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THE STONE SPACE OF A BOOLEAN RING

by Alexander Abian *1)

This is an expository paper reproducing some of the basic results in [1] and [2].

Definition 1. — A ring B is called Boolean, if

$$x^2 = x$$
, for every $x \in B$. (1)

In what follows B shall represent a given Boolean ring.

The following are well known immediate consequences of Definition 1.

$$x + x = 0, (2)$$

$$xy = yx, (3)$$

$$xy\left(x+y\right) \,=\,0\,,\tag{4}$$

for every two elements x and y of B.

Notation. — In what follows, for every non-zero element x of B, p(x) shall represent a prime ideal of B not containing x, i.e., $x \notin p(x)$

and

P(x) shall represent the set of all prime ideals p(x), for a given x.

Lemma 1. — Let I be an ideal of B and x an element of B such that $x \notin I$. Then there exists a prime ideal p(x) such that $I \subset p(x)$.

PROOF. — By Zorn's Lemma, in view of (1) and (3), there exists a largest ideal M of B such that $I \subset M$ and $x \notin M$. It can be easily verified that the ideal M is prime [3].

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Let us observe that since 0 is an element of every ideal of B, hence, in view of Lemma 1,

$$P(x) = \phi \quad \text{if and only if } x = 0. \tag{5}$$

Now, we prove that for every two elements x and y of B,

$$P(xy) = P(x) \cap P(y) \tag{6}$$

To prove (6), let us observe that since p(xy) is an ideal not containing xy, hence, $x \notin p(xy)$ and $y \notin p(xy)$. Thus, $P(xy) \subset (P(x) \cap P(y))$. Conversely, since p(x) is a prime ideal not containing x, hence, if $y \notin p(x)$ then $xy \notin p(x)$. Thus, $(P(x) \cap P(y)) \subset P(xy)$.

From (6) it follows that for every two elements x and y of B,

$$xy = x$$
 implies $P(x) \subset P(y)$ (7)

Since p(x+y) is an ideal not containing x+y, hence $x \in p(x+y)$ implies $y \notin p(x+y)$. Therefore, for every two elements x and y of B,

$$P(x+y) \subset (P(x) \cup P(y)) \tag{8}$$

Furthere, in view of (4), (5) and (6),

$$P(xy) \cap P(x+y) = \phi$$

so that, in view of (8), for every two elements x and y of B,

$$P(x+y) \subset (P(x) \oplus P(y)), \tag{9}$$

where \oplus is the usual set-theoretical symmetric difference operator. Also, let us observe that since p(x) is an ideal not containing x, hence, if $p(x) \notin P(y)$ then $p(x) \in P(x+y)$. Similarly, if $p(y) \notin P(x)$ then $p(y) \in P(x+y)$. Thus,

$$P(x) \oplus P(y) \subset P(x+y)$$

so that in view of (9), for every two elements x and y of B,

$$P(x+y) = P(x) \oplus P(y). \tag{10}$$

Let us observe that since

$$P(y) - P(x) = (P(y) \oplus P(x)) \cap P(y),$$

hence, in view of (10), (6) and (1), for every two elements x and y of B,

$$P(y) - P(x) = P(y + xy).$$
 (11)

Also, in view of (8), for every positive natural number n,

$$x = \sum_{i=1}^{n} c_{i} \quad implies \quad P(x) \subset \bigcup_{i=1}^{n} P(c_{i})$$
 (12)

where c_i is an element of B. Moreover, in view of (1) and (7),

$$P(ca) \subset P(a), \tag{13}$$

where c and a are any two elements of B.

Now, let \mathscr{P} represent the set of all proper prime ideals of B.

Theorem 1. — The Boolean ring B is isomorphic to a subring of the algebra of all subsets of \mathcal{P} .

PROOF. — In view of (5), (6) and (10), the mapping f from B into the power set of \mathcal{P} , given by

$$f(x) = P(x)$$

establishes the desired isomorphism.

Next, in view of (6), we introduce a topology \mathscr{T} in \mathscr{P} such that, for every $x \in B$ the subset P(x) of \mathscr{P} is a basis element of \mathscr{T} .

Definition 2. — The topological space $(\mathcal{P}, \mathcal{T})$ is called the Stone space of B.

Lemma 2. — In the space $(\mathcal{P}, \mathcal{T})$, every basis element is closed.

PROOF. — Let P(x) be a basis element and let $p(y) \notin P(x)$. Clearly, $p(y) \in (P(y) - P(x))$ and hence in view of (11),

$$p(y) \varepsilon P(y+xy)$$
.

Thus, an element p(y), in the complement $\mathcal{P}-P(x)$ of P(x), is contained in a basis element P(y+xy) which is disjoint from P(x). Hence P(x) is closed.

Lemma 3. — The space $(\mathcal{P}, \mathcal{T})$ is totally disconnected.

PROOF. — Let p(x) and p(y) be two distinct elements of \mathscr{P} . Thus, there exists $z \in B$ such that, say, $z \in p(x)$ and $z \notin p(y)$. But

then P(yz) is a basis element containing p(y) and not containing p(x). Consequently, in view of Lemma 2, every two distinct elements p(x) and p(y) of \mathscr{P} are contained in two mutually disjoint closed sets of $(\mathscr{P}, \mathscr{T})$ whose union is \mathscr{P} . Thus, $(\mathscr{P}, \mathscr{T})$ is totally disconnected (and in particular, Hausdorff).

Lemma 4. — The space $(\mathcal{P}, \mathcal{T})$ is locally compact.

PROOF. — It is sufficient to prove that every basis element P(x) of $(\mathscr{P}, \mathscr{T})$ is compact. Now, let $A \subset B$ and $\bigcup_{y \in A} P(y)$ be a covering of P(x), i.e.,

$$P(x) \subset \bigcup_{y \in A} P(y) \tag{14}$$

Let (A) denote the ideal generated by the elements of A. Claim that $x \in (A)$. Assume the contrary that $x \notin (A)$. But then, in view of Lemma 1, there exists a prime ideal p(x) such that $(A) \subset p(x)$, and therefore, $p(x) \notin \bigcup_{y \in A} P(y)$, contradicting (14). Hence, our assumption is false and indeed $x \in (A)$. Conso

Hence, our assumption is false and indeed, $x \varepsilon(A)$. Consequently, there exists a natural number n such that

$$x = \sum_{i=1}^{n} (m_i + b_i) a_i$$

where m_i is an integer, $b_i \varepsilon B$ and $a_i \varepsilon A$. But then, in view of (12) and (13),

$$P(x) \subset \bigcup_{i=1}^{n} P((m_i + b_i) | a_i) \subset \bigcup_{y \in A} P(y)$$

asserting that, in view of (14), $\bigcup_{i=1}^{n} P((m_i + b_i) a_i)$ is a finite sub-

cover of an arbitrary cover $\bigcup_{y \in A} P(y)$ of P(x). Thus, indeed,

P(x) is compact and $(\mathcal{P}, \mathcal{T})$ is locally compact.

Finally, in view of Lemmas 3 and 4, and Theorem 1, we have,

Theorem 2. — Every Boolean ring is isomorphic to a subring of the algebra of all subsets of its Stone space which is totally disconnected and locally compact.

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