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Finally assume that there exists an  $x_1 \geq x_0$  such that we have for all  $x, u$  with  $x \geq u \geq x_1$ :

$$\frac{\Psi'(x)}{\Psi'(u)} \geq \frac{1}{\beta} \quad (x \geq u \geq x_1). \quad (26)$$

Then the series (18) is divergent.

22. Observe that the condition (26) is certainly satisfied from a certain  $x_1$  on, if  $\Psi(x)$  has a finite limit  $\omega$ ,

$$\Psi'(x) \rightarrow \omega < \infty \quad (x \rightarrow \infty). \quad (27)$$

23. *Proof of the Theorem 6.* Since  $x_0$  can be replaced by any greater number we can assume, without loss of generality, that  $x_1 = x_0$ . Then we proceed as in the proof of the Theorem 4 defining  $F(x)$  by (19) and obtain, as in the section 15, using (26):

$$\begin{aligned} F(\Psi(x)) \Psi'(x) &= \lim_{\kappa \rightarrow \infty} f(\Psi(v_\kappa)) \Psi'(x) \geq \frac{1}{\beta} \overline{\lim}_{\kappa \rightarrow \infty} f(\Psi(v_\kappa)) \Psi'(v_\kappa) \\ &\geq \overline{\lim}_{\kappa \rightarrow \infty} f(v_\kappa) \geq F(x). \end{aligned}$$

24. We see that  $F(x)$  satisfies the conditions of the Theorem 2; therefore the integral  $\int_0^\infty F(x) dx$  is divergent and the same holds for the series  $\sum_0^\infty F(\rho)$ , as  $F(x)$  is monotonically decreasing. But then the series (18) is also divergent since  $f(x)$  is a majorant of  $F(x)$ . The Theorem 6 is proved.

#### IV. ANOTHER METHOD IN THE CASE OF DIVERGENCE

25. **THEOREM 7.** *The assertion of the Theorem 4 remains valid if the assumption that  $\Psi'(x)$  is monotonically increasing is replaced by the assumption that  $\Psi'(x)$  is monotonically decreasing.*

26. *Proof.* Since in any case  $\Psi'(x) \geq 0$  there exists a finite  $\omega$  such that

$$\Psi'(x) \downarrow \omega \quad (x \rightarrow \infty)$$

and, as in the sec. 17, we see that this limit is  $\geq 1$ .

Define the  $a_v$  as in the sec. 18. Since we can multiply  $f(x)$  by any fixed constant, we can assume that we have:

$$f(x) \geq 1 \quad (a_0 \leq x \leq a_1).$$

27. Denote the inverse function of  $\Psi(x)$  by  $\sigma(x) \equiv \sigma_1(x)$  and its iterated  $\sigma(\sigma(x))$ ,  $\sigma(\sigma(\sigma(x)))$ , ... by  $\sigma_2(x)$ ,  $\sigma_3(x)$ , ... and define a new function  $F(x)$  in such a way that we have:

$$F(x) = \frac{F(\sigma(x))}{\Psi'(\sigma(x))} \quad (x \geq a_1). \quad (28)$$

For this purpose we put:

$$F(x) = 1 \quad (a_0 \leq x < a_1), \quad F(x) = \frac{1}{\Psi'(\sigma_1(x))} \quad (a_1 \leq x < a_2), \dots$$

$$F(x) = \prod_{v=1}^n \frac{1}{\Psi'(\sigma_v(x))} \quad (a_n \leq x < a_{n+1}), \quad (29)$$

and (28) follows immediately.

28. From (29) we have for  $x = a_n$  and  $x \uparrow a_{n+1}$ :

$$F(a_n) = \prod_{v=0}^{n-1} \frac{1}{\Psi'(a_v)}, \quad F(a_{n+1} - 0) = \prod_{v=1}^n \frac{1}{\Psi'(a_v)},$$

and, putting  $\frac{1}{\Psi'(a_0)} = \sigma_0 \leq 1$ :

$$\frac{F(a_n)}{F(a_n - 0)} = \sigma_0 = \frac{1}{\Psi'(a_0)}. \quad (30)$$

Since we have  $\omega \geq 1$ ,  $\Psi'(x) \geq 1$ , we see that  $\Psi(x) - x$  is non-decreasing, and therefore, the same holds for the length of the  $n$ -th interval between the  $a_v$ ,  $a_{n+1} - a_n$ . The number of the  $a_v$  lying in an interval of the length 1 in the half-line  $x \geq a_0$  has a finite upper bound which may be denoted by  $k$ .

29. From (29) it follows obviously that  $F(x)$  is continuous and monotonically increasing in any half-open interval  $\langle a_n, a_{n+1} \rangle$ . In the points  $a_v$  we have a discontinuity if  $\sigma_0 \neq 1$ . We can therefore write for any  $y \geq a_0$ :

$$F(y) \geq \sigma_0^k F(x) \quad (y - 1 \leq x \leq y). \quad (31)$$

30. Take here as  $y$  an integer  $m \geq a_0$ , multiply by  $dx$  and integrate from  $m - 1$  to  $m$ ; we obtain

$$F(m) \geq \sigma_0^k \int_{m-1}^m F(x) dx$$

and therefore, denoting by  $n_0 - 1$  the first integer  $> a_0$ :

$$\sum_{v=n_0}^n F(v) \geq \sigma_0^k \int_{n_0-1}^n F(x) dx.$$

31. On the other hand, the relation (28) can be written as

$$F(\Psi(x)) \Psi'(x) = F(x), \quad (32)$$

and it follows therefore from the Theorem 2 that  $\int_{n_0-1}^{\infty} F(x) dx$  is divergent. We see that the series  $\sum_{v=n_0}^{\infty} F(v)$  diverges too.

32. In order to prove our Theorem it is therefore sufficient to prove that we have

$$f(x) \geq F(x) \quad (x \geq a_0). \quad (33)$$

But this relation is evident in the interval  $\langle a_0, a_1 \rangle$ . Comparing (17) and (32) this inequality follows also for the interval  $\langle a_1, a_2 \rangle$  and from there on by induction for any  $x \geq a_0$ . The Theorem 7 is proved.

## V. NEW CONDITIONS FOR THE EULER-MACLAURIN THEOREM

33. One of the ideas underlying the proof of the Theorem 6 was the introduction of the condition (26) which is a kind of weakened monotony condition.

We give in what follows the corresponding generalisation of the Euler-Maclaurin convergence criterion, in which we try to weaken the monotony condition even more. Combining the conditions of the Theorem 8 with the assumptions of the Theorems 1 and 2 we obtain then further criteria for the convergence and divergence of the series (18).