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Finally assume that there exists an $x_1 \geq x_0$ such that we have for all x, u with $x \geq u \geq x_1$:

$$\frac{\Psi'(x)}{\Psi'(u)} \geq \frac{1}{\beta} (x \geq u \geq x_1). \quad (26)$$

Then the series (18) is divergent.

22. Observe that the condition (26) is certainly satisfied from a certain x_1 on, if $\Psi(x)$ has a finite limit ω ,

$$\Psi'(x) \rightarrow \omega < \infty (x \rightarrow \infty). \quad (27)$$

23. *Proof of the Theorem 6.* Since x_0 can be replaced by any greater number we can assume, without loss of generality, that $x_1 = x_0$. Then we proceed as in the proof of the Theorem 4 defining $F(x)$ by (19) and obtain, as in the section 15, using (26):

$$\begin{aligned} F(\Psi(x)) \Psi'(x) &= \lim_{\kappa \rightarrow \infty} f(\Psi(v_\kappa)) \Psi'(x) \geq \frac{1}{\beta} \overline{\lim}_{\kappa \rightarrow \infty} f(\Psi(v_\kappa)) \Psi'(v_\kappa) \\ &\geq \overline{\lim}_{\kappa \rightarrow \infty} f(v_\kappa) \geq F(x). \end{aligned}$$

24. We see that $F(x)$ satisfies the conditions of the Theorem 2; therefore the integral $\int_0^\infty F(x) dx$ is divergent and the same holds for the series $\sum_0^\infty F(\nu)$, as $F(x)$ is monotonically decreasing. But then the series (18) is also divergent since $f(x)$ is a majorant of $F(x)$. The Theorem 6 is proved.

IV. ANOTHER METHOD IN THE CASE OF DIVERGENCE

25. **THEOREM 7.** *The assertion of the Theorem 4 remains valid if the assumption that $\Psi'(x)$ is monotonically increasing is replaced by the assumption that $\Psi'(x)$ is monotonically decreasing.*

26. *Proof.* Since in any case $\Psi'(x) \geq 0$ there exists a finite ω such that

$$\Psi'(x) \downarrow \omega \quad (x \rightarrow \infty)$$

and, as in the sec. 17, we see that this limit is ≥ 1 .

Define the a_v as in the sec. 18. Since we can multiply $f(x)$ by any fixed constant, we can assume that we have:

$$f(x) \geq 1 \quad (a_0 \leq x \leq a_1).$$

27. Denote the inverse function of $\Psi(x)$ by $\sigma(x) \equiv \sigma_1(x)$ and its iterated $\sigma(\sigma(x))$, $\sigma(\sigma(\sigma(x)))$, ... by $\sigma_2(x)$, $\sigma_3(x)$, ... and define a new function $F(x)$ in such a way that we have:

$$F(x) = \frac{F(\sigma(x))}{\Psi'(\sigma(x))} \quad (x \geq a_1). \quad (28)$$

For this purpose we put:

$$F(x) = 1 \quad (a_0 \leq x < a_1), \quad F(x) = \frac{1}{\Psi'(\sigma_1(x))} \quad (a_1 \leq x < a_2), \dots$$

$$F(x) = \prod_{v=1}^n \frac{1}{\Psi'(\sigma_v(x))} \quad (a_n \leq x < a_{n+1}), \quad (29)$$

and (28) follows immediately.

28. From (29) we have for $x = a_n$ and $x \uparrow a_{n+1}$:

$$F(a_n) = \prod_{v=0}^{n-1} \frac{1}{\Psi'(a_v)}, \quad F(a_{n+1}-0) = \prod_{v=1}^n \frac{1}{\Psi'(a_v)},$$

and, putting $\frac{1}{\Psi'(a_0)} = \sigma_0 \leq 1$:

$$\frac{F(a_n)}{F(a_n-0)} = \sigma_0 = \frac{1}{\Psi'(a_0)}. \quad (30)$$

Since we have $\omega \geq 1$, $\Psi'(x) \geq 1$, we see that $\Psi(x) - x$ is non-decreasing, and therefore, the same holds for the length of the n -th interval between the a_v , $a_{n+1} - a_n$. The number of the a_v lying in an interval of the length 1 in the half-line $x \geq a_0$ has a finite upper bound which may be denoted by k .

29. From (29) it follows obviously that $F(x)$ is continuous and monotonically increasing in any half-open interval $\langle a_n, a_{n+1} \rangle$. In the points a_v we have a discontinuity if $\sigma_0 \neq 1$. We can therefore write for any $y \geq a_0$:

$$F(y) \geq \sigma_0^k F(x) \quad (y-1 \leq x \leq y). \quad (31)$$

30. Take here as y an integer $m \geq a_0$, multiply by dx and integrate from $m - 1$ to m ; we obtain

$$F(m) \geq \sigma_0^k \int_{m-1}^m F(x) dx$$

and therefore, denoting by $n_0 - 1$ the first integer $> a_0$:

$$\sum_{v=n_0}^n F(v) \geq \sigma_0^k \int_{n_0-1}^n F(x) dx.$$

31. On the other hand, the relation (28) can be written as

$$F(\Psi(x)) \Psi'(x) = F(x), \quad (32)$$

and it follows therefore from the Theorem 2 that $\int_{n_0-1}^{\infty} F(x) dx$ is divergent. We see that the series $\sum_{v=n_0}^{\infty} F(v)$ diverges too.

32. In order to prove our Theorem it is therefore sufficient to prove that we have

$$f(x) \geq F(x) \quad (x \geq a_0). \quad (33)$$

But this relation is evident in the interval $\langle a_0, a_1 \rangle$. Comparing (17) and (32) this inequality follows also for the interval $\langle a_1, a_2 \rangle$ and from there on by induction for any $x \geq a_0$. The Theorem 7 is proved.

V. NEW CONDITIONS FOR THE EULER-MACLAURIN THEOREM

33. One of the ideas underlying the proof of the Theorem 6 was the introduction of the condition (26) which is a kind of weakened monotony condition.

We give in what follows the corresponding generalisation of the Euler-Maclaurin convergence criterion, in which we try to weaken the monotony condition even more. Combining the conditions of the Theorem 8 with the assumptions of the Theorems 1 and 2 we obtain then further criteria for the convergence and divergence of the series (18).