_{p_0 q_0}\, ||k||^*_{p'_0 q_1}$, where $1/p_0 + 1/p'_0 = 1$. But $k = |\bar{g}| * |h|$, where $\bar{g}(x) = g(x^{-1})$. Hence by Lemma 4.8,

$$||k||^*_{p'_0 q_1} \leqq B\, ||\bar{g}||^*_{p_1 q_1}\, ||h||^*_{p' \infty} \leqq B\, ||\bar{g}||^*_{p_1 q_1}\,.$$

Since (G, dm) is unimodular, we have $(\bar{g})^*(t) = g^*(t)$ and the lemma follows.

By applying the strong type interpolation theorem to the end point results of Lemma 4.8 and Lemma 4.9, we obtain

THEOREM 4.10. (Convolution theorem):

$$||f^* g||^*_{pq} \leqq B\, ||f||^*_{p_0 q_0}\, ||g||^*_{p_1 q_1}\,,$$

where $0 < 1/p = 1/p_0 + 1/p_1 - 1 < 1$, $1 < p_0, p_1 < \infty$ *and* $0 \leqq 1/q = 1/q_0 + 1/q_1 \leqq 1$.

Section 5. REFERENCES

Various properties of $L(p, q)$ spaces have appeared in many places, often as special cases of a more general theory. We will mention several places where related results and applications are found. The references given are not necessarily the first or the only place where the indicated result appears.

The principal references are [19] and [20], where G. G. Lorentz defines special cases of $L(p, q)$ spaces and proves many of their properties. The notion of a non-increasing rearrangement of a function was used by Hardy Littlewood and Payley. (See [32].) A simple proof of the inequality $||f||^*_{pq_2} \leqq B\, ||f||^*_{pq_1}$, $q_1 \leqq q_2$, is found in O'Neil [22]. The technique used in the proof of Hardy's inequality is well known. A different proof of Hardy's inequality is found in [9, p. 245] or [32, Vol. I, p. 20]. The author learned the simple proof of inequality (1.9) from class notes from a course given by A. Zygmund in Chicago. A similar proof is given in [11, p. 278].

The $L(p, q)$ spaces which are Banach spaces appear as intermediate spaces in the general interpolation theory of Calderón [1]. Peetre [24] identifies $L(p, q)$ spaces as intermediate spaces for the interpolation theory of Lions and Peetre [18]. Many $L(p, q)$ results are then contained in these general theories. In particular, there are results concerning density, inclusion, separability and duality of the spaces. A ** norm is used in these results. Riviere [25] generalized the results of Calderón [1] to include $L(p, q)$, $p, q > 0$. Similarly, P. Kree and J. Peetre generalized the results of Lions and Peetre [18].

(2.5) is proved by Krein and Semenov [18] and is contained implicitly in Stein and Weiss [30]. Halperin [8] and [9] obtains general results on conjugate spaces and reflexitivity. Results on uniform convexity of some related spaces are found in Halperin [10]. The results concerning linear functionals on $L(p, q)$, $p < 1$, correspond to results of Day [5] for L^p spaces $0 < p < 1$.

The weak type theorem of Section 3 restricted to linear operators on the $L(p, q)$ spaces which are Banach spaces was proved by A. P. Calderón [2]. We learned that E. M Stein also obtain these results. Proof of these cases is found in Lions and Peetre [18], together with Peetre [24]. Also see Calderón [1] and Oklander [21]. Krein and Semonov [17] prove some special cases. The theorem is closely related to results of Stein and Weiss [29] and [30]. The weak type theorem for $L(p, q)$, $p, q > 0$, is proved in Hunt [14]. These cases are also contained in work of P. Kree and J. Peetre.

The strong type theorem of Section 3 for linear operators on the $L(p, q)$ spaces which are Banach spaces is found in Calderón [1]. These results are related to results of Hirschman [13], Stein [26] and Stein and Weiss [27]. The result for sublinear operators follows ideas found in Calderón [1], Calderón and Zygmund [3] and Weiss [31]. Rivière [25] obtains results for linear operators acting on $L(p, q)$ spaces, $p, q > 0$.

Stein [26] proves an analogue of Theorem 4.3 for Fourier coefficients. The multiplication and convolution theorems are proved by a different method in O'Neil [22]. E. M. Stein also obtained these results. (See [22].)

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