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Autor(en): **Abian, Alexander**

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THE STONE SPACE OF A BOOLEAN RING

by Alexander ABIAN *1)

This is an expository paper reproducing some of the basic results in [1] and [2].

DEFINITION 1. — *A ring B is called Boolean, if*

$$x^2 = x, \quad \text{for every } x \in B. \quad (1)$$

In what follows B shall represent a given Boolean ring.

The following are well known immediate consequences of Definition 1.

$$x + x = 0, \quad (2)$$

$$xy = yx, \quad (3)$$

$$xy(x + y) = 0, \quad (4)$$

for every two elements x and y of B .

NOTATION. — *In what follows, for every non-zero element x of B , $p(x)$ shall represent a prime ideal of B not containing x , i.e.,*

$$x \notin p(x)$$

and

$P(x)$ shall represent the set of all prime ideals $p(x)$, for a given x .

LEMMA 1. — *Let I be an ideal of B and x an element of B such that $x \notin I$. Then there exists a prime ideal $p(x)$ such that $I \subset p(x)$.*

PROOF. — By Zorn's Lemma, in view of (1) and (3), there exists a largest ideal M of B such that $I \subset M$ and $x \notin M$. It can be easily verified that the ideal M is prime [3].

1)* Formerly, Smbat Abian.

Let us observe that since 0 is an element of every ideal of B , hence, in view of Lemma 1,

$$P(x) = \phi \quad \text{if and only if } x = 0. \quad (5)$$

Now, we prove that for every two elements x and y of B ,

$$P(xy) = P(x) \cap P(y) \quad (6)$$

To prove (6), let us observe that since $p(xy)$ is an ideal not containing xy , hence, $x \notin p(xy)$ and $y \notin p(xy)$. Thus, $P(xy) \subset (P(x) \cap P(y))$. Conversely, since $p(x)$ is a prime ideal not containing x , hence, if $y \notin p(x)$ then $xy \notin p(x)$. Thus, $(P(x) \cap P(y)) \subset P(xy)$.

From (6) it follows that for every two elements x and y of B ,

$$xy = x \quad \text{implies } P(x) \subset P(y) \quad (7)$$

Since $p(x+y)$ is an ideal not containing $x+y$, hence $x \in p(x+y)$ implies $y \notin p(x+y)$. Therefore, for every two elements x and y of B ,

$$P(x+y) \subset (P(x) \cup P(y)) \quad (8)$$

Further, in view of (4), (5) and (6),

$$P(xy) \cap P(x+y) = \phi$$

so that, in view of (8), for every two elements x and y of B ,

$$P(x+y) \subset (P(x) \oplus P(y)), \quad (9)$$

where \oplus is the usual set-theoretical *symmetric difference* operator. Also, let us observe that since $p(x)$ is an ideal not containing x , hence, if $p(x) \notin P(y)$ then $p(x) \in P(x+y)$. Similarly, if $p(y) \notin P(x)$ then $p(y) \in P(x+y)$. Thus,

$$P(x) \oplus P(y) \subset P(x+y)$$

so that in view of (9), for every two elements x and y of B ,

$$P(x+y) = P(x) \oplus P(y). \quad (10)$$

Let us observe that since

$$P(y) - P(x) = (P(y) \oplus P(x)) \cap P(y),$$

hence, in view of (10), (6) and (1), for every two elements x and y of B ,

$$P(y) - P(x) = P(y + xy). \quad (11)$$

Also, in view of (8), for every positive natural number n ,

$$x = \sum_{i=1}^n c_i \text{ implies } P(x) \subset \bigcup_{i=1}^n P(c_i) \quad (12)$$

where c_i is an element of B . Moreover, in view of (1) and (7),

$$P(ca) \subset P(a), \quad (13)$$

where c and a are any two elements of B .

Now, let \mathcal{P} represent the set of all proper prime ideals of B .

THEOREM 1. — *The Boolean ring B is isomorphic to a subring of the algebra of all subsets of \mathcal{P} .*

PROOF. — In view of (5), (6) and (10), the mapping f from B into the power set of \mathcal{P} , given by

$$f(x) = P(x)$$

establishes the desired isomorphism.

Next, in view of (6), we introduce a topology \mathcal{T} in \mathcal{P} such that, for every $x \in B$ the subset $P(x)$ of \mathcal{P} is a basis element of \mathcal{T} .

DEFINITION 2. — *The topological space $(\mathcal{P}, \mathcal{T})$ is called the Stone space of B .*

LEMMA 2. — *In the space $(\mathcal{P}, \mathcal{T})$, every basis element is closed.*

PROOF. — Let $P(x)$ be a basis element and let $p(y) \notin P(x)$. Clearly, $p(y) \in (P(y) - P(x))$ and hence in view of (11),

$$p(y) \in P(y + xy).$$

Thus, an element $p(y)$, in the complement $\mathcal{P} - P(x)$ of $P(x)$, is contained in a basis element $P(y + xy)$ which is disjoint from $P(x)$. Hence $P(x)$ is closed.

LEMMA 3. — *The space $(\mathcal{P}, \mathcal{T})$ is totally disconnected.*

PROOF. — Let $p(x)$ and $p(y)$ be two distinct elements of \mathcal{P} . Thus, there exists $z \in B$ such that, say, $z \in p(x)$ and $z \notin p(y)$. But

then $P(yz)$ is a basis element containing $p(y)$ and not containing $p(x)$. Consequently, in view of Lemma 2, every two distinct elements $p(x)$ and $p(y)$ of \mathcal{P} are contained in two mutually disjoint closed sets of $(\mathcal{P}, \mathcal{T})$ whose union is \mathcal{P} . Thus, $(\mathcal{P}, \mathcal{T})$ is totally disconnected (and in particular, Hausdorff).

LEMMA 4. — *The space $(\mathcal{P}, \mathcal{T})$ is locally compact.*

PROOF. — It is sufficient to prove that every basis element $P(x)$ of $(\mathcal{P}, \mathcal{T})$ is compact. Now, let $A \subset B$ and $\bigcup_{y \in A} P(y)$ be a covering of $P(x)$, i.e.,

$$P(x) \subset \bigcup_{y \in A} P(y) \tag{14}$$

Let (A) denote the ideal generated by the elements of A . Claim that $x \in (A)$. Assume the contrary that $x \notin (A)$. But then, in view of Lemma 1, there exists a prime ideal $p(x)$ such that $(A) \subset p(x)$, and therefore, $p(x) \not\subset \bigcup_{y \in A} P(y)$, contradicting (14).

Hence, our assumption is false and indeed, $x \in (A)$. Consequently, there exists a natural number n such that

$$x = \sum_{i=1}^n (m_i + b_i) a_i$$

where m_i is an integer, $b_i \in B$ and $a_i \in A$. But then, in view of (12) and (13),

$$P(x) \subset \bigcup_{i=1}^n P((m_i + b_i) a_i) \subset \bigcup_{y \in A} P(y)$$

asserting that, in view of (14), $\bigcup_{i=1}^n P((m_i + b_i) a_i)$ is a finite subcover of an arbitrary cover $\bigcup_{y \in A} P(y)$ of $P(x)$. Thus, indeed,

$P(x)$ is compact and $(\mathcal{P}, \mathcal{T})$ is locally compact.

Finally, in view of Lemmas 3 and 4, and Theorem 1, we have,

THEOREM 2. — *Every Boolean ring is isomorphic to a subring of the algebra of all subsets of its Stone space which is totally disconnected and locally compact.*

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The Ohio State University
Columbus, Ohio.